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Steenrod Operations in the Eilenberg-Moore Spectral Sequence

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1. Introduction

Using relative homological algebra, Eilenberg and Moore [6] have defined a spectral sequence which has proven valuable in studying the homology groups of various fibre spaces. In this paper, we give a more geometric definition and use it to introduce Steenrod operations into the spectral sequence in a natural way.

We study the following geometric situation. Let

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta_0} \\ B & \longrightarrow & B_0 \end{array} \tag{1.1}$$

be a commutative diagram of topological spaces and continuous maps. Assume θ_0 is a (Serre) fibration and θ the induced fibration. Let K be a commutative ring with unit. If B_0 is simply connected, the Eilenberg-Moore spectral sequence, $\{E^r\}$, converges in the usual sense to $H_*(X; K)$. If the homology modules of B_0, B, X_0 satisfy certain flatness conditions, in particular if K is a field, E^2 is naturally isomorphic to $\mathrm{Cotor}^{H_*(B_0;K)}(H_*(X_0; K), H_*(B; K))$. We will prove the following.

1.2. THEOREM. *For $K=\mathbf{Z}/p\mathbf{Z}$, p a prime, $\{E^r\}$ may be given the structure of a spectral sequence of right modules over the* mod-p *Steenrod algebra $\mathscr{A}(p)$ so that*

(i) *Each d^r is $\mathscr{A}(p)$ linear.*

(ii) *For $\alpha\in\mathscr{A}(p)$ of degree i, $\alpha: E^r_{n,q}\to E^r_{n,q-i}$*

(iii) *E^∞ is the graded $\mathscr{A}(p)$ module associated to a filtration of $H_*(X; K)$ by $\mathscr{A}(p)$ submodules.*

(iv) *The coproduct $E^r\to E^r\otimes E^r$ is an $\mathscr{A}(p)$ morphism (Cartan formula).*

In addition we will define a natural $\mathscr{A}(p)$ structure in $\mathrm{Cotor}^{H_*(B_0K)}((H_*(X_0; K), H_*(B; K))$ coinciding with the $\mathscr{A}(p)$ structure in E^2.

1.3. *Remark.* If all spaces have homology modules of finite type, a dual spectral sequence may be defined converging to $H^*(X; K)$ and with $E_2=\mathrm{Tor}_{H^*(B_0)}(H^*(X_0), H^*(B))$, [5], [11], where Tor is given a natural left $\mathscr{A}(p)$ module structure. The Cartan formulae hold in E_r.

1.4. *Note Bene.* If $x\in E_r$ is of homological degree less than zero, then $x^p=0$ [3, 6.4].

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Hence $P^i x$ does not in general equal x^p in E_r unless x has homological degree zero. Thus in $H^*(X, \mathbf{Z}/p\mathbf{Z})$, the filtration of x^p is at least that of x (recall that in the cohomology spectral sequence, filtration is negative and decreasing).

The construction to be given below was inspired by a construction of D. M. Kan yielding an unstable Adams spectral sequence (to appear). The author would like to thank John Moore, D. M. Kan and Larry Smith for several enlightening conversations and useful suggestions.

These results have been proven for the special case of θ_0 the path fibration and B a point independently by L. Smith, A. Clark, and V. Puppe using algebraic methods (unpublished). The full result may also be proven by another geometric construction discovered by Alex Heller (to appear). L. Smith has subsequently discovered another construction [12].

1.5. *The Algebraic Spectral Sequence.* To motivate our construction, we recall the definition of the Eilenberg-Moore spectral sequence. We refer the reader to [6] for details.

Let K be a commutative ring with unit. For a space X, let C_*X be its normalized singular chains with coefficients in K. Then C_*X is a differential graded K-coalgebra by a map

$$\Delta_X\colon C_*X \to C_*X \otimes C_*X$$

given by the Alexander-Whitney formula. If $\varphi\colon Y \to X$ is a map of spaces, then C_*Y is a differential graded C_*X-comodule via the composition

$$C_*Y \xrightarrow{\Delta_Y} C_*Y \otimes C_*Y \xrightarrow{1\otimes\varphi_*} C_*Y \otimes C_*X .$$

For diagram 1.1, Eilenberg and Moore show that

$$H_*(X) = \mathrm{Cotor}_K^{C_*B_0}(C_*X_0, C_*B),$$

where Cotor is a suitable derived functor of cotensor product in the category of C_*B_0 comodules. Cotor may be defined in terms of a relative version of injective resolutions. We need only concern ourselves with

1.6. *The Cobar Construction* [1]. This construction is dual to the bar construction of Eilenberg and MacLane [8, Ch. X]. Let Λ be a differential graded K coalgebra, and let M be a right, N a left Λ-comodule. The cobar construction on M and N over Λ, to be denoted by $\mathbf{F}(M, \Lambda, N)$, is a complex

$$0 \to \mathbf{F}^0 \xrightarrow{\delta^0} \mathbf{F}^1 \xrightarrow{\delta^1} \mathbf{F}^2 \to \cdots$$

of differential graded K-modules defined as follows. Put

$$\mathbf{F}^n = M \otimes \Lambda \otimes \cdots \otimes \Lambda \otimes N,$$

where the factor Λ occurs n-times. The coproducts Δ_M, Δ_Λ, Δ_N induce $(n+1)$-maps $\delta_i\colon \mathbf{F}^{n-1} \to \mathbf{F}^n$ given by

$$\delta_i = \begin{cases} \Delta_M \otimes 1 \otimes \cdots \otimes 1, & i = 0 \\ 1 \otimes 1 \otimes \cdots \otimes \Delta_\Lambda \otimes \cdots \otimes 1 \otimes 1, & 1 \leqslant i \leqslant n-1 \\ 1 \otimes \cdots \otimes 1 \otimes \Delta_N, & i = n \end{cases}$$

where Δ_Λ operates on the i-th factor of Λ. Put

$$\delta^n = \sum_{i=0}^{n} (-1)^i \delta_i .$$

Now $\mathbf{F}$ is a double complex with differentials δ and ∂, where ∂ is the internal differential of $\mathbf{F}^n$. Given such a double complex, we may form a total complex T, with

$$T_n = \prod \mathbf{F}_q^{-p} \quad (p + q = n). \tag{1.7a}$$

The differential is given on the factor $\mathbf{F}_q^{-p}$ by

$$\partial^T = \partial + (-1)^q \delta . \tag{1.7b}$$

We have

$$\operatorname{Cotor}_K^\Lambda(M, N) = H_* T .$$

The Eilenberg-Moore spectral sequence of $\operatorname{Cotor}^\Lambda(M, N)$ is the spectral sequence of a filtration $\{F_r T\}_{r \leqslant 0}$ of T, where

$$(F_r T)_n = \prod \mathbf{F}_q^{-p}, \quad (p + q = n, p \leqslant r).$$

If K is a field, or if $H_*\Lambda$, H_*M and H_*N satisfy appropriate flatness conditions, then the Künnuth theorem may be used to show that

$$E^2 \approx H_*(\mathbf{F}(H_*M, H_*\Lambda, H_*N));$$

so

$$E^2 \approx \operatorname{Cotor}_K^{H_*\Lambda}(H_*M, H_*N).$$

Considerable algebraic difficulties have arisen in attempts to define Steenrod operations in the spectral sequence for the geometric situation. We will avoid those difficulties by carrying out all constructions in the category of spaces. In particular, the filtered complex T will be replaced by the homology exact couple of a tower of cofibrations. Since all homomorphisms in the exact couple will be induced by continuous maps and suspensions, differentials will *a priori* preserve Steenrod operations. The construction to be given will be closely analogous to that above.

2. The Geometric Cobar Construction

Since several constructions below are duals to standard ones in the category of simplicial abelian groups, we will find the following language useful.

2.1. *Cosimplicial Objects.* Let $\mathscr{A}$ be a category. A *cosimplicial $\mathscr{A}$-object* consists of the following data:

(i) Objects $\mathbf{A}^0, \mathbf{A}^1, \mathbf{A}^2, \ldots$, indexed by the non-negative integers.
(ii) For each $n>0$, maps

$$\delta_i: \mathbf{A}^{n-1} \to \mathbf{A}^n, \quad 0 \leqslant i \leqslant n,$$

called *cofaces*, and for each $n \geqslant 0$, maps

$$\sigma_i: \mathbf{A}^{n+1} \to \mathbf{A}^n, \quad 0 \leqslant i \leqslant n,$$

called *codegeneracies*, satisfying the duals to the simplicial identities. Specifically,

$$\left.\begin{array}{ll} \delta_j\delta_i = \delta_i\delta_{j-1}, & i<j \\ \sigma_j\sigma_i = \sigma_i\sigma_{j+1}, & i \leqslant j \\ \sigma_j\delta_i = \delta_i\sigma_{j-1}, & i<j \\ \sigma_j\delta_j = \sigma_j\delta_{j+1} = \text{identity} & \\ \sigma_j\delta_i = \delta_{i-1}\sigma_j, & i>j+1. \end{array}\right\} \tag{2.2}$$

A *cosimplicial map* $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ is a collection of maps $\mathbf{f}^n: \mathbf{A}^n \to \mathbf{B}^n$ commuting with cofaces and codegeneracies. Our primary example is

2.3. *The Geometric Cobar Construction.* Let Λ be a space, and let $A \xrightarrow{\alpha} \Lambda$, $B \xrightarrow{\beta} \Lambda$ be continuous maps. The *geometric cobar construction on A and B over Λ*, to be denoted by $\mathbf{G}(A, \Lambda, B)$, is defined as follows. Put

$$\mathbf{G}^n = A \times \Lambda \times \cdots \times \Lambda \times B,$$

where the factor Λ occurs n-times. As in 1.6, the cofaces are given by

$$\delta_i(a, \lambda_1, \ldots, \lambda_{n-1}, b) = \begin{cases} (a, \alpha a, \lambda_i, \ldots, \lambda_{n-1}, b), & i=0 \\ (a, \lambda_1, \ldots, \lambda_i, \lambda_i, \ldots, \lambda_{n-1}, b), & 1 \leqslant i \leqslant n-1 \\ (a, \lambda_1, \ldots, \lambda_{n-1}, \beta b, b), & i=n \end{cases}$$

$$\sigma_i(a, \lambda_1, \ldots, \lambda_{n+1}, b) = (a, \lambda_1, \ldots, \hat{\lambda}_{i-1}, \ldots, \lambda_{n+1}, b),$$

where "$\wedge$" denotes omission. It is easy to check the cosimplicial identities in $\mathbf{G}$. In a similar way $\mathbf{F}$ is a cosimplicial object in the category of differential graded K-modules. Both $\mathbf{G}$ and $\mathbf{F}$ are functors between the appropriate categories.

2.4. *Notation.* We will denote all cosimplicial objects by boldface. If V is a functor, we denote by $\mathbf{V}$ the induced functor on cosimplicial objects,

$$(\mathbf{VA})^n = V(\mathbf{A}^n).$$

3. The Spectral Sequence of a Fibre Product

Let

$$\begin{array}{ccc} X & \to & B \\ \downarrow & & \downarrow \beta \\ A & \xrightarrow{\alpha} & \Lambda \end{array}$$

be the fibre product of the continuous maps α and β. Let $\mathbf{G}=\mathbf{G}(A, \Lambda, B)$. Since the category of spaces is not additive, we cannot define a total differential δ in $\mathbf{G}$ by an alternating sum such as (1.6). We resort therefore to a construction dual to the Moore normalization of a simplicial group [10].

Let

$$\mathbf{G}_N^p = \mathbf{G}^p/\operatorname{Im}\delta_1 \cup \ldots \cup \operatorname{Im}\delta_p .$$

In particular, let $\mathbf{G}_N^0=\mathbf{G}^0/\phi$, which is $\mathbf{G}^0$ with a disjoint basepoint adjoined. Since $\delta_0\delta_i=\delta_{i+1}\delta_0$, δ_0 induces a map

$$\delta^p\colon \mathbf{G}_N^{p-1} \to \mathbf{G}_N^p$$

so that $\delta^{p+1}\circ\delta^p=*$, where "$*$" will be used to denote all basepoints. We thus have a complex of spaces

$$* \to \mathbf{G}_N^0 \xrightarrow{\delta^1} \mathbf{G}_N^1 \xrightarrow{\delta^2} \mathbf{G}_N^2 \to \cdots .$$

The Eilenberg-Moore spectral sequence of the diagram (3.1) is now derived from the homology exact couple of a sequence of cofibrations

$$\mathbf{G}_N^p \to X^p \to SX^{p-1}, \qquad p \geqslant 0, \tag{3.2}$$

where S denotes reduced suspension. We define X^p inductively using mapping cones.

Let $f\colon Y \to Z$ be a continuous map of pointed spaces. We denote by C_f the mapping cone of f,

$$C_f = Y \times [0,1] \cup Z/(y,0) \sim *, (y,1) \sim f(y), (*,t) \sim * .$$

Then $SY=C_*$, $*\colon Y\to *$, and there is a cofibration $Z\to C_f \to SY$. Put $X^0=\mathbf{G}_N^0$. Suppose X^{p-1} is defined and there is given a map

$$\tilde\delta^p\colon X^{p-1} \to \mathbf{G}_N^p$$

so that $\delta^{p+1}\circ\tilde\delta^p=*$. Then $X^p=C_{\tilde\delta_p}$. The commuting diagram

$$\begin{array}{ccc} X^{p-1} & \longrightarrow & * \\ \downarrow{\scriptstyle \tilde\delta^p} & & \downarrow \\ \mathbf{G}_N^p & \xrightarrow{\delta^{p+1}} & \mathbf{G}_N^{p+1} \end{array}$$

induces the map $\tilde\delta^{p+1}\colon X^p \to \mathbf{G}_N^{p+1}$ so that $\delta^{p+2}\circ\tilde\delta^{p+1}=*$. The cofibrations (3.2) are thus obtained.

Now let K be a commutative ring with unit. Let $\mathscr{C}^K(A, \Lambda, B)$ be the homology exact couple with coefficients in K of the sequence (3.2) of cofibrations. Then

$$D_{p,q} = \begin{cases} \tilde H_q(X^{-p}; K), & p \leqslant 0 \\ 0, & p > 0 \end{cases}$$

$$E_{p,q} = \begin{cases} \tilde H_q(\mathbf{G}_N^{-p}; K), & p \leqslant 0 \\ 0, & p > 0 \end{cases}$$

where, for a pointed space Y, $\tilde{H}_*(Y; K) = H_*(Y, *; K)$. The maps $E_p \to D_p$ are induced by the inclusions $\mathbf{G}_N^{-p} \to X^{-p}$; each map $D_p \to D_{p+1}$ is a composition

$$\tilde{H}_*(X^{-p}) \to \tilde{H}_*(SX^{-p-1}) \overset{s^{-1}}{\cong} \tilde{H}_{*-1}(X^{-p-1})$$

where the first map is induced by the identification and s is the suspension isomorphism. The map $D_{p+1} \to E_p$ is induced by $\tilde{\delta}^{-p}: X^{-p-1} \to \mathbf{G}_N^{-p}$. Thus, $d^1: E_{p+1} \to E_p$ *is induced by* $\delta^{-p}: \mathbf{G}_N^{-p-1} \to \mathbf{G}_N^{-p}$; that is

$$E^1 = \tilde{H}_*(\mathbf{G}_N; K) \tag{3.3}$$

as a complex. Clearly

3.4. PROPOSITION. *If $K = \mathbf{Z}/p\mathbf{Z}$, p a prime, then $\mathscr{C}_K(A, \Lambda, B)$ is an exact couple in the category of right $\mathscr{A}(p)$ modules. So d^r is $\mathscr{A}(p)$ linear.*

To relate the spectral sequence to $H_*(X)$, we want homomorphisms $H_*(X; K) \to D_{-p}$ of degree p so that

$$\begin{array}{ccc} & H_*(X;K) & \\ \swarrow & & \searrow \\ D_{-p} & \longrightarrow & D_{-p+1} \end{array}$$

commutes.

Let $X \xrightarrow{\varepsilon} A \times B$, be the natural inclusion. Then, $\delta_0 \varepsilon = \delta_1 \varepsilon$. Hence $\varepsilon: X/\phi \to \mathbf{G}_N^0$ and $\delta^1 \circ \varepsilon = *$. From the construction of X^p we have maps $\varepsilon^p: S^p(X/\phi) \to X^p$ so that

$$\begin{array}{ccc} S^p(X/\phi) & \xrightarrow{\varepsilon^p} & X^p \\ {\scriptstyle 1}\downarrow & & \downarrow \\ S^p(X/\phi) & \xrightarrow{S\varepsilon^{p-1}} & SX^{p-1} \end{array}$$

commutes. From these maps and suspensions we obtain the maps $H_*(X; K) \to D_{-p}$. In the context of 3.4, the maps preserve $\mathscr{A}(p)$ action.

It remains to consider the functorial properties of $\mathscr{C}_K(A, \Lambda, B)$. Let f be a map of squares. That is, let $f = (f', f'', f''')$ where

$$f': A \to A'$$
$$f'': \Lambda \to \Lambda'$$
$$f''': B \to B'$$

so that $\alpha' f' = f'' \alpha$, $\beta' f''' = f'' \beta$. Then these maps induce a cosimplicial map $\mathbf{f}: \mathbf{G} \to \mathbf{G}'$, where $\mathbf{G}' = \mathbf{G}(A', \Lambda', B')$. This map in turn induces a map of complexes $\mathbf{f}_N: \mathbf{G}_N \to \mathbf{G}'_N$. Since the mapping cone is functorial, there are maps $f^p: X^p \to X'^p$ with the obvious set of commuting diagrams. There is therefore an induced map

$$f_*: \mathscr{C}_K(A, \Lambda, B) \to \mathscr{C}_K(A', \Lambda', B').$$

Suppose two maps f and g of squares are fibre homotopic by a homotopy $F = (F', F'', F''')$, where

$$F' \colon A \times I \to A'$$
$$F'' \colon A \times I \to \Lambda'$$
$$F''' \colon B \times I \to B'$$

so that $\alpha' F' = F''(\alpha \times 1)$ and $\beta' F''' = F''(\beta \times 1)$. There is then a homotopy $\mathbf{F} \colon \mathbf{G} \times \mathbf{I} \to \mathbf{G}'$ where $\mathbf{F}^n \colon \mathbf{G}^n \times \mathbf{I} \to \mathbf{G}'^n$ is given by

$$\mathbf{F}^n(a, \lambda_1, \ldots, \lambda_n, b, t) = (F'(a, t), F''(\lambda_1, t), \cdots, F''(\lambda_n, t), F'''(b, t)).$$

Now cartesian product preserves identifications in one variable, so $(\mathbf{G} \times \mathbf{I})_N = \mathbf{G}_N \times \mathbf{I}$. Hence we have a homotopy $\mathbf{F}_N \colon \mathbf{G}_N \times \mathbf{I} \to \mathbf{G}'_N$ between $\mathbf{f}_N$ and $\mathbf{g}_N$. This in turn yields homotopies between f^p and g^p. So

3.5. PROPOSITION. *If f and g are fibre homotopic maps of squares, then the induced maps f_* and g_* from $\mathscr{C}_K(A, \Lambda, B)$ to $\mathscr{C}_K(A', \Lambda', B')$ are equal.*

4. Calculation of E^2

We shall show that under suitable hypotheses $E^2 = \mathrm{Cotor}^{H_*(\Lambda)}(H_*(A), H_*(B))$.

Let $\mathbf{A}$ be a cosimplicial object in an abelian category. There is associated to $\mathbf{A}$ a complex $t\mathbf{A}$,

$$0 \to t\mathbf{A}^0 \xrightarrow{\delta_t^1} t\mathbf{A}^1 \xrightarrow{\delta_t^2} t\mathbf{A}^2 \to \cdots,$$

where $t\mathbf{A}^n = \mathbf{A}^n$ and

$$\delta_t^n = \sum_{i=0}^{n} (-1)^i \delta_i.$$

There is also a complex $\mathbf{A}_N$, which is dual to the normalization of a simplicial object, where

$$\mathbf{A}_N^p = \mathbf{A}^p / \mathrm{Im}\, \delta_1 + \cdots + \mathrm{Im}\, \delta_p$$

and δ_N^p is induced by δ_0. The quotient map $t\mathbf{A} \to \mathbf{A}_N$ is a chain map. The dual of a standard result is

4.1. PROPOSITION. *The induced map*

$$H_*(t\mathbf{A}) \to H_*(\mathbf{A}_N)$$

is an isomorphism.

Proof. The standard proof (see e.g. [2, 3.6]) dualizes.

Let $\mathbf{G}$ be as in § 3. Then by 3.3, $E^1 = \tilde{H}(\mathbf{G}_N; K)$.

4.2. LEMMA. *$\tilde{H}(\mathbf{G}_N; K) = (\mathbf{H}(\mathbf{G}; K))_N$ as a complex.*

Proof. We prove by induction that, for $p \geqslant 1$,

$$\tilde{H}_*(\mathbf{G}^n/\mathrm{Im}\,\delta_p \cup \ldots \cup \mathrm{Im}\,\delta_n) = H_*(\mathbf{G}^n)/\mathrm{Im}\,\delta_p + \ldots + \mathrm{Im}\,\delta_n.$$

Note that, since $\sigma_{p-1}\delta_i = \delta_{i-1}\sigma_{p-1}$ for $i > p$, σ_{p-1} induces a continuous map

$$\sigma_{p-1}: \mathbf{G}^n/\mathrm{Im}\,\delta_{p+1} \cup \ldots \cup \mathrm{Im}\,\delta_n \to \mathbf{G}^n/\mathrm{Im}\,\delta_p \cup \ldots \cup \mathrm{Im}\,\delta_{n-1}$$

such that $\sigma_{p-1}\delta_p = 1$. Therefore, we have a cofibration

$$\mathbf{G}^{n-1}/\mathrm{Im}\,\delta_p \cup \ldots \cup \mathrm{Im}\,\delta_{n-1} \xrightarrow{\delta_p} \mathbf{G}^n/\mathrm{Im}\,\delta_{p+1} \cup \ldots \cup \mathrm{Im}\,\delta_n \to \mathbf{G}^n/\mathrm{Im}\,\delta_p \cup \ldots \cup \mathrm{Im}\,\delta_n.$$

The homology exact sequence of this cofibration yields a split exact sequence

$$0 \to H_*(\mathbf{G}^{n-1}/\mathrm{Im}\,\delta_p \cup \ldots \cup \mathrm{Im}\,\delta_{n-1}) \xrightarrow{\delta_{p*}} H_*(\mathbf{G}^n/\mathrm{Im}\,\delta_{p+1} \cup \ldots \cup \mathrm{Im}\,\delta_n)$$
$$\to H_*(\mathbf{G}^n/\mathrm{Im}\,\delta_p \cup \ldots \cup \mathrm{Im}\,\delta_n) \to 0.$$

The result now follows easily.

An immediate consequence of 4.1, 4.2 and 3.3 is that, for the exact couple $\mathscr{C}_K(A, \Lambda, B)$ of § 3,

4.3. PROPOSITION. $E^2 = H_*(t\mathbf{H}(\mathbf{G}; K))$.

We have finally,

4.4. THEOREM. *If $H_*(\Lambda; K)$ and either $H_*(A; K)$ or $H_*(B; K)$ are K-flat, in particular if K is a field, then there is a natural isomorphism*

$$E^2 \approx \mathrm{Cotor}_K^{H_*(\Lambda;K)}(H_*(A; K), H_*(B; K)).$$

Proof. Under these conditions we may apply the Künnuth theorem to obtain

$$\mathbf{H}(\mathbf{G}; K) = \mathbf{F}(H_*(A; K), H_*(\Lambda; K), H_*(B; K)).$$

where $\mathbf{F}$ is as in § 1.6. By 4.3 the result follows.

5. Comparison with the Algebraic Construction

Let $\mathscr{C}_K(A, \Lambda, B)$ be as in § 3. We will prove that

5.1. THEOREM. *The spectral sequence derived from $\mathscr{C}_K(A, \Lambda, B)$ is isomorphic to the Eilenberg-Moore spectral sequence of*

$$\mathrm{Cotor}_K^{C_*(\Lambda;K)}(C_*(A; K), C_*(B; K)).$$

From this and the results of [6] it immediately follows that

5.2. COROLLARY. *If Λ is connected and simply connected and $A \to \Lambda$ or $B \to \Lambda$*

is a Serre fibration, then the spectral sequence derived from $\mathscr{C}_K(A, \Lambda, B)$ *converges strongly to* $H_*(X; K)$.

5.3. COROLLARY. *If K is a field, the spectral sequence of* $\mathscr{C}_K(A, \Lambda, B)$ *has a natural coproduct structure and* $E^2 = \mathrm{Cotor}_K^{H_*(\Lambda)}(H_*(A), H_*(B))$ *as a coalgebra.*

To prove 5.1, we first obtain $\mathscr{C}_K(A, \Lambda, B)$ from a filtered differential module. For a pointed space Y, let $\tilde{C}_*Y = C_*(Y; K)$ be its normalized singular chains. From the double complex

$$0 \to \tilde{C}_*\mathbf{G}_N^0 \to \tilde{C}_*\mathbf{G}_N^1 \to \tilde{C}_*\mathbf{G}_N^2 \to \cdots$$

we may form the filtered total complex $\tilde{T} = T(\tilde{C}_*\mathbf{G}_N)$ as in 1.7; the spectral sequence of $\tilde{T}$ may then be derived from the exact couple $\mathscr{C}(\tilde{T})$ with

$$D_{p,q} = H_{p+q}(\tilde{T}/F_{p-1}\tilde{T})$$
$$E_{p,q} = H_{p+q}(F_p\tilde{T}/F_{p-1}\tilde{T}).$$

We want to know that

$$\mathscr{C}_K(A, \Lambda, B) \approx \mathscr{C}(\tilde{T}). \tag{5.4}$$

To see this, let

$$\mathbf{G}_N^p \to X^p \to SX^{p-1}$$

be the cofibrations (3.3). We shall show there is a natural chain homotopy equivalence

$$\tilde{T}/F_{-p-1}\tilde{T} \to \tilde{C}_*X^p \tag{5.5}$$

of degree p. For $\alpha: C \to C'$ a chain map, the *mapping cone* of α is a complex $\hat{C}_\alpha$ with

$$(\hat{C}_\alpha)_n = C_{n-1} \oplus C'_n$$

and

$$\partial(c, c') = (\partial c, \partial c' + (-1)^{n-1}\alpha c)$$

for $c \in C_{n-1}$, $c' \in C'_n$. Put $\hat{S}C = \hat{C}_0$, $0: C \to 0$. Let $f: Y \to Z$ be a map of pointed spaces; then there is a natural chain homotopy equivalence

$$\hat{C}_{f_*} \to \tilde{C}_*(C_f)$$

where $f_*: \tilde{C}_*Y \to \tilde{C}_*Z$ is induced by f. The long exact homology sequence of the cofibration $Z \to C_f \to SY$ is exactly that of the exact sequence of complexes

$$0 \to \tilde{C}_*Z \to \hat{C}_{f_*} \to \hat{S}\tilde{C}_*Y \to 0.$$

Using these facts, an easy induction shows that $\tilde{C}_*X^p$ is naturally chain homotopy equivalent to a complex whose n-chains are

$$\tilde{C}_{n-p}\mathbf{G}_N^0 \oplus \cdots \oplus \tilde{C}_n\mathbf{G}_N^p,$$

and whose boundary is given on $c \in \tilde{C}_{n-i}\mathbf{G}_N^{p-i}$ by

$$\partial c + (-1)^{n-i}\,\delta_*^{p-i+1} c$$

when $i>0$ and ∂c when $i=0$. It may be seen that this complex is isomorphic to $\tilde{T}/F_{-p-1}\tilde{T}$ with a dimension shift of p. Therefore the composition

$$(\tilde{T}/F_{-p-1}\tilde{T})_{q-p} \xrightarrow[\simeq]{(-1)^{pq}} \tilde{C}_{q-p}\mathbf{G}_N^0 \oplus \cdots \oplus \tilde{C}_q\mathbf{G}_N^p \to \tilde{C}_q X^p$$

is the desired chain homotopy equivalence 5.5. It should be noted that *the chain equivalence*

$$(F_{-p}\tilde{T}/F_{-p-1}\tilde{T})_{q-p} \simeq \tilde{C}_q\mathbf{G}_N^p \tag{5.6}$$

of degree p induced by (5.5) *is just multiplication by* $(-1)^{pq}$. A straightforward diagram chase establishes 5.4.

We may now relate the complex $\tilde{C}_*\mathbf{G}_N$ to the algebraic cobar construction. Let $\hat{T}=T(t\mathbf{C}_*\mathbf{G})$ (for notation see § 2.4 and § 4). The identification maps $\mathbf{G}^n \to \mathbf{G}_N^n$ induce a natural transformation $\hat{T} \to \tilde{T}$. By 5.4 and 4.2, this induces an isomorphism $\hat{E}^2 \to \tilde{E}^2$, where these are the E^2 terms of the spectral sequences of $\hat{T}$ and $\tilde{T}$.

By the Eilenberg-Zilber theorem, there is an associative, natural chain equivalence $\xi: C_*(Y\times Z) \to C_*(Y)\otimes C_*(Z)$ for two spaces Y and Z (the Alexander-Whitney map). There is then a cosimplicial map

$$\mathbf{C}_*\mathbf{G} \to \mathbf{F}(C_*(A), C_*(A), C_*(B))$$

which induces an isomorphism

$$\mathbf{HG} \to \mathbf{HF}.$$

Therefore, the spectral sequence of $\hat{T}$ is the Eilenberg-Moore spectral sequence for $\mathrm{Cotor}^{C_*A}(C_*A, C_*B)$ (see construction in § 1).

6. Steenrod Operations in E^2

Assume our ground ring K is a field. We recall some definitions and facts from [6] and [9]. Let Γ be a graded cocommutative Hopf algebra over K. If M and N are graded right Γ-modules, then $M\otimes N$ is a right Γ-module via the diagonal of Γ. A graded K-coalgebra, Σ, which is also a Γ-module is a *Γ-coalgebra* if the structure morphisms

$$\Delta_\Sigma: \Sigma \to \Sigma \otimes \Sigma,$$
$$e_\Sigma: \Sigma \to K$$

are Γ-linear. If M is a left (or right) Σ comodule which is also a Γ-module, we will say it is a $\Sigma-\Gamma$-comodule if the structure morphism

$$\Delta_M: M \to \Sigma \otimes M$$

is Γ-linear. If M is a right and N a left $\Sigma-\Gamma$-comodule, then $M \square_\Sigma N$ has a natural Γ-module structure since $M \square_\Sigma N$ is the kernel of the Γ-morphism

$$M \otimes N \to M \otimes \Sigma \otimes N$$

given by $1 \otimes \Delta_N - \Delta_M \otimes 1$.

We now define $\mathrm{Cotor}_\Gamma^\Sigma(M, N)$ to be the right derived functor of $\square_\Sigma$ in the category of $\Sigma-\Gamma$-modules in the following relative sense [4]. As in [6], let a comodule M be injective if it is the direct summand of an extended $\Sigma-\Gamma$-comodule. Let a sequence of $\Sigma-\Gamma$-comodules

$$0 \to M' \to M \to M'' \to 0$$

be *exact* if is split exact as a sequence of Γ-modules. By arguments identical to those of [6, § 3], *the class of exact sequences in the above sense is an injective class* [4] *relative to the class of injectives.* Thus the right derived functor $\mathrm{Cotor}_\Gamma^\Sigma(M, N)$ exists. We note that

6.1. LEMMA. *$\mathrm{Cotor}_\Gamma^\Sigma(M, N)$ is isomorphic to $\mathrm{Cotor}_K^\Sigma(M, N)$ by a map which is natural with respect to K-morphisms.*

Proof: An injective resolution of a comodule M in the category of $\Sigma-\Gamma$-comodules is an injective resolution in the category of Σ-comodules by forgetting Γ-structure.

6.2. *Example.* The cobar construction $\mathbf{F}(M, \Sigma, N)$ has a Γ-module structure since each $\mathbf{F}^n$ is a tensor product. Thus

$$\mathrm{Cotor}_\Gamma^\Sigma(M, N) = H_* t\mathbf{F}(M, \Sigma, N).$$

It is now easy to see that

6.3. THEOREM. *Let $\{E^r\}$ be the spectral sequence of* § 3. *Let $K = \mathbf{Z}/p\mathbf{Z}$, p a prime. Then*

$$E^2 = \mathrm{Cotor}_{\mathscr{A}(p)}^{H_*A}(H_*A, H_*B).$$

where $H_* = H_*(\ ; K)$.

Proof: By the Cartan formulae,

$$H_*(Y \times Z) = H_*Y \otimes H_*Z$$

as $\mathscr{A}(p)$ modules for any spaces Y and Z. Hence $H_*\mathbf{G}=\mathbf{F}(H_*A, H_*\Lambda, H_*B)$ as $\mathscr{A}(p)$ modules. The rest follows as in 4.4.

It remains to prove

6.4. PROPOSITION (Cartan formula). *If E^r is given the coproduct structure implied by* 5.3, *then the structure morphism*

$$\Delta_{E^r}: E^r \to E^r \otimes E^r$$

is an $\mathscr{A}(p)$ morphism when $K=\mathbf{Z}/p\mathbf{Z}$.

Proof: Since the comultiplication in E^r is induced by that in E^2, it suffices to prove this proposition for E^2. Now $E^2=H_*(t\mathbf{F}(H_*A, H_*\Lambda, H_*B))$. Our coproduct structure in E^2 is defined by identification with that of [6; § 18]. It is straightforward to check that this coproduct may be defined as follows. We first have a map

$$\begin{aligned}\mathbf{F}(\Delta_A, \Delta_\Lambda, \Delta_B)&:\mathbf{F}(H_*A, H_*\Lambda, H_*B)\\ &\to \mathbf{F}(H_*A\otimes H_*A, H_*\Lambda\otimes H_*\Lambda, H_*B\otimes H_*B).\end{aligned}$$

By the Cartan formulae, this map is an $\mathscr{A}(p)$-morphism. Now since $\mathscr{A}(p)$ is cocommutative, for any $\mathscr{A}(p)$ modules M and N the isomorphism

$$T: M\otimes N \to N\otimes M$$

given by $T(x\otimes y)=(-1)^{\dim x\cdot\dim y}(y\otimes x)$ is $\mathscr{A}(p)$-linear. Thus T defines an $\mathscr{A}(p)$ isomorphism

$$\begin{aligned}\tau&:\mathbf{F}(H_*A\otimes H_*A, H_*\Lambda\otimes H_*\Lambda, H_*B\otimes H_*B)\\ &\to \mathbf{F}(H_*A, H_*\Lambda, H_*B)\otimes\mathbf{F}(H_*A, H_*\Lambda, H_*B),\end{aligned}$$

where, for two cosimplicial objects $\mathbf{M}$ and $\mathbf{N}$, $(\mathbf{M}\otimes\mathbf{N})^n=\mathbf{M}^n\otimes\mathbf{N}^n$.

Now if $\mathbf{M}$ and $\mathbf{N}$ are cosimplicial objects in an abelian category, the obvious dual to the Eilenberg-Zilber theorem [2, 2.9] states that *there is a chain equivalence*

$$\zeta: t(\mathbf{M}\otimes\mathbf{N}) \to t\mathbf{M}\otimes t\mathbf{N}.$$

given on $m\otimes n\in(\mathbf{M}\otimes\mathbf{N})_\alpha$ by

$$\zeta(m\otimes n)=\Sigma(-1)^{s(\alpha)}\sigma_{i_1}\dots\sigma_{i_r}m\otimes\sigma_{j_1}\dots\sigma_{j_s}n$$

where the sum is taken over all r, s shuffles $\alpha=(i_1,\dots,i_r,j_1,\dots,j_s)$ of the integers $0, 1,\dots q-1$, $r+s=q$, and $s(\alpha)$ is the usual sign associated to a shuffle. If M and N are cosimplicial $\mathscr{A}(p)$-modules, ζ is clearly $\mathscr{A}(p)$-linear. The coproduct in E^2 is now the $\mathscr{A}(p)$ morphism induced by $\zeta\circ\tau\circ\mathbf{F}(\Delta_A, \Delta_\Lambda, \Delta_B)$.

REFERENCES

[1] ADAMS, J. F., *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A. *42*, 409–412 (1956).
[2] DOLD, A. and PUPPE, D., *Homologie nicht additiver Funktoren. Anwendungen*, Ann. Inst. Fourier, Grenoble *11*, 201–312 (1961).
[3] CARTAN, H., *Algèbre d'Eilenberg-MacLane et Homotopie*, Seminaire Henri Cartan, 7e annee, 1954/55.
[4] EILENBERG, S. and MOORE, J. C.: *Foundations of relative homological algebra*, Memoirs A.M.S. *55* (1965).
[5] EILENBERG, S. and MOORE, J. C., *Homological algebra and fibrations*, Colloque de Topologie, Bruxelles. Sept. 7, 1964. CBRM.
[6] STEENROD, N. E. and EPSTEIN, D. B. A., *Cohomology Operations* (Princeton University Press 1962).
[7] EILENBERG, S. and MOORE, J. C., *Limits and spectral sequences*, Topology *1*, 1–24 (1962).
[8] MACLANE, S., *Homology* (Academic Press 1963).
[9] MILNOR, J. W. and MOORE, J. C., *On the structure of Hopf algebras*, Ann. of Math. *81*, 211–264 (1965).
[10] MOORE, J. C., *Seminar on Algebraic Homotopy*, Lecture notes, Princeton University (1955).
[11] SMITH, L., *Homological Algebra and the Eilenberg-Moore spectral sequence*, Trans. AMS. *129*, 58–93 (1967).
[12] SMITH, L., *On the Künneth Theorem I* (to appear).

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