Zeitschrift:	Commentarii Mathematici Helvetici
Herausgeber:	Schweizerische Mathematische Gesellschaft
Band:	45 (1970)
Artikel:	Analytic Maps Between Tori.
Autor:	Helfenstein, Heinz G.
DOI:	https://doi.org/10.5169/seals-34676

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

### Download PDF: 20.08.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

# Analytic Maps Between Tori

by HEINZ G. HELFENSTEIN (University of Ottawa, Canada)<sup>1</sup>)

## 1. Introduction

It is well known that the conformal 2-dimensional tori fall in two classes, viz. those whose elliptic field of functions admit complex multiplication and those which do not. For short we denote the former as "ample tori", the latter as "non-ample". (For a recent account, with a list of older references, see [1].)

In a different context we show that this dichotomy appears also in the distribution of the complex analytic maps between *two* tori. The source of this behaviour is traced to the structure of certain isotropy groups of hyperbolic motions. It turns out that both the ample and non-ample tori form dense subsets of the manifold of all conformal tori.

We determine necessary and sufficient conditions for the existence of non-constant analytic maps between two tori and classify these maps with respect to homotopy. This amounts to an explicit determination of the bimodule structure of the set of analytic maps over Z and over the rings of complex multiplication of the given tori.

Our methods indicate that similar splittings into disjoint dense classes may be expected for other categories of maps; e.g. affine maps between general flat space forms of arbitrary dimensions, cf. [6].

Some consequences will be discussed elsewhere, cf. [3].

## 2. Conformal Classes of Tori

We make use of the following two groups:

 $GL^+(2, Q)$ =group of all 2×2 matrices with real rational entries and positive determinant; SL(2, Z)=subgroup of all 2×2 matrices with real integral entries and determinant equal to +1 (modular group), and their factor groups:

 $G = GL^+(2, Q) / \{\lambda I: \lambda \neq 0, \text{ rational}\}$  with

$$I = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}; \quad F = SL(2, Z) / \{\pm I\}.$$

*F* can be naturally embedded into *G* as a subgroup. We denote by *H* the Poincaré half-plane  $\dot{H} = \{z = x + iy \in \mathbb{C}: \Im z > 0\}$  with hyperbolic metric  $g = (1/y^2) (dx \otimes dx + dy)$ 

<sup>&</sup>lt;sup>1</sup>) Work supported by the National Research Council of Canada. Cf. research announcement in [2].

 $\otimes$  dy). G and F act effectively on H as subgroups of all isometries by letting

$$g(z) = \frac{az+b}{cz+d}$$
 for  $g = \begin{pmatrix} a, b \\ c, d \end{pmatrix} \in GL^+(2, Q), \quad z \in H.$ 

The conformal classes of tori are in 1:1 correspondence with the surface  $\mathcal{T} = H/F$ , which is homeomorphic to the Euclidean plane, [5]. A point  $\tau \in \mathcal{T}$  represents the conformal equivalence class of the torus  $E^2/\Gamma$ , where  $\Gamma$  is the group of Euclidean motions generated by the two translations  $t_1(z) = z + 1$ , and  $t_2(z) = z + h$ ,  $h \in \tau = F(h) \subset H$ .

All topological statements concerning subsets of the set of conformal equivalence classes of tori are understood with respect to the topology of  $\mathcal{T}$ .

By a "conformal torus" we will mean for short a conformal equivalence class.

### 3. Analytic Immersion Classes

In order to formulate necessary and sufficient conditions for the existence of non-constant analytic maps between two conformal tori we require

DEFINITION 1: Two conformal tori  $\tau_1$  and  $\tau_2$  are called immersion-equivalent,  $\tau_1 \sim \tau_2$ , if there exist  $h_i \in \tau_i$ , i=1, 2, and  $T \in G$  such that  $h_1 = T(h_2)$ .

This definition is justified since it does not depend on the representatives  $h_i$  of the given  $\tau_i$ . The equivalence classes (orbits mod G) into which  $\mathcal{T}$  is partitioned will be called *analytic immersion classes*.

The group G does not act on the surface  $\mathcal{T}$  in the ordinary sense, since its elements do not commute with the group F. We obviously have

THEOREM 1: Every analytic immersion class is dense in the manifold of conformal tori. Every neighbourhood of a conformal torus on  $\mathcal{T}$  contains representatives of all analytic immersion classes infinitely often.

THEOREM 2: Two conformal tori  $\tau_1$  and  $\tau_2$  admit a non-constant analytic map  $f:\tau_1 \rightarrow \tau_2$  if and only if  $\tau_1 \sim \tau$  holds. An analytic map is either a constant or a covering map.

**Proof:** Choose representatives  $h_i \in \tau_i$  and assume that a non-constant analytic  $f: E_1/\Gamma_1 \rightarrow E_2/\Gamma_2$  exists. According to the fibre map theorem (cf. [4]) there exists a lift of f to the universal covering surfaces  $E_i$ , i.e. an entire function  $F: E_1 \rightarrow E_2$  satisfying  $f \circ p_1 = p_2 \circ F$ , where  $p_i: E_i \rightarrow E_i/\Gamma_i$  are the covering projections. Hence there exist two integer-valued functions n(m, m') and n'(m, m') such that

$$F(z_1 + m + m'h_1) = F(z_1) + n + n'h_2$$
(1)

holds identically in  $z_1 \in E_1$ .

Differentiating with respect to z, we find that F' is a constant C. Substituting

F(z) = Cz + D into (1) and letting first m = 1, m' = 0, then m = 0, m' = 1, we recognize that there are 4 integers a = n(1, 0), b = n'(1, 0), c = n(0, 1), d = n'(0, 1), such that

$$h_1 = \frac{ah_2 + b}{ch_2 + d} \tag{2}$$

holds. Since  $\Im(h_i) > 0$ , i=1, 2, we have ad-bc>0. Reversing the arguments we see that the existence of 4 rationals a, b, c, d satisfying ad-bc>0 and (2) is also sufficient for the existence of a non-constant analytic map. Q.E.D.

DEFINITION 2: If f is an analytic map with lift F(z) = Cz + D, the constant C will be called *the complex distortion* of f. The set of admissible values of C for two tori is called the *distortion spectrum* of the (ordered) pair of tori.

The fact that  $\tau_1 \sim \tau_2$  is an equivalence is worth restating as

THEOREM 3: If there exists an analytic immersion  $\tau_1 \rightarrow \tau_2$ , then there exists also an analytic immersion  $\tau_2 \rightarrow \tau_1$ . (In general the inverse  $F^{-1}$  of the lift F of f is, however, not the lift of an analytic map  $\tau_2 \rightarrow \tau_1$ .)

Theorem 1 entails: Given two conformal tori  $\tau_1$ ,  $\tau_2$ , then every neighbourhood of  $\tau_1$  on  $\mathscr{T}$  contains a countable infinity of tori which admit analytic immersions into  $\tau_2$ , and uncountably many tori which do not.

LEMMA 1: If  $\tau_1$ ,  $\tau_2$  are two conformal immersion-equivalent tori then it is possible to choose representatives  $h_i \in \tau$  (i=1, 2) such that  $h_1 = a h_2$ , where a is a positive integer.

**Proof:** If  $h'_i \in \tau_i$  are arbitrary representatives with  $T' \in G$  and  $h'_1 = T'(h'_2)$  then diagonalization of T' leads to three matrices A,  $B \in SL(2, Z)$  and  $T \in GL^+(2, Q)$  with

$$h_1 = (A^{-1}T'B)(h_2) = T(h_2) \text{ and } T = \begin{pmatrix} a, & 0\\ 0, & 1 \end{pmatrix}.$$

Note that none of the three numbers  $h_1$ ,  $h_2$ , a in the relation  $h_1 = ah_2$  is invariantly connected with the pair  $(\tau_1, \tau_2)$ : If e.g.  $h_1 = h_2 = i$ , a = 1,  $h'_1 = (157 + i)/17$ ,  $h'_2 = (157 + i)/170$ , a' = 10, then  $h_1$  and  $h'_1$  represent the same conformal torus  $\tau_1$  because of  $h_1 = P(h'_1)$ ,

$$P(z)=\frac{z-9}{-4z+37}.$$

Similarly

$$h_2 = Q(h'_2)$$
 with  $Q(z) = \frac{-13z + 12}{z - 1}$ .

## 4. Ample Tori

In order to determine the complete set of all analytic maps between two tori we require the following definitions.

DEFINITION 3: The complex number z is called *ample* if  $\Re z$  and  $|z|^2$  are both rational. A conformal torus  $\tau$  is called ample if there exists an ample representative  $h \in \tau$ . A planar lattice

$$\mathscr{L} = \left\{ m\omega + m'\omega' \in \mathbb{C} : m, m' \text{ integers, } \Im \frac{\omega'}{\omega} \neq 0 \right\}$$

is ample if it can be generated by two complex numbers  $\omega$ ,  $\omega'$  with  $\omega'/\omega$  an ample point.

These definitions are justified by their independence from the representative h (invariance under F). The property of a point  $h \in H$  of being ample or non-ample is invariant even under the action of G; hence we can also speak of ample and non-ample analytic immersion classes. There are only countably many ample tori, but uncountably many non-ample ones. Each of the two subsets of  $\mathcal{T}$  corresponding to these two types of tori is dense in  $\mathcal{T}$ , and each consists of whole analytic immersion classes. The ample tori do not form a single immersion class.

### 5. The Isotropy Subgroups of G

The determination of all analytic maps between two conformal tori depends to a large extent on an analysis of the stabilizers of G and their cosets.

DEFINITION 4: Let  $h \in H$ , and let  $I_h$  denote the isotropy subgroup of G with respect to h, i.e. the subgroup of all hyperbolic rotations about h belonging to G.

LEMMA 2: The structure of  $I_h$  is an invariant of the analytic immersion class G(h). G(h) is in 1:1 correspondence with the coset space  $G/I_h$ .

**Proof:** If h runs through an orbit G(h), then  $I_h$  varies in a conjugacy class of G. Since G acts transitively on an orbit, G(h) becomes a homogeneous G-space and is thus representable as  $G/I_h$ . Q.E.D.

The structure of  $I_h$  differs considerably according to whether h is ample or not.

THEOREM 4: If  $h \in H$  is non-ample, then  $I_h$  is trivial. *Proof*: An arbitrary element  $S \in I_h$  can be represented as a matrix  $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$  with relatively prime integral entries, and  $\alpha \delta - \beta \gamma > 0$ . The relation S(h) = h means:

$$\gamma h^2 + (\delta - \alpha) h - \beta = 0.$$
<sup>(3)</sup>

Since  $\Im h > 0$ ,  $\gamma = 0$  entails  $\alpha = \delta$ ,  $\beta = 0$ , i.e. S is the identity. If  $\gamma \neq 0$  then (3) is a quadratic equation for h with real coefficients; hence it is satisfied both by h and

 $h, h \neq h$ . By Vieta's theorem:

γ

$$h + \bar{h} = 2\Re h = \frac{\alpha - \delta}{\gamma},$$

$$(4)$$

$$h\bar{h} = |h|^2 = -\frac{\beta}{\gamma}.$$

$$(5)$$

Since h is not ample, these equations are impossible; hence  $I_h$  contains only the identity.

THEOREM 5: Let  $h \in H$  be ample,  $2 \Re h = p/q$ ,  $|h|^2 = r/s$ , p, q > 0, r > 0, s > 0, integers; g.c.d.(p, q) = g.c.d.(r, s) = 1, g.c.d.(q, s) = g, q' = q/g, s' = s/g.

Define  $M = \begin{pmatrix} ps', -q'r \\ qs', 0 \end{pmatrix}$ Then  $I_h = \{\varrho I + \sigma M : \varrho, \sigma \text{ rational}, \neq (0, 0)\}/\{\lambda I : \lambda \text{ rational}, \neq 0\}.$ 

*Proof:* Let  $S = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$  represent an element of  $I_h$ , with integral entries and  $\alpha\delta - \beta\gamma > 0$ .

As in the proof of Theorem 4,  $\gamma = 0$  leads to the identity map. Assume now  $\gamma \neq 0$ . Then we have again the relations (4) and (5). Since g.c.d.(p, q) = 1, (4) entails the existence of an integer  $\varphi$  such that

$$\alpha - \delta = \varphi p,$$

and

$$\gamma = \varphi q \,. \tag{6}$$

Substituting (6) into (5) we deduce from g.c.d. (r, s) = 1 the existence of an integer  $\psi$  satisfying

 $\beta = -\psi r$ ,

and

 $\varphi q = \psi s \,. \tag{7}$ 

From (7), q = gq', s = gs', and g.c.d. (q', s') = 1, we find an integer v such that  $\psi = vq'$ , and  $\varphi = vs'$ . Solving for  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ , we find

$$S = \delta I + \nu M \,. \tag{8}$$

Conversely, for an arbitrary choice of the integers  $\delta$ , v, except  $\delta = v = 0$ , we gather S(h) = h, and det S becomes a quadratic form in  $\delta$  and v with discriminant  $-4q^2s'^2 \times (\Im h)^2 < 0$ , hence it is positive definite. Thus the above matrix is the most general form representing an element of  $I_h$ . The group operations can be easily read off from the relation  $M^2 = -rs'qq'I + ps'M$ . Q.E.D.

For every ample h the group  $I_h$  is a countably infinite, not finitely generated Abelian group which is dense in the group of all hyperbolic rotations about h. Its finer structure depends on number theoretical properties of h; e.g. every  $I_h$  contains exactly one element of order 2, but for  $h_1 = 2i/\sqrt{3}$  and  $h_2 = i$ ,  $I_{h_1}$  does and  $I_{h_2}$  does not contain elements of order 3.

Combining theorems 4 and 5 we obtain the following characterization of ample points.

THEOREM 6:  $h \in H$  is ample if and only if there exists in G a hyperbolic rotation about h different from the identity.

### 6. Cosets of $G \mod I_h$

In this paragraph a will denote a positive integer which will be identified in §7 with the quantity introduced in lemma 1.

If h is non-ample then each coset of  $I_h$  consists of a single element of G, by theorem 4.

LEMMA 3: Let  $h \in H$  be ample, and define the integers p, q, r, s, g, q', s' as in theorem 5. Furthermore we introduce

$$g' = g.c.d.(a, q), \quad a' = a/g', \quad q'' = q/g'',$$
  

$$g'' = g.c.d.(a', s'), \quad a'' = a'/g'', \quad s'' = s'/g'',$$
  

$$T_1 = \begin{pmatrix} a, & 0 \\ 0, & 1 \end{pmatrix}, \quad T_2 = \frac{1}{g'g''} T_1 M = \begin{pmatrix} a' ps'', & -a''q'r \\ q''s'', & 0 \end{pmatrix}.$$

Then the most general integral matrix which represents an element of the left coset of  $G \mod I_h$  containing the hyperbolic translation  $T = T_1$  is given by

$$L = \kappa_1 T_1 + \kappa_2 T_2, \tag{9}$$

where  $(\kappa_1, \kappa_2) \neq (0, 0)$  denote arbitrary integers.

*Proof:* Using (8) we find  $TS = \delta T + vTM$  with arbitrary integers  $(\delta, v) \neq (0, 0)$ . Since we work in the factor group G of  $GL^+(2, Q)$  we still have to determine all rationals  $\lambda \neq 0$  and all integers  $(\delta, v) \neq (0, 0)$  such that

$$L = \lambda (\delta T + v T M) \tag{10}$$

becomes integral.

Let  $\lambda$ ,  $\delta$ , v be such a triple, and define

$$\Lambda(\delta, v) = \text{g.c.d.} \{a|\delta + ps'v|, aq'r|v|, qs'|v|, |\delta|\}, \quad \xi = \lambda \Lambda(\delta, v).$$

Substituting  $\lambda = \xi/\Lambda(\delta, v)$  into  $\lambda(\delta T + vTM)$  we see that this matrix assumes the

form  $\xi \cdot L'$ , where L' is an integer valued matrix whose four entries are relatively prime. Since  $\xi L'$  must be integer valued and  $\xi$  is rational, it follows that  $\xi$  is an integer  $\neq 0$ .

Conversely, given an integer  $\xi \neq 0$  and a pair of integers  $(\delta, \nu) \neq (0, 0)$ , the quantity  $\lambda = \xi/\Lambda(\delta, \nu)$  is rational,  $\neq 0$ , and  $\lambda(\delta T + \nu TM)$  becomes an integral matrix.

Hence this procedure yields all desired triples  $\lambda$ ,  $\delta$ , v and all matrices L. The correspondence between the triples  $\lambda$ ,  $\delta$ , v and the matrices L is, however, not 1:1, since two different triples can lead to the same matrix. In order to settle this problem we first determine the admissible values for the elements of the last row of L, viz.

$$\kappa'_1 = \xi \frac{\delta}{\Lambda(\delta, \nu)}, \quad \kappa'_2 = \frac{s' q \nu}{\Lambda(\delta, \nu)}.$$
(11)

Assume that  $\xi$ ,  $\delta$ , v are given. If v=0, we have  $\Lambda(\delta, v)=|\delta|$  and  $L=\kappa T$  with an arbitrary integer  $\kappa \neq 0$ .

Let now  $\nu \neq 0$ , hence  $\kappa'_2 \neq 0$ , and

$$\delta = \frac{\kappa_1'}{\kappa_2'} q s' v \,. \tag{12}$$

Noting that for every integer  $j \neq 0$   $\Lambda(j\delta, j\nu) = j\Lambda(\delta, \nu)$  holds, we obtain from (11) and (12):

$$\xi = \frac{\kappa_2'}{s'qr} \Lambda(\delta, v) = \frac{\Lambda(\kappa_1'qs', \kappa_2')}{qs'}$$

Since this must be an integer, inspection of  $\Lambda(\kappa'_1 qs', \kappa'_2)$  reveals that qs' must be a factor of the two quantities

 $a|p|s'|\kappa'_2|$  and  $aq'r|\kappa'_2|$ .

In the following  $g_1, g_2, \dots, g_6$  will denote suitable integers. Then the first condition can be written as

$$\frac{|p|}{q} = \frac{g_1}{a \, |\kappa_2'|},\tag{13}$$

the second as

$$\frac{aq'r|\kappa_2'|}{qs'} = g_2.$$
<sup>(14)</sup>

Since the left hand side of (13) is in the lowest terms there is a  $g_3$  such that

$$g_1 = g_3[p],$$
 (15)

and

$$a|\kappa_2'| = g_3 q \,. \tag{16}$$

### Analytic Maps between Tori 537

Dividing (16) by g' we recognize that there exists  $g_4$  with

$$|\kappa_2'| = g_4 q'', (17)$$

and

$$g_3 = g_4 a'$$
. (18)

Substituting (17) into (14) we obtain

$$\frac{g_2}{a'g_4r} = \frac{q'}{s'} \tag{19}$$

Here the right hand side is in the lowest terms; hence

$$g_2 = g_5 q',$$
 (20)

and

$$a'g_4r = g_5s'$$
. (21)

Dividing (21) by g'' we obtain  $a''g_4r = g_5s''$ . Because of g.c.d.(a'', s'') = g.c.d.(r, s) = 1, we have g.c.d.(a''r, s'') = 1. Thus there is  $g_6$  with

$$g_4 = g_6 s'',$$
 (22)

$$g_5 = g_6 a'' r$$
. (23)

Combination of (17) and (22) yields  $|\kappa'_2| = g_6 q'' s''$ . Finally letting  $\kappa_1 = \kappa'_1$  and  $\kappa_2 = \operatorname{sgn} \kappa'_2 \cdot g_6$ , we obtain  $L = \kappa_1 T_1 + \kappa_2 T_2$  from (10).

Conversely, for an arbitrary choice of the integers  $(\kappa_1, \kappa_2) \neq (0, 0)$ , we can find a corresponding triple  $\xi$ ,  $\delta$ , v; viz.  $\delta = \kappa_1 qs'$ ,  $v = \kappa_2 q''s''$ ,  $\xi = \Lambda(\kappa_1 qs', \kappa_2 q''s'')/qs'$ . (The last expression is easily seen to be an integer.) Q.E.D.

### 7. The Distortion Spectrum

For given tori  $\tau_1, \tau_2$  with representatives  $h_i \in \tau_i$  chosen according to lemma 1 we identify now  $h = h_2$  in lemma 3.

LEMMA 4: Each integral matrix L representing an element of the coset  $TI_{h_2}$ determines an admissible complex distortion for an analytic map  $\tau_1 \rightarrow \tau_2$ , and all analytic maps are obtained in this way.

*Proof:* Writing  $\tau_i = E_i/\Gamma_i$  the lift F(z) = Cz + D must satisfy the commutation relation  $C(z+m+m'h_1) + D = Cz + D + n + n'h_2$ , i.e.  $\Gamma_2$  must contain a subgroup conjugate to  $\Gamma_1$  in the group of all conformal equivalences of the Euclidean plane.

Letting m=1, m'=0, then m=0, m'=1 and dividing we obtain  $h_1=T(h_2)=L(h_2)$ 

with

$$L = \begin{pmatrix} n'(0, 1), n(0, 1) \\ n'(1, 0), n(1, 0) \end{pmatrix} \in GL^+(2, Q).$$

Hence  $T^{-1}L$  represents an element of  $I_{h_2}$ , or L belongs to the left coset of  $G \mod I_h$ , containing T.

Conversely, if we pick all integer valued matrices L representing the same element  $TI_{h_2}$  of  $G/I_{h_2}$  as T, then we obtain all possible complex distortions as

$$C = n(1,0) + n'(1,0) h_2 = n(0,1) + n'(0,1) h_2. \quad \text{Q.E.D.}$$
<sup>(24)</sup>

THEOREM 7: Let  $\tau_1$  and  $\tau_2$  be two conformal tori in the same analytic immersion class A. Then the distortion spectrum is given by

a) the one-dimensional real lattice

$$C(\kappa) = \kappa, \quad \kappa = 0, \pm 1, \pm 2, \tag{25}$$

if A is non-ample;

b) the 2-dimensional lattice

 $C(\kappa_1, \kappa_2) = \kappa_1 + \kappa_2 q'' s'' h_2$ 

for A ample, with  $\kappa_1$  and  $\kappa_2$  running independently through all integers. (Notations of lemma 3 applied to  $h=h_2$ .)

*Proof*: a) By theorem 4 the coset  $TI_{h_2}$  contains only *T*, and *L* can be any integral matrix representing the same element of *G* as *T*; hence  $L = \kappa T$  with an arbitrary integer  $\kappa$ . Thus we obtain (25) from (24).

b) Substitute (9) into (24).

### 8. Some Consequences

A. Although proportional pairs of integers  $(\kappa_1, \kappa_2)$  and  $(\kappa \kappa_1, \kappa \kappa_2)$  yield the same element  $\kappa_1 T_1 + \kappa_2 T_2$  of the coset  $TI_{h_2}$ , the corresponding maps are different.

B. In the ample case there always exist sublattices of real and purely imaginary distortions, but the full distortion spectrum is in general larger than the lattice generated by these two sublattices. The full distortion spectrum is an ample lattice.

C. The distortion spectrum depends on the representatives  $h_i$ , not only on the surfaces  $\tau_i$ . If  $h_i$  with  $h_1 = ah_2$  are used as representatives to compute the maps  $\tau_1 \rightarrow \tau_2$ , then one can use the representatives  $(-1)/h_i$  with the same integer *a* for the determination of the maps  $\tau_2 \rightarrow \tau_1$  (cf. theorem 3). In this case the latter distortion spectrum is the image of the former under reflexion in the imaginary axis.

D. The fact that the distortion spectrum is in both cases a discrete set implies that maps corresponding to different lattice points are not analytically homotopic. It

538

can be shown that they even belong to different ordinary homotopy classes. Only constant maps are analytically null-homotopic, and two tori are of the same analytic homotopy type if and only if they are conformally equivalent.

### BIBLIOGRAPHY

- [1] BOREL, A., CHOWLA, S., HERZ, C. S., IWASAWA, K., and SERRE, J-P.: Seminar on Complex Multiplication, Lecture Notes in Mathematics No. 21 (Springer-Verlag, Berling-Heidelberg-New York, 1966).
- [2] HELFENSTEIN, H. G.: Analytic Maps between Tori, Bull. Am. Math. Soc. 75, No. 4 (1969).
- [3] HELFENSTEIN, H. G.: Local Isometries of Flat Tori, Pacific J. of Math. 32, No. 3 (1970).
- [4] HU, S.T.: Homotopy Theory (Academic Press, New York 1959).
- [5] NEVANLINNA, R.: Uniformisierung (Springer-Verlag, Berlin 1953).
- [6] WOLF, J. A.: Spaces of Constant Curvature (McGraw-Hill, New York 1967).

Received March 6, 1970