Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 45 (1970)

Artikel: Analytic Maps Between Tori.

Autor: Helfenstein, Heinz G.

DOI: https://doi.org/10.5169/seals-34676

Analytic Maps Between Tori

by HEINZ G. HELFENSTEIN (University of Ottawa, Canada)[1]

1. Introduction

It is well known that the conformal 2-dimensional tori fall in two classes, viz. those whose elliptic field of functions admit complex multiplication and those which do not. For short we denote the former as "ample tori", the latter as "non-ample". (For a recent account, with a list of older references, see [1].)

In a different context we show that this dichotomy appears also in the distribution of the complex analytic maps between *two* tori. The source of this behaviour is traced to the structure of certain isotropy groups of hyperbolic motions. It turns out that both the ample and non-ample tori form dense subsets of the manifold of all conformal tori.

We determine necessary and sufficient conditions for the existence of non-constant analytic maps between two tori and classify these maps with respect to homotopy. This amounts to an explicit determination of the bimodule structure of the set of analytic maps over Z and over the rings of complex multiplication of the given tori.

Our methods indicate that similar splittings into disjoint dense classes may be expected for other categories of maps; e.g. affine maps between general flat space forms of arbitrary dimensions, cf. [6].

Some consequences will be discussed elsewhere, cf. [3].

2. Conformal Classes of Tori

We make use of the following two groups:

$GL^+(2, Q)$ = group of all 2×2 matrices with real rational entries and positive determinant; $SL(2, Z)$ = subgroup of all 2×2 matrices with real integral entries and determinant equal to $+1$ (modular group), and their factor groups:

$G = GL^+(2, Q)/\{\lambda I : \lambda \neq 0, \text{ rational}\}$ with

$$I = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}; \quad F = SL(2, Z)/\{\pm I\}.$$

F can be naturally embedded into G as a subgroup. We denote by H the Poincaré half-plane $H = \{z = x + iy \in \mathbf{C} : \Im z > 0\}$ with hyperbolic metric $g = (1/y^2)(dx \otimes dx + dy$

[1]) Work supported by the National Research Council of Canada. Cf. research announcement in [2].

$\otimes\, dy$). G and F act effectively on H as subgroups of all isometries by letting

$$g(z) = \frac{az + b}{cz + d} \quad \text{for} \quad g = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \in GL^+(2, Q), \qquad z \in H.$$

The conformal classes of tori are in 1 : 1 correspondence with the surface $\mathcal{T} = H/F$, which is homeomorphic to the Euclidean plane, [5]. A point $\tau \in \mathcal{T}$ represents the conformal equivalence class of the torus E^2/Γ, where Γ is the group of Euclidean motions generated by the two translations $t_1(z) = z + 1$, and $t_2(z) = z + h$, $h \in \tau = F(h) \subset H$.

All topological statements concerning subsets of the set of conformal equivalence classes of tori are understood with respect to the topology of $\mathcal{T}$.

By a "conformal torus" we will mean for short a conformal equivalence class.

3. Analytic Immersion Classes

In order to formulate necessary and sufficient conditions for the existence of non-constant analytic maps between two conformal tori we require

DEFINITION 1: Two conformal tori τ_1 and τ_2 are called immersion-equivalent, $\tau_1 \sim \tau_2$, if there exist $h_i \in \tau_i$, $i = 1, 2$, and $T \in G$ such that $h_1 = T(h_2)$.

This definition is justified since it does not depend on the representatives h_i of the given τ_i. The equivalence classes (orbits mod G) into which $\mathcal{T}$ is partitioned will be called *analytic immersion classes.*

The group G does not act on the surface $\mathcal{T}$ in the ordinary sense, since its elements do not commute with the group F. We obviously have

THEOREM 1: *Every analytic immersion class is dense in the manifold of conformal tori. Every neighbourhood of a conformal torus on $\mathcal{T}$ contains representatives of all analytic immersion classes infinitely often.*

THEOREM 2: *Two conformal tori τ_1 and τ_2 admit a non-constant analytic map $f : \tau_1 \to \tau_2$ if and only if $\tau_1 \sim \tau$ holds. An analytic map is either a constant or a covering map.*

Proof: Choose representatives $h_i \in \tau_i$ and assume that a non-constant analytic $f : E_1/\Gamma_1 \to E_2/\Gamma_2$ exists. According to the fibre map theorem (cf. [4]) there exists a lift of f to the universal covering surfaces E_i, i.e. an entire function $F: E_1 \to E_2$ satisfying $f \circ p_1 = p_2 \circ F$, where $p_i : E_i \to E_i/\Gamma_i$ are the covering projections. Hence there exist two integer-valued functions $n(m, m')$ and $n'(m, m')$ such that

$$F(z_1 + m + m'h_1) = F(z_1) + n + n'h_2 \tag{1}$$

holds identically in $z_1 \in E_1$.

Differentiating with respect to z, we find that F' is a constant C. Substituting

$F(z)=Cz+D$ into (1) and letting first $m=1$, $m'=0$, then $m=0$, $m'=1$, we recognize that there are 4 integers $a=n(1,0)$, $b=n'(1,0)$, $c=n(0,1)$, $d=n'(0,1)$, such that

$$h_1 = \frac{ah_2 + b}{ch_2 + d} \tag{2}$$

holds. Since $\mathfrak{J}(h_i)>0$, $i=1,2$, we have $ad-bc>0$. Reversing the arguments we see that the existence of 4 rationals a, b, c, d satisfying $ad-bc>0$ and (2) is also sufficient for the existence of a non-constant analytic map. Q.E.D.

DEFINITION 2: If f is an analytic map with lift $F(z)=Cz+D$, the constant C will be called *the complex distortion* of f. The set of admissible values of C for two tori is called the *distortion spectrum* of the (ordered) pair of tori.

The fact that $\tau_1 \sim \tau_2$ is an equivalence is worth restating as

THEOREM 3: *If there exists an analytic immersion $\tau_1 \to \tau_2$, then there exists also an analytic immersion $\tau_2 \to \tau_1$.* (In general the inverse F^{-1} of the lift F of f is, however, not the lift of an analytic map $\tau_2 \to \tau_1$.)

Theorem 1 entails: Given two conformal tori τ_1, τ_2, then every neighbourhood of τ_1 on $\mathcal{T}$ contains a countable infinity of tori which admit analytic immersions into τ_2, and uncountably many tori which do not.

LEMMA 1: *If τ_1, τ_2 are two conformal immersion-equivalent tori then it is possible to choose representatives $h_i \in \tau$ $(i=1,2)$ such that $h_1 = a\, h_2$, where a is a positive integer.*

Proof: If $h_i' \in \tau_i$ are arbitrary representatives with $T' \in G$ and $h_1' = T'(h_2')$ then diagonalization of T' leads to three matrices A, $B \in SL(2, Z)$ and $T \in GL^+(2, Q)$ with $h_1 = (A^{-1}T'B)(h_2) = T(h_2)$ and $T = \begin{pmatrix} a, & 0 \\ 0, & 1 \end{pmatrix}$.

Note that none of the three numbers h_1, h_2, a in the relation $h_1 = ah_2$ is invariantly connected with the pair (τ_1, τ_2): If e.g. $h_1 = h_2 = i$, $a=1$, $h_1' = (157+i)/17$, $h_2' = (157+i)/170$, $a'=10$, then h_1 and h_1' represent the same conformal torus τ_1 because of $h_1 = P(h_1')$,

$$P(z) = \frac{z-9}{-4z+37}.$$

Similarly

$$h_2 = Q(h_2') \quad \text{with} \quad Q(z) = \frac{-13z+12}{z-1}.$$

4. Ample Tori

In order to determine the complete set of all analytic maps between two tori we require the following definitions.

DEFINITION 3: The complex number z is called *ample* if $\Re z$ and $|z|^2$ are both rational. A conformal torus τ is called ample if there exists an ample representative $h \in \tau$. A planar lattice

$$\mathscr{L} = \left\{ m\omega + m'\omega' \in \mathbf{C}\colon m, m' \text{ integers}, \quad \Im \frac{\omega'}{\omega} \neq 0 \right\}$$

is ample if it can be generated by two complex numbers ω, ω' with ω'/ω an ample point.

These definitions are justified by their independence from the representative h (invariance under F). The property of a point $h \in H$ of being ample or non-ample is invariant even under the action of G; hence we can also speak of ample and non-ample analytic immersion classes. There are only countably many ample tori, but uncountably many non-ample ones. Each of the two subsets of $\mathscr{T}$ corresponding to these two types of tori is dense in $\mathscr{T}$, and each consists of whole analytic immersion classes. The ample tori do not form a single immersion class.

5. The Isotropy Subgroups of G

The determination of all analytic maps between two conformal tori depends to a large extent on an analysis of the stabilizers of G and their cosets.

DEFINITION 4: Let $h \in H$, and let I_h denote the isotropy subgroup of G with respect to h, i.e. the subgroup of all hyperbolic rotations about h belonging to G.

LEMMA 2: *The structure of I_h is an invariant of the analytic immersion class $G(h)$. $G(h)$ is in 1:1 correspondence with the coset space G/I_h.*

Proof: If h runs through an orbit $G(h)$, then I_h varies in a conjugacy class of G. Since G acts transitively on an orbit, $G(h)$ becomes a homogeneous G-space and is thus representable as G/I_h. Q.E.D.

The structure of I_h differs considerably according to whether h is ample or not.

THEOREM 4: *If $h \in H$ is non-ample, then I_h is trivial.*

Proof: An arbitrary element $S \in I_h$ can be represented as a matrix $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ with relatively prime integral entries, and $\alpha\delta - \beta\gamma > 0$. The relation $S(h) = h$ means:

$$\gamma h^2 + (\delta - \alpha)\, h - \beta = 0. \tag{3}$$

Since $\Im h > 0$, $\gamma = 0$ entails $\alpha = \delta$, $\beta = 0$, i.e. S is the identity. If $\gamma \neq 0$ then (3) is a quadratic equation for h with real coefficients; hence it is satisfied both by h and

$\bar{h}$, $h \neq \bar{h}$. By Vieta's theorem:

$$h + \bar{h} = 2\Re h = \frac{\alpha - \delta}{\gamma}, \tag{4}$$

$$h\bar{h} = |h|^2 = -\frac{\beta}{\gamma}. \tag{5}$$

Since h is not ample, these equations are impossible; hence I_h contains only the identity.

THEOREM 5: *Let $h \in H$ be ample, $2\Re h = p/q$, $|h|^2 = r/s$, p, $q > 0$, $r > 0$, $s > 0$, integers;* g.c.d.$(p, q) =$ g.c.d.$(r, s) = 1$, g.c.d.$(q, s) = g$, $q' = q/g$, $s' = s/g$.

Define $M = \begin{pmatrix} ps', & -q'r \\ qs', & 0 \end{pmatrix}$

Then $I_h = \{\varrho I + \sigma M : \varrho, \sigma$ rational, $\neq (0, 0)\} / \{\lambda I : \lambda$ rational, $\neq 0\}$.

Proof: Let $S = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ represent an element of I_h, with integral entries and $\alpha\delta - \beta\gamma > 0$.

As in the proof of Theorem 4, $\gamma = 0$ leads to the identity map. Assume now $\gamma \neq 0$. Then we have again the relations (4) and (5). Since g.c.d.$(p, q) = 1$, (4) entails the existence of an integer φ such that

$$\alpha - \delta = \varphi p,$$

and

$$\gamma = \varphi q. \tag{6}$$

Substituting (6) into (5) we deduce from g.c.d.$(r, s) = 1$ the existence of an integer ψ satisfying

$$\beta = -\psi r,$$

and

$$\varphi q = \psi s. \tag{7}$$

From (7), $q = gq'$, $s = gs'$, and g.c.d.$(q', s') = 1$, we find an integer v such that $\psi = vq'$, and $\varphi = vs'$. Solving for α, β, γ, δ, we find

$$S = \delta I + vM. \tag{8}$$

Conversely, for an arbitrary choice of the integers δ, v, except $\delta = v = 0$, we gather $S(h) = h$, and $\det S$ becomes a quadratic form in δ and v with discriminant $-4q^2 s'^2 \times (\Im h)^2 < 0$, hence it is positive definite. Thus the above matrix is the most general form representing an element of I_h. The group operations can be easily read off from the relation $M^2 = -rs'qq'I + ps'M$. Q.E.D.

For every ample h the group I_h is a countably infinite, not finitely generated Abelian group which is dense in the group of all hyperbolic rotations about h. Its finer structure depends on number theoretical properties of h; e.g. every I_h contains exactly one element of order 2, but for $h_1 = 2i/\sqrt{3}$ and $h_2 = i$, I_{h_1} does and I_{h_2} does not contain elements of order 3.

Combining theorems 4 and 5 we obtain the following characterization of ample points.

THEOREM 6: *$h \in H$ is ample if and only if there exists in G a hyperbolic rotation about h different from the identity.*

6. Cosets of G mod I_h

In this paragraph a will denote a positive integer which will be identified in § 7 with the quantity introduced in lemma 1.

If h is non-ample then each coset of I_h consists of a single element of G, by theorem 4.

LEMMA 3: *Let $h \in H$ be ample, and define the integers p, q, r, s, g, q', s' as in theorem 5. Furthermore we introduce*

$$g' = \text{g.c.d.}(a, q), \quad a' = a/g', \quad q'' = q/g'',$$
$$g'' = \text{g.c.d.}(a', s'), \quad a'' = a'/g'', \quad s'' = s'/g'',$$
$$T_1 = \begin{pmatrix} a, & 0 \\ 0, & 1 \end{pmatrix}, \quad T_2 = \frac{1}{g'g''} T_1 M = \begin{pmatrix} a'ps'', & -a''q'r \\ q''s'', & 0 \end{pmatrix}.$$

Then the most general integral matrix which represents an element of the left coset of G mod I_h containing the hyperbolic translation $T = T_1$ is given by

$$L = \kappa_1 T_1 + \kappa_2 T_2, \tag{9}$$

where $(\kappa_1, \kappa_2) \neq (0, 0)$ denote arbitrary integers.

Proof: Using (8) we find $TS = \delta T + \nu TM$ with arbitrary integers $(\delta, \nu) \neq (0, 0)$. Since we work in the factor group G of $GL^+(2, Q)$ we still have to determine all rationals $\lambda \neq 0$ and all integers $(\delta, \nu) \neq (0, 0)$ such that

$$L = \lambda(\delta T + \nu TM) \tag{10}$$

becomes integral.

Let λ, δ, ν be such a triple, and define

$$\Lambda(\delta, \nu) = \text{g.c.d.}\{a|\delta + ps'\nu|, aq'r|\nu|, qs'|\nu|, |\delta|\}, \quad \xi = \lambda\Lambda(\delta, \nu).$$

Substituting $\lambda = \xi/\Lambda(\delta, \nu)$ into $\lambda(\delta T + \nu TM)$ we see that this matrix assumes the form $\xi \cdot L'$, where L' is an integer valued matrix whose four entries are relatively prime. Since $\xi L'$ must be integer valued and ξ is rational, it follows that ξ is an integer $\neq 0$.

Conversely, given an integer $\xi \neq 0$ and a pair of integers $(\delta, \nu) \neq (0, 0)$, the quantity $\lambda = \xi / \Lambda(\delta, \nu)$ is rational, $\neq 0$, and $\lambda(\delta T + \nu TM)$ becomes an integral matrix.

Hence this procedure yields all desired triples λ, δ, ν and all matrices L. The correspondence between the triples λ, δ, ν and the matrices L is, however, not 1:1, since two different triples can lead to the same matrix. In order to settle this problem we first determine the admissible values for the elements of the last row of L, viz.

$$\kappa_1' = \xi \frac{\delta}{\Lambda(\delta, \nu)}, \quad \kappa_2' = \frac{s'q\nu}{\Lambda(\delta, \nu)}. \tag{11}$$

Assume that ξ, δ, ν are given. If $\nu = 0$, we have $\Lambda(\delta, \nu) = |\delta|$ and $L = \kappa T$ with an arbitrary integer $\kappa \neq 0$.

Let now $\nu \neq 0$, hence $\kappa_2' \neq 0$, and

$$\delta = \frac{\kappa_1'}{\kappa_2'} qs'\nu. \tag{12}$$

Noting that for every integer $j \neq 0$ $\Lambda(j\delta, j\nu) = j\Lambda(\delta, \nu)$ holds, we obtain from (11) and (12):

$$\xi = \frac{\kappa_2'}{s'qr} \Lambda(\delta, \nu) = \frac{\Lambda(\kappa_1' qs', \kappa_2')}{qs'}$$

Since this must be an integer, inspection of $\Lambda(\kappa_1{}' qs', \kappa_2')$ reveals that qs' must be a factor of the two quantities

$$a|p|\, s'|\kappa_2'| \quad \text{and} \quad aq'r|\kappa_2'|.$$

In the following $g_1, g_2, \dots g_6$ will denote suitable integers. Then the first condition can be written as

$$\frac{|p|}{q} = \frac{g_1}{a\,|\kappa_2'|}, \tag{13}$$

the second as

$$\frac{aq'r\,|\kappa_2'|}{qs'} = g_2. \tag{14}$$

Since the left hand side of (13) is in the lowest terms there is a g_3 such that

$$g_1 = g_3|p|, \tag{15}$$

and

$$a|\kappa_2'| = g_3 q. \tag{16}$$

Dividing (16) by g' we recognize that there exists g_4 with

$$|\kappa_2'| = g_4 q'' , \tag{17}$$

and

$$g_3 = g_4 a' . \tag{18}$$

Substituting (17) into (14) we obtain

$$\frac{g_2}{a' g_4 r} = \frac{q'}{s'} \tag{19}$$

Here the right hand side is in the lowest terms; hence

$$g_2 = g_5 q' , \tag{20}$$

and

$$a' g_4 r = g_5 s' . \tag{21}$$

Dividing (21) by g'' we obtain $a'' g_4 r = g_5 s''$. Because of g.c.d.(a'', s'') = g.c.d.$(r, s) = 1$, we have g.c.d.$(a'' r, s'') = 1$. Thus there is g_6 with

$$g_4 = g_6 s'' , \tag{22}$$

$$g_5 = g_6 a'' r . \tag{23}$$

Combination of (17) and (22) yields $|\kappa_2'| = g_6 q'' s''$. Finally letting $\kappa_1 = \kappa_1'$ and $\kappa_2 = \operatorname{sgn} \kappa_2' \cdot g_6$, we obtain $L = \kappa_1 T_1 + \kappa_2 T_2$ from (10).

Conversely, for an arbitrary choice of the integers $(\kappa_1, \kappa_2) \neq (0, 0)$, we can find a corresponding triple ξ, δ, ν; viz. $\delta = \kappa_1 q s'$, $\nu = \kappa_2 q'' s''$, $\xi = \Lambda(\kappa_1 q s', \kappa_2 q'' s'')/q s'$. (The last expression is easily seen to be an integer.) Q.E.D.

7. The Distortion Spectrum

For given tori τ_1, τ_2 with representatives $h_i \in \tau_i$ chosen according to lemma 1 we identify now $h = h_2$ in lemma 3.

LEMMA 4: *Each integral matrix L representing an element of the coset TI_{h_2} determines an admissible complex distortion for an analytic map $\tau_1 \to \tau_2$, and all analytic maps are obtained in this way.*

Proof: Writing $\tau_i = E_i / \Gamma_i$ the lift $F(z) = Cz + D$ must satisfy the commutation relation $C(z + m + m' h_1) + D = Cz + D + n + n' h_2$, i.e. Γ_2 must contain a subgroup conjugate to Γ_1 in the group of all conformal equivalences of the Euclidean plane.

Letting $m = 1$, $m' = 0$, then $m = 0$, $m' = 1$ and dividing we obtain $h_1 = T(h_2) = L(h_2)$

with

$$L = \begin{pmatrix} n'(0, & 1), & n(0, & 1) \\ n'(1, & 0), & n(1, & 0) \end{pmatrix} \in GL^+(2, Q).$$

Hence $T^{-1}L$ represents an element of I_{h_2}, or L belongs to the left coset of $G \bmod I_{h_2}$ containing T.

Conversely, if we pick all integer valued matrices L representing the same element TI_{h_2} of G/I_{h_2} as T, then we obtain all possible complex distortions as

$$C = n(1,0) + n'(1,0)\, h_2 = n(0,1) + n'(0,1)\, h_2 . \quad \text{Q.E.D.} \tag{24}$$

THEOREM 7: *Let τ_1 and τ_2 be two conformal tori in the same analytic immersion class A. Then the distortion spectrum is given by*

a) *the one-dimensional real lattice*

$$C(\kappa) = \kappa, \quad \kappa = 0, \pm 1, \pm 2, \tag{25}$$

if A is non-ample;

b) *the 2-dimensional lattice*

$$C(\kappa_1, \kappa_2) = \kappa_1 + \kappa_2 q'' s'' h_2$$

for A ample, with κ_1 and κ_2 running independently through all integers. (Notations of lemma 3 applied to $h = h_2$.)

Proof: a) By theorem 4 the coset TI_{h_2} contains only T, and L can be any integral matrix representing the same element of G as T; hence $L = \kappa T$ with an arbitrary integer κ. Thus we obtain (25) from (24).

b) Substitute (9) into (24).

8. Some Consequences

A. Although proportional pairs of integers (κ_1, κ_2) and $(\kappa\kappa_1, \kappa\kappa_2)$ yield the same element $\kappa_1 T_1 + \kappa_2 T_2$ of the coset TI_{h_2}, the corresponding maps are different.

B. In the ample case there always exist sublattices of real and purely imaginary distortions, but the full distortion spectrum is in general larger than the lattice generated by these two sublattices. The full distortion spectrum is an ample lattice.

C. The distortion spectrum depends on the representatives h_i, not only on the surfaces τ_i. If h_i with $h_1 = ah_2$ are used as representatives to compute the maps $\tau_1 \to \tau_2$, then one can use the representatives $(-1)/h_i$ with the same integer a for the determination of the maps $\tau_2 \to \tau_1$ (cf. theorem 3). In this case the latter distortion spectrum is the image of the former under reflexion in the imaginary axis.

D. The fact that the distortion spectrum is in both cases a discrete set implies that maps corresponding to different lattice points are not analytically homotopic. It can be shown that they even belong to different ordinary homotopy classes. Only constant maps are analytically null-homotopic, and two tori are of the same analytic homotopy type if and only if they are conformally equivalent.

BIBLIOGRAPHY

[1] BOREL, A., CHOWLA, S., HERZ, C. S., IWASAWA, K., and SERRE, J-P.: *Seminar on Complex Multiplication*, Lecture Notes in Mathematics No. 21 (Springer-Verlag, Berling-Heidelberg-New York, 1966).
[2] HELFENSTEIN, H. G.: *Analytic Maps between Tori*, Bull. Am. Math. Soc. *75*, No. 4 (1969).
[3] HELFENSTEIN, H. G.: *Local Isometries of Flat Tori*, Pacific J. of Math. *32*, No. 3 (1970).
[4] HU, S.T.: *Homotopy Theory* (Academic Press, New York 1959).
[5] NEVANLINNA, R.: *Uniformisierung* (Springer-Verlag, Berlin 1953).
[6] WOLF, J. A.: *Spaces of Constant Curvature* (McGraw-Hill, New York 1967).

Received March 6, 1970