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Perturbation of Closed Operators and Their Adjoints

by PETER HESS and TOSIO KATO

1. Introduction

Let T denote a densely defined linear operator of a Banach space X into a Banach space Y , and let the linear operator A be bounded from X into Y with domain $D(A) = X$. Then the adjoint of $T + A$ is $T^* + A^*$. In general, if both T and A are densely defined, but unbounded, it is only known that $(T + A)^* \supset T^* + A^*$, provided $D(T) \cap D(A)$ is dense in X . In this note we show that $(T + A)^* = T^* + A^*$ for perturbations A sufficiently small with respect to T measured in the gap topology on the set $C(X, Y)$ of closed linear operators with domain in X and range contained in Y .

THEOREM. *Let X, Y be Banach spaces, $\{T(t)\}_{0 \leq t \leq 1}$ a (in the gap topology) continuous family of densely defined operators in $C(X, Y)$, and $\{S(t)\}_{0 \leq t \leq 1}$ a continuous family of operators in $C(Y^*, X^*)$. Assume $S(t) \subset T(t)^*$ for all t . If $S(t) = T(t)^*$ for some t , then the equality holds for all t .*

If T and A are two linear operators from X into Y , with $D(A) \supset D(T)$, then A is said to be T -bounded if there exist constants $a, b \geq 0$ such that $\|Au\| \leq a\|u\| + b\|Tu\|$ for all $u \in D(T)$. The greatest lower bound of all possible values $b \geq 0$ is called T -bound of A . For $T \in C(X, Y)$ and A a T -bounded operator with relative bound smaller than one, the family $\{T(t) = T + tA\}_{0 \leq t \leq 1}$ of operators in $C(X, Y)$ is continuous in the gap topology ([4], theorem IV. 2.14). Thus, we get as a consequence the following result, which is of interest in the applications:

COROLLARY 1. *Let X, Y denote Banach spaces, and let T be a densely defined operator in $C(X, Y)$. Suppose A is a T -bounded operator such that A^* is T^* -bounded, with both relative bounds smaller than one. Then $T + A$ is closed, and $(T + A)^* = T^* + A^*$.*

This Corollary leads to a new proof of the interior regularity of weak solutions of linear elliptic partial differential equations.

We finally generalize Corollary 1 in a way to observe that the property of linear manifolds to be cores¹⁾ of operators T, T^* is invariant under small perturbations of these operators. This result extends a similar assertion by F. E. Browder ([1], Theorem 22) inasmuch as we do not assume any existence of regular points of the operator T .

It was pointed out by Professor Browder that our results are strongly connected to those in [2] and [3].

¹⁾ A linear manifold D_0 contained in the domain of a closed linear operator T is called a *core* of T if the closure of the restriction of T to D_0 is again T .

2. Proof of the Theorem

We consider first closed linear manifolds M, N, \dots of a Banach space Z . Let S_M denote the unit sphere of M . For any two closed linear manifolds M, N of Z , we set²⁾

$$\delta(M, N) = \sup_{u \in S_M} \text{dist}(u, N), \tag{1}$$

$$\hat{\delta}(M, N) = \max(\delta(M, N), \delta(N, M)). \tag{2}$$

If $M=0$, (1) has no meaning; in this case we define $\delta(0, N)=0$ for any N . $\hat{\delta}(M, N)$ is called the *gap* between M and N . Though $\hat{\delta}$ is in general not a proper distance function, it is equivalent to such a function. In [4], Lemma IV.2.2, it is proved that if M, N are two closed linear manifolds of Z , and if $u \in Z$, then

$$(1 + \delta(M, N)) \text{dist}(u, M) \geq \text{dist}(u, N) - \|u\| \cdot \delta(M, N). \tag{3}$$

LEMMA 1. *Let Z be a Banach space, M, N, N' closed linear manifolds in Z , where $N' \subset N$. If*

$$\delta(N, M) < \frac{1}{3}, \quad \delta(M, N') < \frac{1}{3}, \tag{4}$$

then $N' = N$.

Proof. Suppose $N' \neq N$. For any $\varepsilon > 0$, there exists an element $u \in N$ such that $\|u\| = 1$ and $\text{dist}(u, N') > 1 - \varepsilon$ (see [4], Lemma III.1.12). Replacing N by N' in (3) we get

$$(1 + \delta(M, N')) \text{dist}(u, M) \geq \text{dist}(u, N') - \|u\| \delta(M, N') > 1 - \varepsilon - \delta(M, N').$$

Since $\delta(N, M) \geq \text{dist}(u, M)$ and $\varepsilon > 0$ is arbitrary, we obtain

$$(1 + \delta(M, N')) \delta(N, M) \geq 1 - \delta(M, N'),$$

which is a contradiction because of (4), q.e.d.

For closed linear manifolds M, N of Z , with M^\perp, N^\perp denoting the orthogonal complement in Z^* of M and N respectively, we have ([4], Theorem IV.2.9)

$$\hat{\delta}(M, N) = \hat{\delta}(M^\perp, N^\perp). \tag{5}$$

LEMMA 2. *Let $\{M(t)\}_{0 \leq t \leq 1}$ be a (in the gap topology) continuous family of closed linear manifolds of a Banach space Z , and let $\{N(t)\}_{0 \leq t \leq 1}$ be a similar family in the dual space Z^* . Assume $N(t) \subset M(t)^\perp$ for all t . If $N(t) = M(t)^\perp$ holds for some t , then it holds for all t .*

²⁾ see [4], chapt. IV, § 2.

Proof. The families $\{M(t)\}$ and $\{N(t)\}$ are uniformly continuous on $[0, 1]$. Also $\{M(t)^\perp\}$ has the same property by (5).

Suppose $N(t_0) = M(t_0)^\perp$ for some $t_0 \in [0, 1]$. By the stated uniform continuity, there is an $\varepsilon > 0$ (independent of t_0) such that $\hat{\delta}(N(t), M(t_0)^\perp) = \hat{\delta}(N(t), N(t_0)) < \frac{1}{3}$ and $\hat{\delta}(M(t)^\perp, M(t_0)^\perp) < \frac{1}{3}$ if $|t - t_0| < \varepsilon$. Then $N(t) = M(t)^\perp$ by Lemma 1. Thus the desired property for t propagates to all $t \in [0, 1]$ in a finite number of steps, q.e.d.

Let us now consider the set $C(X, Y)$ of all closed linear operators from X into Y . If $S, T \in C(X, Y)$, their graphs $G(S), G(T)$ are closed linear manifolds of the Banach space $Z = X \times Y$, where the norm is chosen to be $\|\{u, v\}\| = (\|u\|^2 + \|v\|^2)^{1/2}$ for $u \in X, v \in Y$, implying that $Z^* = X^* \times Y^*$. We set

$$\delta(S, T) = \delta(G(S), G(T)), \quad \hat{\delta}(S, T) = \hat{\delta}(G(S), G(T)); \quad (6)$$

$\hat{\delta}(S, T)$ is then called the *gap* between S and T .

Proof of the Theorem. We apply Lemma 2 with $Z = X \times Y, Z^* = X^* \times Y^*, M(t) = G(T(t))$ and $N(t) = G'(-S(t))$ (the inverse graph of $-S(t)$), noting that $M(t)^\perp = G(T(t))^\perp = G'(-T(t)^*) \supset G'(-S(t)) = N(t)$ for all t , and that $\hat{\delta}(G(S_1), G(S_2)) = \hat{\delta}(G'(S_1), G'(S_2))$, q.e.d.

3. Interior Regularity of Solutions of Linear Elliptic Equations

In the Sobolev spaces $H^s(\mathbb{R}^n)$ (s an integer) being the completion of the set $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_s^2 = \int (1 + |k|^2)^s |\hat{u}(k)|^2 dk$ (where \hat{u} denotes the Fourier transformed of the function u), we consider linear elliptic differential expressions³⁾ $\mathcal{S} = \sum_{|p| \leq m} a_p(x) D^p$ of order m with coefficients $a_p \in C^\infty(\mathbb{R}^n)$ which are bounded together with all their derivatives. Under these assumptions, \mathcal{S} maps the space $H^{s+m}(\mathbb{R}^n)$ continuously into $H^s(\mathbb{R}^n)$. Let \mathcal{S}^* denote the formal adjoint expression, $\mathcal{S}^* = \sum_{|p| \leq m} (-1)^{|p|} D^p(\bar{a}_p(x))$. Then⁴⁾ $(\mathcal{S}^*u, v)_0 = (u, \mathcal{S}v)_0$ for all $u, v \in C_0^\infty(\mathbb{R}^n)$ and, by a limiting process, for all $u \in H^s(\mathbb{R}^n), v \in H^t(\mathbb{R}^n)$ with $s+t \geq m$.

The following theorem on interior regularity of weak solutions of linear elliptic equations is well known:

THEOREM. *Let \mathcal{S} be a linear elliptic differential expression of order m , defined in an open domain $\Omega \subset \mathbb{R}^n$, with coefficients in $C^\infty(\Omega)$. Suppose $u \in L_{\text{loc}}^2$ is a weak solution of the equation $\mathcal{S}u = f, f \in H_{\text{loc}}^t$ ($t \geq 0$), i.e.*

$$(u, \mathcal{S}^*\varphi)_0 = (f, \varphi)_0$$

for all $\varphi \in C_0^\infty(\Omega)$. Then $u \in H_{\text{loc}}^{t+m}$.

³⁾ Script letters denote formal differential expressions, latin letters the induced differential operators acting in some Sobolev space $H^s(\mathbb{R}^n)$, with prescribed domain of definition.

⁴⁾ $(\cdot, \cdot)_0$ denotes the L^2 scalar product.

The theorem can be reduced to a similar *global* assertion on the solutions of elliptic equations involving expressions with “almost constant” coefficients, defined on the whole of R^n . It is rather easy to show interior regularity of solutions of elliptic equations $\mathcal{T}u=f$ with \mathcal{T} having constant coefficients. Based on these properties of elliptic expressions with constant coefficients which we assume to be known, we prove as an application of Corollary 1 the following

PROPOSITION. *Let $\mathcal{S} = \sum_{|p| \leq m} a_p(x) D^p$ be a linear elliptic differential expression with coefficients $a_p \in C^\infty(R^n)$ which are bounded together with all their derivatives. Suppose there exists $x_0 \in R^n$ such that with the notation $a_p^0 = a_p(x_0)$,*

$$\left| \sum_{|p|=m} a_p^0 \xi^p \right| \geq \alpha |\xi|^m, \quad \left| \sum_{|p|=m} (a_p(x) - a_p^0) \xi^p \right| \leq \beta |\xi|^m \tag{7}$$

for all $\xi \in R^n$, where $\beta < \alpha$.

If $u \in H^s(R^n)$ and $\mathcal{S}u=f \in H^s(R^n)$, then $u \in H^{s+m}(R^n)$.

Proof. Let $\mathcal{T} = \sum_{|p| \leq m} a_p^0 D^p$ denote the elliptic differential expression with constant coefficients, and set $\mathcal{A} = \sum_{|p| \leq m} (a_p(x) - a_p^0) D^p$. Let T be the operator induced by \mathcal{T} in the space $H^s(R^n)$, with $D(T) = H^{s+m}(R^n)$. The first assumption in (7) implies that to each $\varepsilon > 0$ there exists a constant $b(\varepsilon)$ such that

$$\|Tu\|_s \geq (\alpha - \varepsilon) \|u\|_{s+m} - b(\varepsilon) \|u\|_s \tag{8}$$

for all $u \in D(T)$, as is seen by applying the Gårding inequality to the elliptic differential expression $\mathcal{T}^* \mathcal{T} - (\alpha - \varepsilon)^2 (1 + \Delta)^m$. We infer that the operator T is closed. The spaces $H^s(R^n)$ and $H^{-s}(R^n)$ being mutually adjoint by the scalar product $(\cdot, \cdot)_0$, T^* is a linear operator in $H^{-s}(R^n)$, induced by \mathcal{T}^* . Because of the regularity properties of elliptic mappings with constant coefficients, T^* has domain $H^{-s+m}(R^n)$. Since \mathcal{T} and \mathcal{T}^* have conjugate complex principal parts, for each $\varepsilon > 0$ an estimate of the form

$$\|T^*v\|_{-s} \geq (\alpha - \varepsilon) \|v\|_{-s+m} - b^*(\varepsilon) \|v\|_{-s} \tag{8a}$$

holds for all $v \in D(T^*)$.

Let A and A^* be the operators induced by \mathcal{A} and \mathcal{A}^* , respectively, acting in the same spaces as T and T^* , and with domains $D(A) = D(T)$, $D(A^*) = D(T^*)$. As a consequence of the second estimate in (7) and of Gårding's inequality, to each $\varepsilon > 0$ there exist constants $c(\varepsilon, s)$ and $c^*(\varepsilon, s)$ such that

$$\|Au\|_s \leq (\beta + \varepsilon) \|u\|_{s+m} + c(\varepsilon, s) \|u\|_s, \quad u \in D(T), \tag{9}$$

$$\|A^*v\|_{-s} \leq (\beta + \varepsilon) \|v\|_{-s+m} + c^*(\varepsilon, s) \|v\|_{-s}, \quad v \in D(T^*). \tag{9a}$$

It follows from the inequalities (8), (9) as well as (8a), (9a) that A and A^* are relatively bounded with respect to T and T^* , with bounds smaller than one.

For all $v \in D(T^*)$,

$$(v, f)_0 = (v, (\mathcal{T} + \mathcal{A})u)_0 = ((\mathcal{T}^* + \mathcal{A}^*)v, u)_0 = ((T^* + A^*)v, u)_0.$$

Therefore, applying Corollary 1 (with $X=Y=H^{-s}(R^n)$) and making use of the reflexivity of the Banach spaces $H^s(R^n)$,

$$\begin{aligned} u \in D((T^* + A^*)^*) &= D(T^{**} + A^{**}) = D(T + A^{**}) = D(T + A) \\ &= D(T) = H^{s+m}(R^n), \quad \text{q.e.d.} \end{aligned}$$

4. Invariance of cores under perturbations. The following slight generalization of Corollary 1 extends a result by Browder [1].

COROLLARY 2. *Let X, Y be Banach spaces, $T \in C(X, Y)$ densely defined, and suppose D_0 is a core of T , D_1 a core of T^* . Let further $A \in C(X, Y)$ with $D(A) \supset D_0$, $D(A^*) \supset D_1$, such that*

$$\|Au\| \leq a\|u\| + b\|Tu\|, \quad u \in D_0,$$

$$\|A^*v\| \leq a\|v\| + b\|T^*v\|, \quad v \in D_1,$$

with $0 \leq b < 1$. Then $S = T + A$ is closed with $D(S) = D(T)$, $S^* = T^* + A^*$ and has domain $D(S^*) = D(T^*)$, and D_0 and D_1 are cores of S and S^* , respectively.

Proof. By T_0 we denote the restriction of T to D_0 . Similar notations are used for restrictions of other operators. The T_0 -boundedness of A_0 implies the T -boundedness of the closure \tilde{A}_0 of A_0 with conservation of the relative bound. Hence $S = T + A = T + \tilde{A}_0$ is closed, and $D(S) = D(T)$. By the same argument, A^* is T^* -bounded with bound smaller one, and consequently $S^* = (T + A)^* = T^* + A^*$ with $D(S^*) = D(T^*)$ by Corollary 1. It remains to show that D_0 and D_1 are cores of S and S^* , respectively.

Evidently $\tilde{S}_0 \subset S$. Conversely suppose $u \in D(S) = D(T)$. Then it exists a sequence $\{u_n\} \subset D_0$ with $u_n \rightarrow u$, $T_0 u_n \rightarrow Tu$. Therefore $A_0 u_n \rightarrow Au$, and we conclude that $u_n \rightarrow u$ and $S_0 u_n \rightarrow Su$. Consequently $u \in D(\tilde{S}_0)$ and $\tilde{S}_0 u = Su$, i.e. $\tilde{S}_0 \supset S$. It is analogously seen that $(S_1^*)^\sim \subset S^*$ and $(S_1^*)^\sim \supset T^* + A^* = S^*$, q.e.d.

Example. Let T be an elliptic differential operator of order m with constant coefficients, acting in some Sobolev space $H^s(R^n)$, with domain $H^{s+m}(R^n)$. Its adjoint T^* in $H^{-s}(R^n)$ has domain $H^{-s+m}(R^n)$. It is obvious that $C_0^\infty(R^n)$ is a core of the operators T and T^* .

Consider now an elliptic differential expression \mathcal{S} of order m satisfying the hypotheses of the Proposition. Let S be the operator induced by \mathcal{S} in $H^s(R^n)$, with $D(S) = H^{s+m}(R^n)$. Then S^* acts in the space $H^{-s}(R^n)$ and has domain $H^{-s+m}(R^n)$. Further $C_0^\infty(R^n)$ is core of S and S^* .

REFERENCES

- [1] BROWDER, F. E.: *On the spectral theory of elliptic differential operators I*. Math. Ann. 142, 22–130 (1961).
- [2] ———, *A continuity property for adjoints of closed operators in Banach spaces, and its application to elliptic boundary value problems*. Duke Math. Journal 28, 157–182 (1961).
- [3] ———, *Non-local elliptic boundary value problems*. Amer. Journ. Math. 86, 735–750 (1964).
- [4] KATO, T.: *Perturbation theory for linear operators*. Springer-Verlag Berlin/Heidelberg/New York (1966).

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Added in Proof. Making use of Corollary 1 and an argument on holomorphic operator families of type (A) ([4], chapt. VII, § 2) closely related to a method applied by R. Wüst (*Stabilität der Selbstadjungiertheit gegenüber Störungen*; Dissertation, Rhein.-Westfäl. Techn. Hochschule Aachen, 1970), one can prove the following.

PROPOSITION. *Let $T \in C(X, Y)$ be densely defined, and let A be a linear operator from X into Y having the property that $D(A) \supset D(T)$ and $D(A^*) \supset D(T^*)$. Suppose that for each $t \in [0, 1]$, the operators $T + tA$ and $T^* + tA^*$ are closed. Then $(T + A)^* = T^* + A^*$.*

This result allows especially to discuss the limit case of Corollary 1 where the relative bounds of A and A^* with respect to T and T^* equal one.

LEMMA 3. *Let $T \in C(X, Y)$, and let A be T -bounded with T -bound one. Then the following assertions are equivalent:*

(i) $T + A$ is closed;

(ii) There exist constants $a, b > 0$ such that for all $u \in D(T)$,

$$\|Au\| \leq a\|u\| + b\|(T + A)u\|.$$