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A Model of Intuitionistic Analysis

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Introduction

The aim of the present paper is to present a new model of intuitionistic analysis where by "intuitionistic analysis" we mean essentially the formal system codified in [2]. The intuitive background of the model (explained in section 3.3) is technical in nature and has nothing to do with any philosophy of free choice sequences. More precisely the model is derived from known proof theoretic properties of intuitionistic formal systems as will be explained below. In chapters I, II we present the preliminaries needed in order to understand the intuitive background and the model, described in chapters III and IV respectively. In IV, the main chapter, we develop the full model. Chapter V contains additional comments and conclusive remarks.

1. Continuity functions

1.1. *Sequence numbers.*

Let $p_0, p_1, p_2, \ldots$ be the list of primes, listed in increasing order and starting with 2. With the finite sequence of natural numbers $a_0, a_1, \ldots, a_{s-1}$ we associate the natural number $m=\prod_0^{s-1} p_i^{a_i+1}$. We call m the sequence number associated with $a_0, \ldots, a_{s-1}$ and denote it by $\langle a_0, \ldots, a_{s-1}\rangle$. Its length, denoted by length (m), is s. With the empty sequence we associate the number 1, also denoted by $\langle\ \rangle$; by definition length $(1)=0$. N is the set of natural numbers and N_m denotes the set of m-place number theoretic functions; N_0 is identified with N. For $f\in N_1$ we put $\bar{f}(0)=1$ and $\bar{f}(n)=$ $=\langle f(0), \ldots, f(n-1)\rangle$ for $n>0$. In order to denote sequence numbers we use italic letters $u, v, w, u_1, u_2, \ldots$ etc. We say that $w=\langle a_0, \ldots, a_{s-1}\rangle$ is an extension of $v=$ $=\langle b_0, \ldots, b_{t-1}\rangle$ if $t<s$ and $a_i=b_i$ for $i<t$; in this case we write $v\subset w$. $v\subseteq w$ is short for "$v\subset w$ or $v=w$". We also write $v\subset f$ if $v=\bar{f}(n)$ for some n. With v, w as before we denote by $w*v$ the sequence number $\langle a_0, \ldots, a_{s-1}, b_0, b_{t-1}\rangle$; for $f\in N_1$ we denote by $w*f$ the element of N_1 given by $w*f(i)=a_i$ for $i<s$ and $w*f(i)=f(i-s)$ for $i\geqslant s$.

1.2. *Continuity functions.*

An element $\tau\in N_s$ is called a continuity function if a) if $\tau(n_1, \ldots, n_s)\neq 0$ then all n_i are sequence numbers all having one and the same length, b) for elements $f_i\in N_1$ $(i\leqslant s)$, if $\tau(\bar{f}_1(n), \ldots, \bar{f}_s(n))\neq 0$ then $\tau(\bar{f}_1(n), \ldots, \bar{f}_s(n))=\tau(\bar{f}_1(m), \ldots, \bar{f}_s(m))$ for all $n<m$, c) for every s-tupel $f_i\in N_1$ $(i\leqslant s)$ there is an n with $\tau(\bar{f}_1(n), \ldots, \bar{f}_s(n))\neq 0$. More generally an element $\tau(x_1, \ldots, x_s, y_1, \ldots, y_t)\in N_{s+t}$ is said to be a continuity function of type (s, t) if

$\tau(x_1, \ldots, x_s, n_1, \ldots, n_t)$ is a continuity function with respect to $x_1, \ldots, x_s$ for all t-tupel $n_1, \ldots, n_t$. Clearly, every continuity function $\tau(x_1, \ldots, x_s)$ is a continuity function of type (s, 0); every element from N_t whose range does not contain 0 can be considered as a continuity function of type (0, t), and every $n \in N$ different from 0 is a continuity function of type (0, 0).

1.3. *Continuous functionals.*

If A is any set then A^s denotes the s-fold cartesian product of A. A functional of type (s, t) is a mapping of $N_1^s \times N^t$ into N. If e is a functional of type (s, t), if $f_1, \ldots, f_s$ and $n_1, \ldots, n_t$ are elements of N_1 and N respectively then we denote the value of e for $f_1, \ldots, f_s$ and $n_1, \ldots, n_t$ by $e(f_1, \ldots, f_s, n_1, \ldots, n_t)$. As functionals of type (0, 0) we simply take the natural numbers. With every continuity function τ of type (s, t) we associate in a unique way a functional of type (s, t) denoted by e_τ as follows: if $\tau(\bar{f}_1(m), \ldots, \bar{f}_s(m), n_1, \ldots, n_t)$ is $\neq 0$ then $e_\tau(f_1, \ldots, f_s, n_1, \cdots, n_t) = \tau(\bar{f}_1(m), \cdots, \bar{f}_s(m), n_1, \cdots, n_t) - 1$. Every functional of type (s, t) which is of the form e_τ is said to be continuous. We call e_τ the functional induced by τ. Among the continuous functionals we mention four of type (0,1), (0,2), (0,2) and (0,2) respectively. The first is the successor function ′, the second is addition +, the third is multiplication · and the fourth is concatenation *.

1.4. *Continuous operators.*

A mapping F from $N_1^s \times N^t$ into N_1 is called an operator of type (s, t). As operators of type (0,0) we simply take the elements of N_1. If F is an operator of type (s, t), if $f_1, \ldots, f_s$ and $n_1, \ldots, n_t$ are elements of N_1 and N respectively then we denote the value of F for these arguments by $F[f_1, \ldots, f_s, n_1, \ldots, n_t]$. With every continuity function τ of type (s, t+1) we can associate a functor F_τ as follows: if $\tau(\bar{f}_1(m), \ldots, \bar{f}_s(m), n_1, \ldots, n_t, q) \neq 0$ then $F_\tau[f_1, \ldots, f_s, n_1, \ldots, n_t](q) = \tau(\bar{f}_1(m), \ldots, \bar{f}_s(m), n_1, \ldots, n_t, q) - 1$. An operator which is of the form F_τ is said to be continuous. We call F_τ the functor induced by τ. Among the continuous operators we mention a particular one, of type (1,1), denoted by C. The definition of C is as follows: a) if n is not a sequence number then $C[f, n](i) = 0$ for all i, b) if $u = \langle u_0, \ldots, u_{s-1} \rangle$ then $C[f, u](i) = u_i$ for $i < s$ and $= f(i-s)$ if $i \geqq s$. We also write more suggestively $u * f$ in place of $C[f, u]$.

2. A formal language

2.1. *The alphabet.*

Let e and G be two distinct and fixed symbols. With every continuity function τ of type (s, t) we associate the expression $e^\tau_{s,t}$, with every continuity function σ of type (s, t+1) we associate the expression $G^\sigma_{s,t}$. In order to avoid complex notations we often omit the indices s, t and merely write e^τ and G^σ respectively. Now we introduce a formal language L, which contains a nondenumerable set of constants. The alphabet of L contains the following symbols: a) the logical connectives $\wedge$, $\vee$, $\neg$, $\supset$, $\forall$, E, b) the equality sign $=$ and the abstraction operator λ, c) for every natural number m an individual constant m_0, d) a denumerable list $a_1, a_2, \ldots$, a, b, c... of free individual variables, e) a denumerable list $x_1, x_2, \ldots$, x, y, z,... of bound individual variables, f) a denumerable list $\xi_1, \xi_2, \ldots, \xi, \eta, \zeta, \ldots$ of bound function variables, g) for every sequence number $u = \langle u_0, \ldots, u_{s-1} \rangle$ ($u = \langle\ \rangle$ included) a denumerable list $\alpha_u^1, \alpha_u^2, \ldots, \alpha_u$, $\beta_u, \ldots$ of free choice variables, h) for every continuity function τ of type (s, t) the symbol $e_{s,t}^\tau$, i) for every continuity function σ of type (s, t+1) the symbol $G_{s,t}^\sigma$, j) two pairs of brackets [,] and (,). The symbols $e_{s,t}^\tau$ are called functional constants of type (s, t), the symbols $G_{s,t}^\sigma$ are called functor constants of type (s, t). The free choice variables α_u^i, α_u etc. are assumed to range over number theoretic functions f such that $\bar{f}(s) = \langle u_0, \ldots, u_{s-1} \rangle = u$. The alphabet of L is highly nonconstructive in that it contains a nondenumerable set of constants. It would not be difficult below to avoid the use of uncountable many constants, however their use turns out to be very convenient in that we can save quite a bit of notation. In the cases where τ is $'$, $+$, $\cdot$, $*$ we obtain corresponding functional constants $e_{0,1}^\tau$ and $e_{0,2}^\tau$ respectively, which for simplicity will also be denoted by $'$, $+$, $\cdot$, $*$ respectively. The operator C (see end of 1.4) is of course continuous, that is of the form F_σ for some σ of type (1,2). The operator constant $G_{1,4}^\sigma$ corresponding to this σ will also be denoted by C. Without confusion we often omit the index o in m_0 and simply write m.

2.2. *Terms, functors and formulas.*

Starting with the alphabet we build up terms and functors by simultaneous inductive definition as follows: a) the m_0's and all free individual variables are terms, b) free choice variables and functor symbols of type (0,1) are functors, c) if e is a functional constant of type (s, t), if $F_1, \ldots, F_s$ are functors and $q_1, \ldots, q_t$ terms then $e(F_1, \ldots, F_s, q_1, \ldots, q_t)$ is a term, d) if G is a functor symbol of type (s, t), if $F_1, \ldots, F_s$ are functors and $q_1, \ldots, q_t$ terms then $G[F_1, \ldots, F_s, q_1, \ldots, q_t]$ is a functor, e) if F is a functor and t a term then F(t) is a term, f) if t(a) is a term and a free individual variable then $(\lambda x t(x))$ is a functor, where x is a bound inividual variable not occurring in t(a). Prime formulas are those of the form p=q with p, q terms. Formulas are given as follows: 1) prime formulas are formulas, 2) if A, B are formulas, then so are $A \wedge B$, $A \vee B$, $\neg A$, $A \supset B$, 3) if A(a) is a formula and a a free individual variable then $(\forall x)\, A(x)$ and $(Ex)\, A(x)$ are formulas, where x is a bound variable not occurring in A(a), 4) if $A(\alpha_{\langle\ \rangle}^i)$ is a formula and $\alpha_{\langle\ \rangle}^i$ a free choice variable associated with the empty sequence then $(\forall \xi)\, A(\xi)$ and $(E\xi)\, A(\xi)$ are formulas where ξ is a bound function variable not occurring in $A(\alpha_{\langle\ \rangle}^i)$. Universal quantification is often written more briefly as $(x)\, A(x)$, $(\xi)\, A(\xi)$ instead of $(\forall x)\, A(x)$, $(\forall \xi)\, A(\xi)$ respectively.

2.3. *Other languages.*

In one place below we will use the language of second order analysis used in [2];

we denote it by L_K. In [1] a certain formalisation of number theory is presented. The language on which this formalisation is based is denoted by L_N.

2.4. *Saturation.*

There are two important notions, namely that of a saturated term and that of its value. By definition only closed terms, that is terms without free individual variables will be saturated. If t is a saturated term then we denote its value by $|t|$. The definitions are given by induction with respect to the number of symbols contained in t, observing thereby the following conventions: a) the symbols λ, [,], =, (,), m, x_1, x_2,... are counted once, b) the symbols α_u^i, e_τ, G_τ are counted twice. We say that t is saturated and that its value is $|t|$ if one of the clauses below is satisfied.

Clause 0: t is m_0. Then t is saturated and $|t| = m$.

Case 1: t is $\alpha_u^k(p)$, p is a saturated term, $|p| = j$, $u = \langle a_0, \ldots, a_{s-1} \rangle$ and $j < s$. Then t is saturated and $|t| = a_j$.

Case 2: t is $G_\tau(p)$, p is saturated, $|p| = n$ and is of type (0,1). Then $G_\tau(p)$ is saturated and $|G_\tau(p)| = \tau(n) - 1$.

Case 2: t is $e_\tau(G_1, \ldots, G_s, q_1, \ldots, q_t)$. Then t is called saturated if the following conditions are satisfied: 1) all q_i's are saturated and $|q_i| = n_i$, 2) for every $i \leqq s$ there is a $k_i > 0$ such that $G_i(j_0)$ is saturated for all $j < k_i$, 3) $\tau(u_1, \ldots, u_s, n_1, \ldots, n_t) \neq 0$ where u_i is the sequence number $\langle |G_i(0_0)|, \ldots, |G_i((k_i - 1)_0)| \rangle$. We put $|t| = \tau(u_1, \ldots, u_s, n_1, \ldots, n_t) - 1$. We note that in view of our convention $G_i(j_0)$ contains less symbols than t.

Case 3: t is $G_\tau[G_1, \ldots, G_s, q_1, \ldots, q_t](p)$. Then t is called saturated if the following conditions are satisfied: 1) all q_i's are saturated and $|q_i| = n_i$, 2) p is saturated and $|p| = m$, 3) for every $i \leqslant s$ there is a $k_i > 0$ such that $G_i(j_0)$ is saturated for all $j < k_i$, 4) $\tau(u_1, \ldots, u_s, n_1, \ldots, n_t, m) \neq 0$ where u_i is $\langle |G_i(0_0)|, \ldots, |G_i((k_i - 1)_0)| \rangle$. We put $|t| = \tau(u_1, \ldots, u_s, n_1, \ldots, n_t, m) - 1$. As before, $G_i(j_0)$ contains less symbols than t in view of our counting convention.

Case 4: t is $(\lambda x p(x))(q)$. Then t is saturated if 1) q is saturated, 2) $p(m_0)$ is saturated where $m = |q|$. We put $|t| = |p(m_0)|$.

In connection with this definition we introduce a notation. Let $u = \langle a_0, \ldots, a_{s-1} \rangle$ be a sequence number and G a constant functor having the property: if $i < s$ then $G(i_0)$ is saturated and $|G(i_0)| = a_i$. Then we write $u \subseteq G$.

A few properties of saturation have to be known. To this end we need a lemma on continuity functions.

LEMMA 1: *Let τ have type* (p, q). *Let $\sigma_1, \ldots, \sigma_p$ and $\mu_1, \ldots, \mu_q$ have type* (s, 1) *and* (s, 0) *respectively. Then there is a continuity function v of type* (s, 0) *having the following property: if $v(w_1, \ldots, w_s) \neq 0$ then* 1) $\mu_j(w_1, \ldots, w_s) \neq 0$ *for all* $j = 1, \ldots, q$, 2) *there is an* $i \leqslant$ *length*(w_1) *such that* $\sigma_j(\vec{w}, k) \neq 0$ *for all* $k < i$ *and all* $j \leqq s$ *and such that* $\tau(u_1, \ldots, u_p,$

$n_1, \ldots, n_q) \neq 0$ *where* $u_j = \langle \sigma_j(\vec{w}, 0) - 1, \ldots, \sigma_j(\vec{w}, i-1) - 1 \rangle$, $n_j = \mu_j(\vec{w}) - 1$ *and where* $\vec{w}$ *is an abbreviation for* $w_1, \ldots, w_s$.

Proof: Call an s-tupel $w_1, \ldots, w_s$ of sequence numbers secured if they all have the same length and if in addition 1) and 2) of the lemma are satisfied. Put $\nu(w_1, \ldots, w_s) = 1$ iff $w_1, \ldots, w_s$ is secured and 0 otherwise. It remains to show: if $f_1, \ldots, f_s$ are numbertheoretic functions then there is an N such that $\nu(\bar{f}_1(N), \ldots, \bar{f}_s(N)) = 1$. To this end we define numbertheoretic functions g_j, $j = 1, \ldots, p$ as follows: $g_j(n) = m$ iff there is a k with $\sigma_j(\bar{f}_1(k), \ldots, \bar{f}_s(k), n) = m + 1$. Next we determine an M so large that $\mu_j(\bar{f}_1(M), \ldots, \bar{f}_s(M)) = n_j + 1$ for $j = 1, \ldots, q$. Then we clearly find an i such that $\tau(\bar{g}_1(i), \ldots, \bar{g}_p(i), n_1, \ldots, n_q) \neq 0$. With this i given we finally determine an $N \geqq M$ so large that $\sigma_j(\bar{f}_1(N), \ldots, \bar{f}_s(N), k) \neq 0$ for all $j = 1, \ldots, p$ and all $k < i$. Necessarily

$$\bar{g}_j(i) = \langle \sigma_j(\bar{f}_1(N), \ldots, \bar{f}_s(N), 0) - 1, \ldots, \sigma_j(\bar{f}_1(N), \ldots, \bar{f}_s(N), i - 1) - 1 \rangle$$

and $n_j + 1 = \mu_j(\bar{f}_1(N), \ldots, \bar{f}_s(N))$ holds. Then clearly $\nu(\bar{f}_1(N), \ldots, \bar{f}_s(N)) = 1$ what concludes the proof.

LEMMA 2: *Let t be a term without free individual variables whose choice variables are among* $\alpha^{i_1}_{u_1}, \ldots, \alpha^{i_s}_{u_s}$; *we indicate this by writing* $t(\alpha^{i_1}_{u_1}, \ldots, \alpha^{i_s}_{u_s})$. *Then there is a continuity function of type* (s, 0), *say* τ, *with the property*: *if* $\tau(v_1, \ldots, v_s) = m + 1$ *then* $t(\alpha^{i_1}_{u_1 * v_1}, \ldots, \alpha^{i_s}_{u_s * v_s})$ *is saturated and its value is* m.

Proof: We proceed by induction with respect to the number of symbols in t, respecting thereby the counting convention. In order to avoid heavy symbolism we content ourself with the discussion of some typical cases. *Case* 1: t is $e_\tau(G[\alpha_u, \beta_v], q(\alpha_u, \beta_v))$ where G and q do not contain other choice variables than α_u, β_v. For every n, $G[\alpha_u, \beta_v](n)$ has less symbols than t. By induction there are continuity functions of type (2,0), say σ_n, with the property: if $\sigma_n(u_1, v_1) = m + 1$ then $G[\alpha_{u * u_1}, \beta_{v * v_1}](n)$ is saturated and has value m. We piece the σ_n's together in order to obtain a continuity function σ of type (2,1) with the property: if $\sigma(u_1, v_1, n) = m + 1$ then $G[\alpha_{u * u_1}, \beta_{v * v_1}](n)$ is saturated and has value m. On the other hand $q(\alpha_u, \beta_v)$ has less symbols than t. Hence there is a continuity function μ of type (2,0) with the property: if $\mu(u_1, v_1) = m + 1$ then $q(\alpha_{u * u_1}, \beta_{v * v_1})$ is saturated and has value m. According to the previous lemma there is a continuity function ν of type (2,0) with the property: if $\nu(u_1, v_1) \neq 0$ then 1) $q(\alpha_{u * u_1}, \beta_{v * v_1})$ is saturated with value say m, 2) there is an $i \leqq \mathrm{length}(u_1)$ such that $G[\alpha_{u * u_1}, \beta_{v * v_1}](k)$ is saturated for all $k < i$ with value say m_k and in addition $\tau(\langle m_0, \ldots, m_{i-1} \rangle, m) \neq 0$. But according to the definition of saturation this means that $e_\tau(G[\alpha_{u * u_1}, \beta_{v * v_1}], q(\alpha_{u * u_1}, \beta_{v * v_1}))$ is saturated and has value $\tau(\langle m_0, \ldots, m_{i-1} \rangle, m) - 1$. The continuity function whose existence is postulated by the lemma is now given by $\nu *$ whose definition goes as follows: 1) $\nu * (u_1, v_1) \neq 0$ iff

$\nu(u_1, v_1) \neq 0$, 2) if $\nu(u_1, v_1) \neq 0$ then $\nu*(u_1, v_1) = \tau(\langle m_0, \ldots, m_{i-1}\rangle, m)$ with $m_0, \ldots, m_{i-1}$ and m as above.

Case 2: t is $F_\tau[G[\alpha_u, \beta_v], q(\alpha_u, \beta_v)](p(\alpha_u, \beta_v))$. This is treated in exactly the same way as the previous case; the fact that τ is of type (1,2) has no influence on the argument. *Case* 3: t is $\alpha_u(p(\alpha_u, \beta_v))$. We leave the easy construction of ν (on the basis of μ associated with $p(\alpha_u, \beta_v)$ according to the induction hypothesis) to the reader. *Case* 4: t is $(\lambda x p(\alpha_u, \beta_v, x))(q(\alpha_u, \beta_v))$. This case can be reduced to an application of lemma 1, however we prefer to sketch a direct argument. According to the induction hypothesis there exists a continuity function τ of type (2,1) with the property: if $\tau(u_1, v_1, n) = m+1$ then $p(\alpha_{u*u_1}, \beta_{v*v_1}, n)$ is saturated and its value is m. Again according to the induction hypothesis there is a continuity function μ of type (2,0) such that $q(\alpha_{u*u_1}, \beta_{v*v_1})$ is saturated with value n whenever $\mu(u_1, v_1) = n+1$. Call u_1, v_1 secured if it is a pair of sequence numbers, both of equal length, with the property: 1) $\mu(u_1, v_1) = n+1$, 2) $\tau(u_1, v_1, n) = m+1$. If u_1, v_1 is secured, then $(\lambda x p(\alpha_{u*u_1}, \beta_{v*v_1}, x))(q(\alpha_{u*u_1}, \beta_{v*v_1}))$ is saturated by definition and its value is m with m as above. Define ν as follows: $\nu(u_1, v_1) = m+1$ if $\tau(u_1, v_1, n) = m+1$ (with $n = \mu(u_1, v_1) - 1$) where u_1, v_1 is secured, and 0 otherwise. In order to show that ν is indeed a continuity function (of type (2,0)) one proceeds in the same way as in the proof of lemma 1.

LEMMA 3: *Let* $p(n_0)$ *be a saturated term and let* t *be a saturated term whose value is* n. *Then* $p(t)$ *is saturated and* $|p(n_0)| = |p(t)|$.

Proof: We proceed by induction with respect to the number of symbols in p. We content ourself to treat one typical case among the induction steps. All other cases are similar but simpler to treat. Let p be $F_\tau[G[n_0], q(n_0)](r(n_0))$. Since p is saturated, the following holds: 1) there is an u with $u \subseteq G[n_0]$, 2) $|q(n_0)| = m_1$, 3) $|r(n_0)| = m_2$, 4) $\tau(u, m_1, m_2) = |p(n_0)| + 1$. From 1) we conclude that $G[n_0](i)$ is saturated for all $i < \text{length}(u) = s$ and that $|G[n_0](i)| = u_i$ where $u = \langle u_0, \ldots, u_{s-1}\rangle$. $G[n](i)$ has less symbols than p, hence we can apply the induction hypothesis: $G[t](i) = u_i$ for all $i < s$. Similarly $|q(t)| = m_1$, $|r(t)| = m_2$ according to the induction hypothesis. In view of the definition of saturation it follows that $F_\tau[G(t), q(t)](r(t))$ is saturated with value $\tau(u, m_1, m_2) - 1 = |p(n_0)|$; this proves the statement in this case.

LEMMA 4: *Let* $p(\alpha_u)$ *be a saturated term and assume* $u \subseteq G$. *Then* $p(G)$ *is saturated and* $|p(\alpha_u)| = |p(G)|$.

Proof: We proceed by induction with respect to the number of symbols in p. We content ourself by treating two typical cases among the induction steps. The remaining cases are similar but simpler to treat. *Case* 1: $p(\alpha_u)$ is $\alpha_u(q(\alpha_u))$. Let u be $\langle u_0, \ldots, u_{s-1}\rangle$. Since p is saturated, it follows by definition that $q(\alpha_u)$ is saturated with value $i<s$. Since q has less symbols than p we can apply the induction hypothesis and conclude: $q(G)$ is saturated and $|q(G)|=i$. On the other hand $u\subseteq G$, that is, $G(i_0)$ is saturated and $|G(i_0)|=u_i$ for $i<s$. In virtue of the previous lemma we find that $G(q(G))$ is saturated too and that its value is again u_i. Hence $|p(G)|=u_i$. *Case* 2: $p(\alpha_u)$ is $F_\tau[H[\alpha_u], q(\alpha_u)](r(\alpha_u))$. Since $p(\alpha_u)$ is saturated, the following holds by definition: 1) there is a v with $v\subseteq H[\alpha_u]$, 2) $q(\alpha_u)$ and $r(\alpha_u)$ are saturated and have values say m_1 and m_2, 3) $\tau(v, m_1, m_2)-1=|p(\alpha_u)|$. Let v be $\langle v_0,\dots, v_{s-1}\rangle$. Since $v\subseteq H$ it follows that $H[\alpha_u](i_0)$ is saturated for $i<s$ and that its value is v_i. $H[\alpha_u](i_0)$ has less symbols than p. Hence it follows from the induction hypothesis that $H[G](i_0)$ is saturated for all $i<s$ and that its value is v_i. Similarly we find $|q(G)|=m_1$, $|r(G)|=$ $=m_2$. From the definition of saturation it follows that $F_\tau[H[G], q(G)](r(G))$ is saturated and has value $\tau(v, m_1, m_2)-1$, that is $|p(\alpha_u)|$. Hence $|p(G)|=|p(\alpha_u)|$.

LEMMA 5: *If* $p(\alpha_{u_1}^{i_1},\dots,\alpha_{u_s}^{i_s})$ *is saturated with value* m, *if* $u_1\subseteq G_1,\dots, u_s\subseteq G_s$ *then* $p(G_1,\dots, G_s)$ *is saturated with value* m.

Proof: This is obtained by a repeated application of the previous lemma.

DEFINITION 1: Let $i_1,\dots, i_s$ and $k_1,\dots, k_s$ be two sets of pairwise distinct numbers. Let $u_1,\dots, u_s$ be a list of sequence numbers. Then the two lists $\alpha_{u_1}^{i_1},\dots,\alpha_{u_s}^{i_s}$ and $\alpha_{u_1}^{k_1},\dots,\alpha_{u_s}^{k_s}$ are said to be of the same type.

LEMMA 6: *Let* $\alpha_{u_1}^{i_1},\dots,\alpha_{u_s}^{i_s}$ *and* $\alpha_{u_1}^{k_1},\dots,\alpha_{u_s}^{k_s}$ *be of the same type. Let* $p(\alpha_{u_1}^{i_1},\dots,\alpha_{u_s}^{i_s})$ *be a saturated term whose choice variables are among* $\alpha_{u_1}^{i_1},\dots,\alpha_{u_s}^{i_s}$ *and whose value is* m. *Then* $p(\alpha_{u_1}^{k_1},\dots,\alpha_{u_s}^{k_s})$ *is saturated and has value* m.

Proof: The proof proceeds by an easy induction with respect to the length of p.

LEMMA 7: *Let* $p(\alpha_{u_1}^{i_1},\dots,\alpha_{u_s}^{i_s})$, $\alpha_{u_1}^{i_1},\dots,\alpha_{u_s}^{i_s}$ and $\alpha_{u_1}^{k_1},\dots,\alpha_{u_s}^{k_s}$ *be as in lemma* 6. *Let* τ *have the property*: *if* $\tau(v_1,\dots, v_s)\neq 0$ *then* $p(\alpha_{u_1*v_1}^{i_1},\dots,\alpha_{u_s*v_s}^{i_s})$ *is saturated. Then*: *if* $\tau(v_1,\dots, v_s)\neq 0$ *then* $p(\alpha_{u_1*v_1}^{k_1},\dots,\alpha_{u_s*v_s}^{k_s})$ *is satured with value* $p(\alpha_{u_1*v_1}^{i_1},\dots,\alpha_{u_s*v_s}^{i_s})$.

Proof: $\alpha_{u_1*v_1}^{i_1},\dots,\alpha_{u_s*v_s}^{i_s}$ and $\alpha_{u_1*v_1}^{k_1},\dots,\alpha_{u_s*v_s}^{k_s}$ have the same type. Apply lemma 6.

LEMMA 8: *Let* $u_1,\dots, u_s$ *be a list of sequence numbers and* $F_i[\alpha_{v_1}^{j_1},\dots,\alpha_{v_t}^{j_t}]$, $i=1,\cdots, s$ *a list of functors containing no other free variables than those indicated and such that* $u_i\supseteq F_i[\alpha_{v_1}^{j_1},\dots,\alpha_{v_t}^{j_t}]$ *holds. Let* τ *be a continuity function of type* (s, o). *Then there exists a continuity function* ν *of type* (t, 0) *with the property*: *if* $\nu(w_1,\dots, w_t)\neq 0$ *then there exist* $u_1',\dots, u_s'$ *such that* $u_i * u_i'\subseteq F_i[\alpha_{v_1*w_1}^{j_1},\dots,\alpha_{v_t*w_t}^{j_t}]$ *and such that* $\tau(u_1',\dots, u_s')\neq 0$.

Proof: Let σ_i be the continuity function of type (t, 1) with the property: if $\sigma_i(w_1, \cdots, w_t, n) = k+1$ then $F_i\,[\alpha_{j_1}^{v_1 * w_1}, \ldots, \alpha_{v_t * w_t}^{j_t}]\,(n)$ is saturated and has value k. Specializing lemma 1 to the present case we get a v of type (t, 0) with the property: if $v(w_1, \ldots, w_t) \neq 0$ there is an $i \leqslant \text{length}(w_1)$ such that $F^k\,[\alpha_{v_1 * w_1}^{j_1}, \ldots, \alpha_{v_t * w_t}^{j_t}]\,(m)$ is saturated with value a_m^k for all $m < i$, $k = 1, \ldots, s$ and $\tau(u_1^*, \cdots, u_s^*) \neq 0$ where $u_k^* = \langle a_0^k, \ldots, a_{i-1}^k \rangle$. If $i \leqslant \text{length}(u_1) = p$ then we put $u_k' = \langle\ \rangle$, otherwise we put $u_k' = \langle a_p^k, \cdots, a_{i-1}^k \rangle$.

DEFINITION 2: A formula $p(\alpha_{u_1}^{i_1}, \ldots, \alpha_{u_s}^{i_s}) = q(\alpha_{u_1}^{i_1}, \ldots, \alpha_{u_s}^{i_s})$ is called true if there is a continuity function τ of type (s, 0) such that $p(\alpha_{u_1 * v_1}^{i_1}, \ldots, \alpha_{u_s * v_s}^{i_s})$, $q(\alpha_{u_1 * v_1}^{i_1}, \ldots, \alpha_{u_1 * v_1}^{i_1})$ are both saturated and have the same value whenever $\tau(v_1, \ldots, v_s) \neq 0$.

3. Intuitive background of the model

3.1. *Three systems of number theory*

Below we use three systems of intuitionistic number theory in order to explain the intuitive background of the model. The first system, to be denoted by Z_1, is that one described in [1]. The second, to be denoted by Z_2, is obtained by omitting the axiom of choice, of bar induction and continuity from the system of intuitionistic analysis described in [2]. The third one, to be denoted by Z_3, is based on the language L and can roughly be described as follows : 1) it contains suitable axioms for $+$, $'$, $\cdot$, 2) it contains all formulas $m_0' = (m+1)_0$ as axioms, 3) the usual axioms of intuitionistic propositional calculus are in Z_3, 4) for every i and for every nonempty sequence number $u = \langle u_0, \ldots, u_{s-1} \rangle$ with $i < s$ it contains all the formulas $G(i)_0 = u_i$ as axioms, provided $u \subseteq G$ holds, 5) it contains modus ponens, 6) it contains the axioms $(x)\,A(x) \supset A(t)$, $A(t) \supset (Ex)\,A(x)$, $(\xi)\,A(\xi) \supset A(F)$, $A(F) \supset (E\xi)\,A(\xi)$ where x and ξ are bound variables not occurring in A(a) and $A(\alpha)$ respectively, 7) it contains the four rules $A \supset B(a) / A \supset (x)\,B(x)$, $A \supset B(\alpha_{\langle\,\rangle}^i) / A \supset (\xi)\,B(\xi)$, $B(a) \supset A / (Ex)\,B(x) \supset A$ and $B(\alpha_{\langle\,\rangle}^i) \supset A / (E\xi)\,B(\xi) \supset A$ where a and $\alpha_{\langle\,\rangle}^i$ do not occur in A respectively and where x and ξ do not occur in B(a) and $B(\alpha_{\langle\,\rangle}^i)$ respectively, 8) it contains the induction schema. The system Z_3 contains in addition the functor symbol $*$ and suitable axioms for it.

3.2. *A realizability notion of Kleene*

In [3] Kleene introduced a realizability notion "realizable and provable" for formulas without free variables belonging to the language L_N; below we denote the statement "A is realizable" by /A. This notion, in its simplest version, is defined as follows: 1) $/p = q$ iff $Z_1 \vdash p = q$, 2) $/A \wedge B$ iff /A and /B, 3) $/(x)\,A(x)$ iff $/A(n)$ for all n, 4) $/A \vee B$ iff either /A and $Z_1 \vdash A$ or else /B and $Z_1 \vdash B$, 5) $/(Ex)\,A(x)$ iff there is an n such that $/A(n)$ and $Z_1 \vdash A(n)$, 6) $/A \supset B$ iff /A and $Z_1 \vdash A$ imply /B, 7) $/\neg A$ iff $/A \supset 0 = 1$.

The main result in [3] says: if $Z_1 \vdash A$ then $/A$. From this one can deduce the following familiar properties of intuitionistic number theory: I) if $Z_1 \vdash A \vee B$ then $Z_1 \vdash A$ or $Z_1 \vdash B$, if $Z_1 \vdash (Ex)\, A(x)$ then there is an n with $Z_1 \vdash A(n)$. Here, A, B and (Ex) A(x) are formulas without free variables. We can distort the above definition slightly as follows: 1) $/p=q$ iff $p=q$ is true, 2) $/A \wedge B$ iff $/A$ and $/B$, 3) $/(x)\, A(x)$ iff $/A(n)$ for all n, 4) $/A \vee B$ iff we can effectively affirm either $/A$ or $/B$, 5) $/(Ex)\, A(x)$ if we find effectively an n such that $/A(n)$ holds. This definition is of course somewhat vague in that the precise meaning of "effective" is not clear. But it would be easy to sharpen "effectively" by using Gödelnumbers of certain recursive functions; in this way we would end up with Kleenes realizability notion introduced in [1]. Corresponding to the main result above one can show: if $Z_1 \vdash A$ then $/A$. This result is of course wheaker than the first one and its only immediate consequence is that Z_1 is consistent.

3.3. *The intuitive motivation for the model*

We now come to the description of the intuitive motivation of the model. We must point out that this motivation is by no means stringent; it has rather the character of an "Ansatz" and there is no philosophical basis for it. To begin with, let us briefly look at Kleenes realizability notion, presented in [3]. The situation is this: one starts with a suitable realizability notion and ends up with proof theoretic properties (result I) in 3.2.) of a certain intuitionistic system, namely Z_1. A closer look at the definition of the first realizability notion in 3.2. shows that the properties of Z_1, described by result I) are built in a certain sense into this definition. Now let us proceed in the converse direction. To this end let P be an intuitionistic formal system, whose detailed structure is not relevant at the moment. Assume that for some reason or other we know the proof theoretic properties of P. Assume that these proof theoretic properties are described by a theorem I* which is of the same kind as result I in 3.2.; again it is not relevant at the moment to know the detailed form of I*.

Then we might be tempted to define a certain notion "realizable" by incorporating in the definition the properties of P, described by I*, in the same way as the properties of Z_1 have been incorporated in the first realizability notion in 3.2. Let us denote this new realizability notion by R1. In virtue of the relation between first and second realizability notion presented in 3.2. it is not unreasonable to try a second step: we drop every reference to provability which might occur in R1 and hope to end up with a new realizability notion R2 which is a "model" of P. This is more or less the way we will proceed below. More precisely, we will simplify the procedure a little bit by going from P directly to R2 instead of making the detour via R1. This is not unreasonable since a closer look at the definition of the second realizability notion in 3.2. shows that even there the properties of Z_1 given by I) are in some sense contained in this definition. Hence let us try to start with Z_2 in place of P. In order to work out the above program we have to know the proof theoretic properties of Z_2, more precisely we have to know the behaviour of Z_2 with respect to disjunctial and existential statements, that is we have to know what to take for I*. In order to list the essential properties of Z_2 let $A(\alpha) \vee B(\alpha)$, $(Ex)\, A(\alpha, x)$ and $(E\xi)\, D(\alpha, \xi)$ be formulas from the language L_K, which for simplicity do not contain other free variables then possibly eventually α. Then the following is true: 1) if $Z_2 \vdash A(\alpha) \vee B(\alpha)$ then there exists a prim. rec. continuity function $\tau(x)$ such that for every u with $\tau(u) \neq 0$ either $Z_2 \vdash A(u * \alpha)$ or else $Z_2 \vdash B(u * \alpha)$ holds, 2) if $Z_2 \vdash (Ex)\, A(\alpha, x)$ then there is a prim. rec. continuity function $\tau(x)$ such that for every u with $\tau(u) \neq 0$ there is a term t_u which does not contain other free variables than eventually α for which $Z_2 \vdash A(u * \alpha, t_u)$ holds, 3) if $Z_2 \vdash (E\xi)\, D(\alpha, \xi)$ then there is a prim. rec. continuity function $\tau(x)$ such that for every u with $\tau(u) \neq 0$ there is a functor F_u which contains at most α free and for which $Z_2 \vdash D(u * \alpha, F_u)$ holds. On the basis of these properties it would be possible to work out our programm outlined above for the language L_K. However it has turned out that it is easier to work with the language L. Therefore let us list the properties of Z_3 which correspond to the properties 1)-3) of Z_2 just listed. To this end let $A(\alpha_u) \vee B(\alpha_u)$, $(Ex)\, C(\alpha_u, x)$ and $(E\xi)\, D(\alpha_u, \xi)$ be formulas from the language L which do not contain other free variables than eventually α_u. Then the following is true: 1) if $Z_3 \vdash A(\alpha_u) \vee B(\alpha_u)$ then there is a prim. rec. continuity function $\tau(x)$ such that $\tau(v) \neq 0$ implies $Z_3 \vdash A(\alpha_{u*v})$ or $Z_3 \vdash B(\alpha_{u*v})$, 2) if $Z_3 \vdash (Ex)\, C(\alpha_u, x)$ then there is a prim. rec. continuity function $\tau(x)$ such that for v with $\tau(v) \neq 0$ there is a term t_v which contains at most α_{u*v} free for which $Z_3 \vdash C(_{u*v}, t_v)$ holds, 3) if $Z_3 \vdash (E\xi)\, D(\alpha_u, \xi)$ then there is a prim. rec. continuity function $\tau(x)$ such that for every v with $\tau(v) \neq 0$ there is a functor F_v containing at most α_{u*v} free and for which $Z_3 \vdash D(\alpha_{u*v}, F_v)$ holds. These properties can be proved in many ways; one possibility e.g. is to use the methods described in [4]. The general form of a realizability notion based on 1) – 3) just listed will look roughly speaking as follows: "A is realizable iff there is a continuity function τ such that...". Another possibility is to take as general schema of definition the following: "The continuity function τ realizes A iff ...". Both forms of definition are fully equivalent and we choose the second one because it has some technical advantages. We express the fact that τ realizes A notationally by τ/A.

We now present a first, provisional and incomplete definition of τ/A. In this definition we omit every reference to recursiveness. For simplicity we assume that the formula A has exactly one free variable, namely the free choice variable α_u. The definition we have in mind is as follows: 1) $\tau/p(\alpha_u) = q(\alpha_u)$ off $\tau(v) \neq 0$ implies $p(\alpha_{u*v})$, $q(\alpha_{u*v})$ saturated and $|p(\alpha_{u*v})| = |q(\alpha_{u*v})|$,2) $\tau/A \wedge B$ iff there are continuity functions τ_1, τ_2 with τ_1/A and τ_2/B, 3) $\tau/(x)\, A(x)$ iff for every term t there is a continuity function σ_t with $\sigma_t/A(t)$, 4) $\tau/(\xi)\, A(\xi)$ iff for every functor F there is a continuity function σ_F with $\sigma_F/A(F)$, 5) $\tau/A \supset B$ iff for every τ_1 with τ_1/A there is a τ_2 with τ_2/B, 6) $\tau/\neg A$ iff $\tau/A \supset 0 = 1$, 7) $\tau/A(\alpha_u) \vee B(\alpha_u)$ iff $\tau(v) \neq 0$ implies the existence of a continuity function σ_v such that either $\sigma_v/A(\alpha_{u*v})$ or else $\sigma_v/B(\alpha_{u*v})$ holds, 8) $\tau/(Ex)\, A(x,$

α_u) iff $\tau(v) \neq 0$ implies the existence of a σ_v and of a term t_v containing at most α_{u*v} free such that $\sigma_v/A(t_v, \alpha_{u*v})$ holds, 9) $\tau/(E\xi)\, A(\xi, \alpha_u)$ iff $\tau(v) \neq 0$ implies the existence of a σ_v and of a functor F_v containing at most α_{u*v} free such that $\sigma_v/A(F_v, \alpha_{u*v})$ holds. Let us say that A is realizable if there is a τ with τ/A. If we try to verify that every axiom of intuitionistic analysis is realizable, then everything works well with the exception of the axiom of continuity. The full continuity axiom could only be proved to be realizable if the following were true: if $A(\alpha_u)$ is realizable and if $u \subseteq F$ then A(F) is realizable. However there are simple counterexamples which show that the latter statement is not true in general. In order to include the continuity axioms we have to change the above definition, which will be denoted by D1, in one essential point. In order to explain this point, let us reconsider Kleenes realizability definition in [3]. In this definition the notion of provability takes part. Now let P be an arbitrary but fixed property of formulas; P(A) indicates that A has the property P. Now let us alter Kleenes definition by replacing $Z_1 \vdash A$ wherever it occurs by P(A): 1) $/n = m$ iff $P(n = m)$, 2) $/A \wedge B$ iff $/A$ and $/B$, 3) $(x)\, A(x)$ iff $/A(n)$ for all n, 4) $/A \supset B$ iff $/B$ whenever $/A$ and P(A) hold, 5) $/A \vee B$ iff $/A$ and P(A) or $/B$ and P(B), 6) $/(Ex)\, A(x)$ iff there is an n such that $/A(n)$ and $P(A(n))$ hold, 7) $/\neg A$ iff $/A \supset 0 = 1$. Denote this definition by D_p. Now assume that a realizability notion has been defined for all formulas from L having at most n logical symbols, that is that the meaning of τ/A is known for all such formulas. Let $A(\alpha_u)$ be a formula containing at most n logical symbols and assume for simplicity that its only free variable is α_u. Let S be the following property: for all functors F with $u \subseteq F$ there is a continuity function τ_F such that $\tau/A(F)$ holds. S is called the substitutivity property. Now we derive from definition D1 a new definition D2 by building the property S into D2 in a way which is very similar to the way the property P has been built into the definition D_P. The definition D2 introduces the binary relation τ/A and the one place predicate S by simultaneous inductive definition: if τ/A has already been defined for all formulas with at most n logical symbols for a certain n then S is the substitutivity property described above. The incomplete and provisorial definition D2 goes as follows: 1) $\tau/p(\alpha_u) = q(\alpha_u)$ iff for all v with $\tau(v) \neq 0$ both $p(\alpha_{u*v})$ and $q(\alpha_{u*v})$ are saturated and have the same value, 2) $\tau/A \wedge B$ iff there are τ_1, τ_2 with τ_1/A and τ_2/B, 3) $\tau/(x)\, A(x)$ iff for all terms t there is a τ_t with $\tau_t/A(t)$, 4) $\tau/(\xi)\, A(\xi)$ iff for all functors F there is a τ_F with $\tau_F/A(F)$, 5) $\tau/A(\alpha_u) \vee B(\alpha_u)$ iff for all v with $\tau(v) \neq 0$ either $S(A(\alpha_{u*v}))$ or $S(B(\alpha_{u*v}))$ holds, 6) $\tau/(Ex)\, A(\alpha_u, x)$ iff for all v with $\tau(v) \neq 0$ there is a term t, containing at most α_{u*v} free such that $S(A(\alpha_{u*v}, t))$ holds, 7) $\tau/(E\xi)\, A(\alpha_u, \xi)$ iff for all v with $\tau(v) \neq 0$ there is a functor F containing at most α_{u*v} free such that $S(A(\alpha_{u*v}, F))$ holds, 8) $\tau/A(\alpha_u) \supset \supset B(\alpha_u)$ iff there is a σ with σ/B whenever $S(A(\alpha_u))$ holds, 9) $\tau/\neg A$ iff $\tau/A \supset 0 = 1$. It is clear that we could formulate D2 without the aid of S. Clause 7) eg. would then be read as follows: $\tau/(E\xi)\, A(\alpha_u, \xi)$ iff for all v with $\tau(v) \neq 0$ there is a functor $F_v[\alpha_{u*v}]$ containing at most α_{u*v} free such that for every functor G with $u*v \subseteq G$ there is a σ

with $\sigma/A(G, F[G])$. For the definition D2 one can indeed show that if a formula A is provable in intuitionistic analysis then it is realizable (that is there exists a τ with τ/A).

4. A model of intuitionistic analysis

4.1. *A system of intuitionistic analysis*

By adding to Z_3 a number of new axioms we obtain a system of intuitionistic analysis IA. The list of these new axioms is given as follows: 1) the set of true prime formulas, 2) the axiom of choice for quantifierfree formulas, 3) the continuity axioms as given in [2], 4) the axioms of transfinite induction for quantifierfree partial orderings. The precise form of the axioms 2), 3), 4) will be given below. Our system is only seemingly weaker than that of [2]; by making heavy use of the continuity axiom and the fact that transfinite induction and bar induction for decidable formulas are equivalent we can reduce the system of [2] to I.A (consult [5] for this respect).

4.2. *Some notations*

For easy reading below we introduce some notational conventions. Boldface letters **u**, **v**, **w** denote lists of sequence numbers, say $u_1, \ldots, u_s$ where all u_i are supposed to have the same length. We call s the length of **u** (or **v** or **w**). Boldface letters **a**, **b**, **c**, **d** denote lists of pairwise distinct free choise variables, say $\alpha_{u_1}^{i_1}, \ldots, \alpha_{u_s}^{i_s}$ etc; there the u_i's are not required to have all the same length. In both cases the lists may be empty (s=0). Let **u**, **v**, **a** denote $u_1, \ldots, u_s$ and $v_1, \ldots, v_s$ and $\alpha_{w_1}^{i_1}, \ldots, \alpha_{w_s}^{i_s}$ respectively; then **u*****v** and **a*****u** denote $u_1 * v_1, \ldots, u_s * v_s$ and $\alpha_{w_1 * u_1}^{i_1}, \ldots, \alpha_{w_s * u_s}^{i_s}$ respectively. If τ is a continuity function from N_{s+t}, if **u** and **v** denote $u_1, \ldots, u_s$ and $v_1, \ldots, v_t$ respectively then $\tau(\mathbf{u}, \mathbf{v})$ is a short way of writing $\tau(u_1, \ldots, u_s, v_1, \ldots, v_t)$. Lists of functors are denoted by boldface letters **G**, **H**, **F** and lists of terms by boldface letters **t**, **p**, **q**. If **a**, **G** denote $\alpha_{u_1}^{i_1}, \ldots, \alpha_{u_s}^{i_s}$ and $G_1, \ldots, G_s$ respectively then $\mathbf{a} \subseteq \mathbf{G}$ expresses that for every i the relation $u_i \subseteq G_i$ holds. If **G** denotes $G_1, \ldots, G_s$, if **a**, **b** denote $\alpha_{u_1}^{i_1}, \ldots, \alpha_{u_t}^{i_t}$ and $\beta_{v_1}^{j_1}, \ldots, \beta_{v_r}^{j_r}$ respectively and have no member in common, if moreover every free choice variable which occurs in some G_i is a member of **a** or **b** then we express this by writing G [**a**, **b**]. If in addition **H**, **F** denote $H_1, \ldots, H_t$ and $F_1, \ldots, F_r$ then **G** [**H**, **F**] denotes the list $G'_1, \ldots, G'_s$ where G'_i is obtained from G_i by replacing each occurence of $\alpha_{u_k}^{i_k}$ and $\beta_{v_k}^{j_k}$ in G_i by H_k and F_k respectively, for all k. In case of a single functor G we write correspondingly G [**a**, **b**] and G [**H**, **F**]. An analogous notation is used in case of a single list **a** or three pairwise disjoint lists **a**, **b**, **c** of free choice sequences; similarly with terms and lists of terms. If however A is a formula, if **a**, **b** are two disjoint lists of free choice sequences then A (**a**, **b**) expresses the fact that every free choice variable occurring in A is a member of **a** or **b** and that conversely every member of **a** or **b** occurs somewhere in A. The notation A (**H**, **F**) has the same meaning as before; similarly in case of a single list **a** or three pairwise disjoint lists **a**, **b**, **c**. If two lists L_1, L_2 of objects have the same length we express this by writing $L_1 \sim L_2$.

4.3. *The model*

The model, to be defined below, is a two place relation τ/A whose domain of definition is the set of ordered pairs (τ, A) satisfying the following conditions: 1) τ is a continuity function of type (s, 0), 2) A is a formula containing precisely s distinct free choice variables and otherwise no other free variables, 3) if $s=0$ then τ is the natural number 1. In this connection we adopt the following notation: the continuity function of type (s, 0) ($s=0$ included) whose value is always 1 will be denoted by τ_s^ϕ and without danger of confusion we omit the index s and write simply τ^ϕ. In the definition below $//A(\mathbf{a})$ is an abbreviation of the following statement: for every list $\mathbf{F}$ such that $\mathbf{a}\sim\mathbf{F}$ and $\mathbf{a}\subseteq\mathbf{F}$ there is a continuity function τ such that $\tau/A(\mathbf{F})$ holds. The sign // in the definition below plays exactly the same role as S in D2. We define τ/A by induction with respect to the number of logical symbols in A. All formulas, terms and functors appearing in the definition do not contain free number variables; in addition we always assume that $\mathbf{a}\sim\mathbf{u}$, $\mathbf{b}\sim\mathbf{v}$, $\mathbf{c}\sim\mathbf{w}$, $\mathbf{a}\sim\mathbf{u}'$, $\mathbf{b}\sim\mathbf{v}'$ holds whenever these symbols appear below.

1. $\tau/p(\mathbf{a}, \mathbf{b})=q(\mathbf{a}, \mathbf{c})$ iff for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ with $\tau(\mathbf{u}, \mathbf{v}, \mathbf{w})\neq 0$ both $p(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v})$, $q(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$ are saturated and have the same value.

2. $\tau/A(\mathbf{a}, \mathbf{b})\wedge B(\mathbf{a}, \mathbf{c})$ iff there exist τ_1 and τ_2 with $\tau_1/A(\mathbf{a}, \mathbf{b})$ and $\tau_2/B(\mathbf{a}, \mathbf{c})$.

3. $\tau/(x)\, A(x, \mathbf{a})$ iff for every term t there is a σ with $\sigma/A(t, \mathbf{a})$.

4. $\tau/(\xi)\, A(\xi, \mathbf{a})$ iff for every functor F there exists a σ with $\sigma/A(F, \mathbf{a})$.

5. $\tau/A(\mathbf{a}, \mathbf{b})\vee B(\mathbf{a}, \mathbf{c})$ iff for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ with $\tau(\mathbf{u}, \mathbf{v}, \mathbf{w})\neq 0$ either $//A(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v})$ or $//B(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$ holds.

6. $\tau(/Ex)\, A(\mathbf{a}, x)$ iff for every $\mathbf{u}$ with $\tau(\mathbf{u})\neq 0$ there exists a term t containing no other free variables than those occurring in $\mathbf{a}*\mathbf{u}$, such that $//A(\mathbf{a}*\mathbf{u}, t)$ holds.

7. $\tau/(E\xi)\, A(\mathbf{a}, \xi)$ iff for every $\mathbf{u}$ with $\tau(\mathbf{u})\neq 0$ there exists a functor F containing no other free variables than those occurring in $\mathbf{a}*\mathbf{u}$ such that $//A(\mathbf{a}*\mathbf{u}, F)$ holds.

8. $\tau/A(\mathbf{a}, \mathbf{b})\supset B(\mathbf{a}, \mathbf{c})$ iff there exists a σ with $\sigma/B(\mathbf{a}, \mathbf{c})$ whenever $//A(\mathbf{a}, \mathbf{b})$ holds.

9. $\tau/\neg A$ iff $\tau/A\supset 0=1$.

One easily recognizes that the above definition is nothing else than an elaboration of D2.

DEFINITION 3: If $A(a_1,\ldots, a_s)$ is a formula whose free number variables are among $a_1,\ldots, a_s$ then we call $A(a_1,\ldots, a_s)$ strongly realizable if $//(x_1,\ldots, x_s)\, A$ holds. A formula without free individual variables, say A, is called realizable if there is a τ with τ/A.

Our main effort is devoted to the proof of

THEOREM 1: *If* $IA\vdash A$ *then* A *is strongly realizable.*

COROLLARY: IA *is consistent.*

Proof: Evident from the theorem since $0=1$ is not realizable.

4.4. *Some preliminary lemmas*

In order to prove theorem 1 we need some preliminary lemmas. They are easy consequences of the definition of the model. Proofs will therefore only be sketched. In order to distinguish those lemmas from the lemmas in the next section we denote them by H1, H2, etc.

H1: If $//A(\mathbf{a})$ and $\mathbf{a}\subseteq\mathbf{F}$ then $//A(\mathbf{F})$.

Proof: Let $\mathbf{d}$ be the list of those variables which occur in $\mathbf{F}$; we express this by writing $F[\mathbf{d}]$. Assume $\mathbf{d}\subseteq\mathbf{G}$. Then $\mathbf{a}\subseteq F[\mathbf{G}]$ according to lemma 5; hence $\sigma/A(\mathbf{F}[\mathbf{G}])$ for some σ. Since $\mathbf{G}$ has only to satisfy $\mathbf{d}\subseteq\mathbf{G}$ and is otherwise arbitrary we have $//A(\mathbf{F}[\mathbf{d}])$ by definition.

H2: If $P(\mathbf{a})$ is a prime formula and if $\sigma/P(\mathbf{a})$ then $//P(\mathbf{a})$.

Proof: Let $P(\mathbf{a})$ be $p(\mathbf{a})=q(\mathbf{a})$ and assume $\mathbf{a}\subseteq\mathbf{F}[\mathbf{b}]$. According to lemma 8 there is a τ with the property: if $\tau(\mathbf{v})\neq 0$ then there is an $\mathbf{u}$ such that $\mathbf{a}*\mathbf{u}\subseteq\mathbf{F}[\mathbf{b}*\mathbf{v}]$ and such that $\sigma(\mathbf{u})\neq 0$. But then $p(\mathbf{a}*\mathbf{u})$ and $q(\mathbf{a}*\mathbf{u})$ are saturated and have the same value, hence $|p(\mathbf{F}[\mathbf{b}*\mathbf{v}])|=|q(\mathbf{F}[\mathbf{b}*\mathbf{v}])|$ according to lemma 4. Hence $\tau/p(\mathbf{F}[\mathbf{b}])=q(\mathbf{F}[\mathbf{b}])$. Since $\mathbf{F}[\mathbf{b}]$ was arbitrary apart from $\mathbf{a}\subseteq\mathbf{F}[\mathbf{b}]$ we conclude $//P(\mathbf{a})$.

H3: Let $\mathbf{a}$ and $\mathbf{b}$ be two lists of the same type. Then: 1) if $\tau/A(\mathbf{a})$ then $\tau/A(\mathbf{b})$, 2) if $//A(\mathbf{a})$ then $//A(\mathbf{b})$.

Proof: For prime formulas the statement is an easy consequence of lemma 7 and H2. For arbitrary formulas A we prove 1) and 2) by an easy simultaneous induction with respect to the number of logical symbols in A.

H4: $\tau^{\Phi}/(x)\,A(x)$ iff $\sigma/(x)\,A(x)$. Similarly with $(\xi)\,A(\xi)$, $A\wedge B$ and $A\supset B$.

Proof: The statement is evident since in all these cases the definition of $\sigma/(x)\,A(x)$ etc. does not depend on σ.

H5: Let $(x_1,\ldots,x_s)\,A(x_1,\ldots,x_s)$ not contain free individual variables. Then $//(x_1,\ldots,x_s)\,A(x_1,\ldots,x_s)$ iff $//A(q_1,\ldots,q_s)$ for all s-tuples $q_1,\ldots,q_s$ of terms not containing free individual variables.

Proof: Case 1: $s=1$. Let $(x)\,A(x)$ be more explicitly $(x)\,A(\mathbf{a},x)$. Assume $//(x)\,A(\mathbf{a},x)$ and let $q(\mathbf{a},\mathbf{b})$ be given. We have to show: $//A(\mathbf{a},q(\mathbf{a},\mathbf{b}))$. Assume $\mathbf{a}\subseteq\mathbf{G}$, $\mathbf{b}\subseteq\mathbf{H}$. According to H1 we have $//(x)\,A(\mathbf{G},x)$ and hence $\sigma/A((\mathbf{G},q(\mathbf{G},\mathbf{H}))$ for some σ. Since $\mathbf{G}$, $\mathbf{H}$ where essentially arbitrary we have $//A(\mathbf{a},q(\mathbf{a},\mathbf{b}))$. Now assume conversely $//A(\mathbf{a},q)$ for all q. Let $\mathbf{G}[\mathbf{b}]$ be such that $\mathbf{a}\subseteq\mathbf{G}[\mathbf{b}]$ holds. We have to show $\tau^{\Phi}/(x)\,A(\mathbf{G}[\mathbf{b}],x)$ (see H4). This amounts to show: for any $q(\mathbf{b},\mathbf{c})$ there is a σ with

$\sigma/A(\mathbf{G}[\mathbf{b}], q(\mathbf{b}, \mathbf{c}))$. Let $\mathbf{b}'$, $\mathbf{c}'$ be of the same type as $\mathbf{b}$, $\mathbf{c}$ and such that $\mathbf{a}$ has no element in common with $\mathbf{b}'$, $\mathbf{c}'$. Then still $\mathbf{a} \subseteq \mathbf{G}[\mathbf{b}']$ by lemmas 4,5. Therefore $\mathbf{a} \subseteq \mathbf{G}[\mathbf{b}']$, $\mathbf{b}' \subseteq \mathbf{b}'$ and $\mathbf{c}' \subseteq \mathbf{c}'$. On the other hand $//A(\mathbf{a}, q(\mathbf{b}', \mathbf{c}'))$ by assumption and hence $\sigma/A(\mathbf{G}[\mathbf{b}'], q(\mathbf{b}', \mathbf{c}'))$ by definition. By H3 we obtain $\sigma/A(\mathbf{G}[\mathbf{b}], q(\mathbf{b}, \mathbf{c}))$, what proves the converse direction of H5 in case $s=1$. Case 2: $s>1$. One proceeds by induction with respect to s. Since the inductive step is rather trivial, we omit it.

In order to prove the next lemma we need

DEFINITION 4: Let τ be a continuity function of type $(s, 0)$ having the property: with every $\mathbf{v}$ such that $\tau(\mathbf{v}) \neq 0$ there is associated a continuity function $\tau_{\mathbf{v}}$ of type $(s, 0)$. Then there is evidently a continuity function σ with the property: if $\sigma(\mathbf{w}) \neq 0$ then there is a decomposition $\mathbf{w}=\mathbf{v}*\mathbf{v}'$ such that $\tau(\mathbf{v}) \neq 0$ and $\tau_{\mathbf{v}}(\mathbf{v}') \neq 0$. The well-determined σ will be denoted by $\tau*$.

H6: Let τ be as in the above definition. Assume that the following holds: if $\tau(\mathbf{v}) \neq 0$ then $\tau_{\mathbf{v}}/A(\mathbf{a}*\mathbf{v})$. Then there is a τ' with $\tau'/A(\mathbf{a})$.

Proof: The proof proceeds by induction with respect to the number of logical symbols in A. *Case* 1: A is prime, say $p(\mathbf{a})=q(\mathbf{a})$. Then it is trivial to verify $\tau*/p(\mathbf{a})=$ $=q(\mathbf{a})$ where $\tau*$ is derived from τ according to the previous definition. *Case* 2: A is $B(\mathbf{a}, \mathbf{b}) \wedge C(\mathbf{a}, \mathbf{c})$. According to H4 we have $\tau^{\Phi}/B(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v}) \wedge C(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$ whenever $\tau(\mathbf{u}, \mathbf{v}, \mathbf{w}) \neq 0$ holds. From this one easily gets two continuity functions τ_1 and τ_2 having the property: 1) if $\tau_1(\mathbf{u}, \mathbf{v}) \neq 0$ then there exists a τ_1' with $\tau_1'/B(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v})$, 2) if $\tau_2(\mathbf{u}, \mathbf{w}) \neq 0$ then there exists τ_2' with $\tau_2'/C(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$. From the induction hypothesis we conclude that there exist σ_1, σ_2 with $\sigma_1/B(\mathbf{a}, \mathbf{b})$ and $\sigma_2/C(\mathbf{a}, \mathbf{c})$. But this implies $\tau^{\Phi}/B(\mathbf{a}, \mathbf{b}) \wedge C(\mathbf{a}, \mathbf{c})$. *Case* 3: A is $(\xi)\, B(\mathbf{a}, \xi)$. By definition and H4 we have $\tau^{\Phi}/(\xi)\, B(\mathbf{a}*\mathbf{u}, \xi)$ whenever $\tau(\mathbf{u}) \neq 0$ holds. Let $\mathbf{F}[\mathbf{a}, \mathbf{b}]$ be arbitrary. We are through if we can find a σ^{F} with $\sigma^{F}/B(\mathbf{a}, \mathbf{F}[\mathbf{a}, \mathbf{b}])$. Define σ_0 as follows: $\sigma_0(\mathbf{u}, \mathbf{v}) \neq 0$ iff $\tau(\mathbf{u}) \neq 0$. If $\sigma_0(\mathbf{u}, \mathbf{v}) \neq 0$ then $\tau(\mathbf{u}) \neq 0$, hence $\tau^{\Phi}/(\xi)\, B(\mathbf{a}*\mathbf{u}, \xi)$ by H4, hence there is a $\sigma^{F}_{\mathbf{u},\mathbf{v}}$ with $\sigma^{F}_{\mathbf{u},\mathbf{v}}/B(\mathbf{a}*\mathbf{u}, \mathbf{F}[\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v}])$. According to the induction hypothesis there is a σ^{F} such that $\sigma^{F}/B(\mathbf{a}, \mathbf{F}[\mathbf{a}, \mathbf{b}])$.

Case 4: A is $(x)\, B(\mathbf{a}, x)$. We proceed in the same way as in case 3.

Case 5: A is $(E\xi)\, B(\mathbf{a}, \xi)$. Let $\tau*$ be associated with τ according to def. 4. If $\tau*(\mathbf{u}) \neq 0$ then there is a decomposition $\mathbf{u}=\mathbf{u}'*\mathbf{u}''$ such that $\tau(\mathbf{u}') \neq 0$ and $\tau_{\mathbf{u}'}(\mathbf{u}'') \neq 0$. Since $\tau_{\mathbf{u}'}/(E\xi)\, B(\mathbf{a}*\mathbf{u}', \xi)$ by assumption there is a $\mathbf{F}[\mathbf{a}*\mathbf{u}'*\mathbf{u}'']$ (depending on $\mathbf{u}$ of course) such that $//B(\mathbf{a}*\mathbf{u}'*\mathbf{u}'', \mathbf{F}[\mathbf{a}*\mathbf{u}'*\mathbf{u}''])$, or what amounts to the same, such that $//B(\mathbf{a}*\mathbf{u}, \mathbf{F}[\mathbf{a}*\mathbf{u}])$ holds. Hence $\tau*/(E\xi)\, B(\mathbf{a}, \xi)$ by definition.

Case 6: A is $(Ex)\, B(\mathbf{a}, x)$. We proceed in the same way as in case 5.

Case 7: A is $B(\mathbf{a}, \mathbf{b}) \supset C(\mathbf{a}, \mathbf{c})$. We have by assumption and H4: if $\tau(\mathbf{u}, \mathbf{v}, \mathbf{w}) \neq 0$ then $\tau^{\Phi}/B(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v}) \supset C(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$. Now assume $//B(\mathbf{a}, \mathbf{b})$. By H1 we have $//B(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$. Obviously there is a continuity function σ having the property: if

$\sigma(\mathbf{u}, \mathbf{w}) \neq 0$ then $\tau(\mathbf{u}, \mathbf{v}_0, \mathbf{w}) \neq 0$ where $\mathbf{v}_0$ is a list of sequence numbers of the form $\langle 0, \ldots, 0 \rangle$. If $\sigma(\mathbf{u}, \mathbf{w}) \neq 0$ then $\tau(\mathbf{u}, \mathbf{v}_0, \mathbf{w}) \neq 0$ and hence $\tau^{\Phi}/\mathrm{B}(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v}_0) \supset \mathrm{C}(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$. Clearly $//\mathrm{B}(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v}_0)$. Therefore there is a $\sigma_{\mathbf{u},\mathbf{w}}$ such that $\sigma_{\mathbf{u},\mathbf{w}}/\mathrm{C}(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$. From the induction hypothesis it follows that there is a σ_0 such that $\sigma_0/\mathrm{C}(\mathbf{a}, \mathbf{c})$, what proves the statement, also in this case.

H7: $//\mathrm{A} \wedge \mathrm{B}$ iff $//\mathrm{A}$ and $//\mathrm{B}$.

We omit the rather trivial proof.

H8: Let t be saturated and assume $|t| = n$. Then: 1) if $\tau/\mathrm{A}(t)$ then $\tau'/(n)$ for some τ' and conversely, 2) if $//\mathrm{A}(t)$ then $//\mathrm{A}(n)$ and conversely.

Proof: The statement is obtained by an easy simultaneous induction with respect to the number of logical symbols in A, making thereby use of lemmas 4, 5 in case where A is prime. At each step of the induction we first prove 1) with the aid of the inductive assumption and afterward 2) with the aid of 1).

H9: Assume that for every $\mathbf{u}$ with $\tau(\mathbf{u}) \neq 0$ and every m there is a $\tau_{\mathrm{u}}^{\mathrm{m}}$ with $\tau_{\mathrm{u}}^{\mathrm{m}}/\mathrm{A}(\mathbf{a}*\mathbf{u}, \mathbf{m})$. Then $\tau^{\Phi}/(x)\,\mathrm{A}(\mathbf{a}, x)$.

Proof: We have to show: for any term t (without free individual variables) there is a σ_t such that $\sigma_t/\mathrm{A}(\mathbf{a}, t)$ holds. Let $t(\mathbf{a}, \mathbf{b})$ be such a term. Evidently there is a continuity function σ_0 having the following properties: if $\sigma_0(\mathbf{u}, \mathbf{v}) \neq 0$ then 1) $t(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v})$ is saturated, 2) $\tau(\mathbf{u}) \neq 0$. Assume $\sigma_0(\mathbf{u}, \mathbf{v}) \neq 0$ and let $|t(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v})|$ be m. Since $\tau(\mathbf{u}) \neq 0$ there is a $\tau_{\mathrm{u}}^{\mathrm{m}}$ with $\tau_{\mathrm{u}}^{\mathrm{m}}/\mathrm{A}(\mathbf{a}*\mathbf{u}, \mathrm{m})$. According to H8 there is a $\hat{\tau}_{\mathrm{u}}^{\mathrm{m}}$ with $\hat{\tau}_{\mathrm{u}}^{\mathrm{m}}/\mathrm{A}(\mathbf{a}*\mathbf{u}, t(\mathbf{a}*\mathbf{u}, \mathbf{v}*\mathbf{b}))$. By combining this fact with H6 we obtain the desired σ_t with $\sigma_t/\mathrm{A}(\mathbf{a}, t)$.

H10: $//(x)\,\mathrm{A}(\mathbf{a}, x)$ iff $//\mathrm{A}(\mathbf{a}, n)$ for all n.

Proof: a) Assume $\mathbf{a} \subseteq \mathbf{F}[\mathbf{b}]$ and $//(x)\,\mathrm{A}(\mathbf{a}, x)$. We conclude $\tau^{\Phi}/(x)\,\mathrm{A}(\mathbf{F}[\mathbf{b}], x)$, that is $\sigma_n/\mathrm{A}(\mathbf{F}[\mathbf{b}], n)$ for every n and some suitable σ_n. Since $\mathbf{F}[\mathbf{b}]$ was essentially arbitrary we obtain $//\mathrm{A}(\mathbf{a}, n)$ for all n. b) Assume conversely $//\mathrm{A}(\mathbf{a}, n)$ for all n and let $\mathbf{F}[\mathbf{b}]$ be such that $\mathbf{a} \subseteq \mathbf{F}[\mathbf{b}]$ holds. Then $//\mathrm{A}(\mathbf{F}[\mathbf{b}], n)$ for all n by H1. This implies that for every u and every n there is a $\sigma_{\mathrm{u}}^{\mathrm{n}}$ with $\sigma_{\mathrm{u}}^{\mathrm{n}}/\mathrm{A}(\mathbf{F}[\mathbf{b}*\mathbf{u}], n)$. That is we are in the situation of lemma H9 with τ^{Φ} in place of τ. By H9 we have $\tau^{\Phi}/(x)\,\mathrm{A}(\mathbf{F}[\mathbf{b}], x)$. Since $\mathbf{F}[\mathbf{b}]$ was essentially arbitrary we conclude $//(x)\,\mathrm{A}(\mathbf{a}, x)$.

H11: Assume $//\mathrm{p}(\mathbf{a}) = \mathrm{q}(\mathbf{a}) \supset 0 = 1$ and let σ be a continuity function with the property: if $\sigma(\mathbf{u}) \neq 0$ then $\mathrm{p}(\mathbf{a}*\mathbf{u})$ and $\mathrm{q}(\mathbf{a}*\mathbf{u})$ are both saturated. For such an $\mathbf{u}$: $|\mathrm{p}(\mathbf{a}*\mathbf{u})| \neq |\mathrm{q}(\mathbf{a}*\mathbf{u})|$.

Proof: From the assumption we conclude $\tau^{\Phi}/\mathrm{p}(\mathbf{a}*\mathbf{u}) = \mathrm{q}(\mathbf{a}*\mathbf{u}) \supset 0 = 1$ for all $\mathbf{u}$. Let σ be as stated by the lemma. For an $\mathbf{u}$ with $\sigma(\mathbf{u}) \neq |0\,\mathrm{p}(\mathbf{a}*\mathbf{u})|$ and $|\mathrm{q}(\mathbf{a}*\mathbf{u})|$ are clearly different, since otherwise $//\mathrm{p}(\mathbf{a}*\mathbf{u}) = \mathrm{q}(\mathbf{a}*\mathbf{u})$ would hold implying $|0| = |1|$.

H12: $//(\xi_1, \ldots, \xi_s)\,\mathrm{A}(\mathbf{a}_1, \xi \ldots, \xi_s)$ iff for all $\mathrm{G}_1, \ldots, \mathrm{G}_s$ and all $\mathbf{F}$ with $\mathbf{a} \subseteq \mathbf{F}$ we have $//\mathrm{A}(\mathbf{F}, \mathrm{G}_1, \ldots, \mathrm{G}_s)$.

The proof is essentially the same as that of the previous lemma and is therefore omitted.

H13: Let σ and p(**a**) prime be such that the following holds: if $\sigma(\mathbf{u})\neq 0$ then p(**a**∗**u**) is saturated and $|p(\mathbf{a}*\mathbf{u})|\neq 0$. Then $//p(\mathbf{a})\supset 0=1$, that is $//p(\mathbf{a})\neq 0$ holds.

Proof: It is sufficient to observe that $//p(\mathbf{F}[\mathbf{b}])=0$ never holds provided $\mathbf{a}\subseteq\mathbf{F}[\mathbf{b}]$ is true. The observation is an easy consequence of lemmas 2,8.

H14: Assume $//p(\mathbf{a})\neq 0$. Let σ be such that $\sigma(\mathbf{u})\neq 0$ implies p(**a**∗**u**) saturated. Then $|p(\mathbf{a}*\mathbf{u})|\neq 0$ for such an **u**.

Proof: This is a special case of H11.

H15: $//A(\mathbf{a}, \mathbf{F}[\mathbf{a}])$ implies $\tau^{\Phi}/(E\xi)\, A(\mathbf{a}, \xi)$.

Proof: From $//A(\mathbf{a}, F[\mathbf{a}])$ and H1 we conclude $//A(\mathbf{a}*\mathbf{u}, F[\mathbf{a}*\mathbf{u}])$ for all **u**. Therefore $\tau^{\Phi}/(E\xi)\, A(\mathbf{a}, \xi)$ by definition.

H18: $//A(\mathbf{a}, F[\mathbf{a}])$ implies $//(E\xi)\, A(\mathbf{a}, \xi)$.

Proof: Assume $\mathbf{a}\subseteq\mathbf{G}$. By H1 we have $//A(\mathbf{G}, F[\mathbf{G}])$ and from H17 we obtain $\tau^{\Phi}/(E\xi)\, A(\mathbf{G}, \xi)$. In other words, whenever $\mathbf{a}\subseteq\mathbf{G}$ then $\tau^{\Phi}/(E\xi)\, A(\mathbf{G}, \xi)$. This implies $//(E\xi)\, A(\mathbf{a}, \xi)$.

H19: $//A(\mathbf{a}, t(\mathbf{a}))$ implies $//(Ex)\, A(\mathbf{a}, x)$.

Proof: Exactly the same as that of H18.

4.5. *The rules*

Now we can pass to the proof of theorem 1. This is done by showing that each axiom of IA is strongly realizable and that the rules preserve strong realizability. We start with the rules. The lemmas below will be denoted by L1, L2, etc.

L1: If $//A(\mathbf{a}, \mathbf{b})$, $//A(\mathbf{a}, \mathbf{b})\supset B(\mathbf{a}, \mathbf{c})$ then $//B(\mathbf{a}, \mathbf{c})$.

Proof: Let **F**, **H** be lists of functors such that $\mathbf{a}\sim\mathbf{F}$, $\mathbf{c}\sim\mathbf{H}$, $\mathbf{a}\subseteq\mathbf{F}$, $\mathbf{c}\subseteq\mathbf{H}$. Take any list **G** such that $\mathbf{b}\sim\mathbf{G}$, $\mathbf{b}\subseteq\mathbf{G}$. Such a list **G** can be found in many ways. Then there exists a τ with $\tau/A(\mathbf{F}, \mathbf{G})\supset B(\mathbf{F}, \mathbf{H})$. On the other hand $//A(\mathbf{F}, \mathbf{G})$ by H1 and the assumption $//A(\mathbf{a}, \mathbf{b})$. Hence $\sigma/B(\mathbf{F}, \mathbf{H})$ for some σ. Hence $//B(\mathbf{a}, \mathbf{b})$.

L2: If A and $A\supset B$ are strongly realizable then B is strongly realizable.

Proof: Let the free choice variables of A, B be those of the lists **a**, **b** and **a**, **c** respectively, let the free number variables of A, B be among $a_1, \ldots, a_s$, to be abbreviated by $\vec{a}$. We express this by writing $A(\mathbf{a}, \mathbf{b}, \vec{a})$ and $B(\mathbf{a}, \mathbf{c}, \vec{a})$. According to the definition of strong realizability we have $//(x_1,\ldots, x_s)\, A(\mathbf{a}, \mathbf{b}, \vec{x})$ and $//(x_1,\ldots, x_s)\, (A(\mathbf{a}, \mathbf{b}, \vec{x})\supset$ $\supset B(\mathbf{a}, \mathbf{c}, \vec{x}))$ (with $\vec{x}$ short for $x_1,\ldots, x_s$). H2 implies $//A(\mathbf{a}, \mathbf{b}, \mathbf{q})$ and $//A(\mathbf{a}, \mathbf{b}, \mathbf{q})\supset$ $\supset B(\mathbf{a}, \mathbf{c}, \mathbf{q})$ for any list $\mathbf{q}$ of terms not containing free number variables. From L1 we conclude that $//B(\mathbf{a}, \mathbf{c}, \mathbf{q})$ holds for any such list, that is $//(\vec{x})\, B(\mathbf{a}, \mathbf{c}, \vec{x})$ holds by H2 what proves the statement.

L3: If $//(x)\, (A(\mathbf{a}, \mathbf{b})\supset B(\mathbf{a}, \mathbf{c}, x))$ then $//A(\mathbf{a}, \mathbf{b})\supset (x)\, B(\mathbf{a}, \mathbf{c}, x)$ where x does not occur in $A(\mathbf{a}, \mathbf{b})$.

Proof: Assume $//(x)\, (A(\mathbf{a}, \mathbf{b})\supset B(\mathbf{a}, \mathbf{c}, x))$. The statement is proved if we can show: if $\mathbf{a}\subseteq\mathbf{F}$, $\mathbf{b}\subseteq\mathbf{G}$, $\mathbf{c}\subseteq\mathbf{H}$ and $//A(\mathbf{F}, \mathbf{G})$ then for every term q without free individual variables there is a σ_q with $\sigma_q/B(\mathbf{F}, \mathbf{H}, q)$. Hence let q be such a term and assume $\mathbf{a}\subseteq\mathbf{F}$, $\mathbf{b}\subseteq\mathbf{G}$, $\mathbf{c}\subseteq\mathbf{H}$ and $//A(\mathbf{F}, \mathbf{G})$. From $//(x)\, (A(\mathbf{a}, \mathbf{b})\supset B(\mathbf{a}, \mathbf{c}, x))$ we conclude $\tau^{\Phi}/(x)\, (A(\mathbf{F}, \mathbf{G})\supset B(\mathbf{F}, \mathbf{H}, x))$ that is $\tau^{\Phi}/A(\mathbf{F}, \mathbf{G})\supset B(\mathbf{F}, \mathbf{H}, q)$. From $//A(\mathbf{F}, \mathbf{G})$ it follows that there is a σ_q with $\sigma_q/B(\mathbf{F}, \mathbf{H}, q)$, concluding the proof.

L4: If $(x)\, (A\supset B(x))$ is strongly realizable then $A\supset (x)\, B(x)$ is strongly realizable (where x does not occur in $A(\mathbf{a}, \mathbf{b})$).

Proof: The reduction of L4 to L3 is essentially the same as that of L2 to L1.

L5: If $//(x)\, (A(\mathbf{a}, \mathbf{b}, x)\supset B(\mathbf{a}, \mathbf{c}))$ then $\tau^{\Phi}/(Ex)\, A(\mathbf{a}, \mathbf{b}, x)\supset B(\mathbf{a}, \mathbf{c})$ (where x does not occur in $B(\mathbf{a}, \mathbf{c})$).

Proof: Assume $//(Ex)\, A(\mathbf{a}, \mathbf{b}, x)$. Then there exists a τ with $\tau/(Ex)\, A(\mathbf{a}, \mathbf{b}, x)$. Let $\mathbf{a}$ and $\mathbf{b}$ have length s and t respectively and let $\mathbf{u}$ be the list $u_1,\ldots, u_s$ of sequence numbers, all having the same length, say n. Let r_n be the sequence number $\langle 0,\ldots, 0\rangle$ of length n and let $\vec{\mathbf{r}}_n$ be the list $r_n,\ldots, r_n$ of sequence numbers having t members. Define $\tau*$ as follows: $\tau*(\mathbf{u}, \mathbf{w})\neq 0$ iff $\tau(\mathbf{u}, \vec{\mathbf{r}}_n)\neq 0$ where $\vec{\mathbf{r}}_n$ is determined by $\mathbf{u}$ in the way just described. $\tau*$ is of course a continuity function. Assume $\tau*(\mathbf{u}, \mathbf{v})\neq 0$. Then $\tau(\mathbf{u}, \vec{\mathbf{r}}_n)\neq 0$ and hence there is a term q without free individual variables such that $//A(\mathbf{a}*\mathbf{u}, \mathbf{b}*\vec{\mathbf{r}}_n, q)$ holds. From H1 on the other hand we get $\tau^{\Phi}/(x)\, (A(\mathbf{a}*\mathbf{u}, \mathbf{b}*\vec{\mathbf{r}}_n, x)\supset B(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w}))$ and hence $\tau^{\Phi}/A(\mathbf{a}*\mathbf{u}, \mathbf{b}*\vec{\mathbf{r}}_n, q)\supset B(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$. But $//A(\mathbf{a}*\mathbf{u}, \mathbf{b}*\vec{\mathbf{r}}_n, q)$. Hence there is a σ' with $\sigma'/B(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$. That is: if $\tau*(\mathbf{u}, \mathbf{w})\neq 0$ then there is a σ' with $\sigma'/B(\mathbf{a}*\mathbf{u}, \mathbf{c}*\mathbf{w})$. According to H4 this implies the existence of a σ with $\sigma/B(\mathbf{a}, \mathbf{c})$ what proves the statement.

L6: If $//(x)\, (A(\mathbf{a}, \mathbf{b}, x)\supset B(\mathbf{a}, \mathbf{c}))$ then $//(Ex)\, A(\mathbf{a}, \mathbf{b}, x)\supset B(\mathbf{a}, \mathbf{c})$ (with x not in A).

Proof: Assume $\mathbf{a}\subseteq\mathbf{F}$, $\mathbf{b}\subseteq\mathbf{G}$, $\mathbf{c}\subseteq\mathbf{H}$. Then $//(x)\, (A(\mathbf{F}, \mathbf{G}, x)\supset B(\mathbf{F}, \mathbf{H}))$ according to

H1, hence $\tau^{\Phi}/(Ex)\, A(\mathbf{F}, \mathbf{G}, x) \supset B(\mathbf{F}, \mathbf{H})$ according to L5, what proves the statement.

L7: If $(x)\,(A(x) \supset B)$ then $(Ex)\, A(x) \supset B$ is strongly realizable (x not occurring in B).

Proof: L7 is reduced to L6 as L2 to L1.

L8: $//(\xi)\,(A \supset B(\xi))$ then $//A \supset (\xi)\, B(\xi)$ (with ξ not in A).

Proof: Exactly the same as that of L3.

L9: If $(\xi)\,(A \supset B(\xi))$ is strongly realizable then $A \supset (\xi)\, B(\xi)$ is strongly realizable (with ξ not in A).

Proof: We reduce L9 to L8 in the same way as L2 to L1.

L10: If $//(\xi)\,(A(\mathbf{a}, \mathbf{b}, \xi) \supset B(\mathbf{a}, \mathbf{c}))$ then $//(E\xi)\, A(\mathbf{a}, \mathbf{b}, \xi) \supset B(\mathbf{a}, \mathbf{c})$ (with ξ not in $B(\mathbf{a}, \mathbf{c})$).

Proof: Exactly the same as that of L6.

L11: If $(\xi)\,(A(\xi) \supset B)$ is strongly realizable then so is $(E\xi)\, A(\xi) \supset B$ (with ξ not in B).

Proof: Reduction to L10 in the same way as L2 is reduced to L1. Lemmas L1-L11 settle the questions connected with the rules of IA, which come up in the proof of theorem 1.

4.6. *The true prime formulas*

By definition, if P is a true prime formula (without free number variables) then σ/P for some continuity function σ. From H2 we obtain $//P$ for such a prime formula P. If finally $P(a_1, \ldots, a_s)$ is a true prime formula whose free number variables are among $a_1, \ldots, a_s$ then by definition $P(q_1, \ldots q_s)$ is true for all terms not containing free number variables. Hence again by H_2 $//P(q_1, \ldots, q_s)$ for all such terms. According to H5 this implies $//(x_1, \ldots, x_s)\, P(x_1, \ldots, x_s)$, that is P is strongly realizable. We note in this connection that IA contains the whole body of primitive recursive arithmetic: formulas such as $(p + q') = (p + q)'$, $p \cdot q' = p\, q + p$, $(\lambda x t(x))\,(q) = t(q)$ etc. are obviously all true. For later use we also note

L12: For every quantifierfree formula Q there is a term t_Q containing exactly the same free variables as Q such that the following formulas are provable in Z_3 and hence in IA: 1) $t_Q = 0 \vee t_Q = 1$, 2) $t_Q = 0 \supset Q$, 3) $Q \supset t_Q = 0$.

We omit the routine proof which is based on the fact that IA contains primit. rec. Arithmetic.

4.7. *The axioms of propositional calculus.*

L13: If A is an axiom of intuitionistic propositional calculus which does not contain free number variables then $//A$.

Proof: We content ourself with proving the statement for those two axioms of propositional calculus which contain disjuntions. All other axioms are trivial to treat.

a) The formula in question is $A(\mathbf{a}, \mathbf{b}) \supset A(\mathbf{a}, \mathbf{b}) \vee B(\mathbf{a}, \mathbf{c})$. Assume $//A(\mathbf{a}, \mathbf{b})$. We claim $\tau^{\Phi}/A(\mathbf{a}, \mathbf{b}) \vee B(\mathbf{a}, \mathbf{c})$. Indeed, since $//A(\mathbf{a}, \mathbf{b})$ it follows from H1 that $//A(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v})$ holds for all $\mathbf{u}$, $\mathbf{v}$ what proves the claim. That is we have proved $\tau^{\Phi}/A(\mathbf{a}, \mathbf{b}) \supset \supset A(\mathbf{a}, \mathbf{b}) \vee B(\mathbf{a}, \mathbf{c})$ for all formulas $A(\mathbf{a}, \mathbf{b})$, $B(\mathbf{a}, \mathbf{c})$. If $\mathbf{a} \subseteq \mathbf{F}$, $\mathbf{b} \subseteq \mathbf{G}$, $\mathbf{c} \subseteq \mathbf{H}$ then $A(\mathbf{F}, \mathbf{G}) \supset \supset A(\mathbf{F}, \mathbf{G}) \vee B(\mathbf{F}, \mathbf{H})$ is again such an axiom hence $\tau^{\Phi}/A(\mathbf{F}, \mathbf{G}) \supset A(\mathbf{F}, \mathbf{G}) \vee B(\mathbf{F}, \mathbf{H})$ according to the arguments just given. Therefore $//A(\mathbf{a}, \mathbf{b}) \supset A(\mathbf{a}, \mathbf{b}) \vee B(\mathbf{a}, \mathbf{c})$.

b) The formula in question is $(A \supset C \wedge B \supset C) \supset (A \vee B \supset C)$. Let us assume for simplicity that every member of the list $\mathbf{a}$ of free choice variables occurs in each of the formulas A, B, C and that the formulas A, B, C contain only free choice variables from the list $\mathbf{a}$; we indicate this by writing $A(\mathbf{a})$, $B(\mathbf{a})$, $C(\mathbf{a})$. The case where the distribution of variables is more general is treated in exactly the same way as this particular case. First we show: $\tau^{\Phi}/(A \supset C \wedge B \supset C) \supset (A \vee B \supset C)$. Hence assume $//A \supset C \wedge B \supset C$. We have to prove $\tau^{\Phi}/A \vee B \supset C$, or what amounts to the same, that $//A(\mathbf{a}) \vee B(\mathbf{a})$ implies $\sigma/C(\mathbf{a})$ for some σ. Hence assume in addition $//A(\mathbf{a}) \vee B(\mathbf{a})$. Then there is clearly a τ with $\tau/A(\mathbf{a}) \vee B(\mathbf{a})$. Assume $\tau(\mathbf{u}) \neq 0$. Then either $//A(\mathbf{a}*\mathbf{u})$ or $//B(\mathbf{a}*\mathbf{u})$; assume eg. $//A(\mathbf{a}*\mathbf{u})$. From $//A \supset C \wedge B \supset C$ we obtain $//A \supset C$ and $//B \supset C$ by H7. From this we get in particular $\tau^{\Phi}/A(\mathbf{a}*\mathbf{u}) \supset C(\mathbf{a}*\mathbf{u})$. Since $//A(\mathbf{a}*\mathbf{u})$ by assumption it follows that there is a σ' with $\sigma'/C(\mathbf{a}*\mathbf{u})$. Similarly if $//B(\mathbf{a}*\mathbf{u})$ holds in place of $//A(\mathbf{a}*\mathbf{u})$. Hence: if $\tau(\mathbf{u}) \neq 0$ then there is a σ' with $\sigma'/C(\mathbf{a}*\mathbf{u})$. But then it follows from H6 that there is a σ with $\sigma/C(\mathbf{a})$. Hence $\tau^{\Phi}/(A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C)$ has been proved. From this the stronger statement $//(A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C)$ immediately follows; we only have to use the fact that performing a substitution on such an axiom transforms it into another axiom of the same form, say $(A' \supset C') \wedge (B' \supset C') \supset \supset (A' \vee B' \supset C')$.

L14: If A is an axiom of intuitionistic propositional calculus then A is strongly realizable.

Proof: L15 follows from L14 in the same way as L2 from L1.

4.8. *The quantification axioms*

L15: $//(\xi)\, A(\mathbf{a}, \xi) \supset A(\mathbf{a}, F)$.

Proof: It is sufficient to prove $\tau^{\Phi}/(\xi)\, A(\mathbf{a}, \xi) \supset A(\mathbf{a}, F)$. The statement then follows from the observation that substitution does not change the form of the axiom. Assume $//(\xi)\, A(\mathbf{a}, \xi)$. Then we get $\tau^{\Phi}/(\xi)\, A(\mathbf{a}, \xi)$ as a particular case, hence $\sigma/A(\mathbf{a}, F)$ for some σ what proves the statement.

L16: $(\xi)\, A(\xi) \supset A(F)$ is strongly realizable.

Proof: This follows from L15 as L2 from L1.

L17: (x) A(x)$\supset$A(t) is strongly realizable.

Proof: Exactly the same as that of L16.

In the lemma 19 below let α be a free choice variable not occurring in the list **a** and **c** the list of those free choice variables which occur in the functor F but not in **a**; we express this by writing F [**a**, **c**].

L18: τ^{Φ}/A(**a**, F [**a**, **c**))$\supset$(Eξ) A(**a**, ξ).

Proof: Assume //A(**a**, F [**a**, **c**]). Let in addition **G** be a list of functors, none containing free variables and such that **c**$\subseteq$**G**. From the assumption we obtain //A(**a**, F [**a**, **G**)) according to H1. Clearly the free variables of F [**a**, **G**] all belong to the list **a**. Moreover //A(**a*****u**, F [**a*****u**, **G**]) again by H1. Therefore τ^{Φ}/(Eξ) A(**a**, ξ) since for every list **u** with **a**$\sim$**u** there is a functor **H**, namely F [**a*****u**, **G**] with //A(**a*****u**, F [**a*****u**, **G**]).

L19: With A(**a**, α) and F [**a**, **c**] as before, //A(**a**, F [**a**, **c**])$\supset$(Eξ) A(**a**, ξ).

Proof: Substitution transforms A(**a**, F [**a**, **c**])$\supset$(Eξ) A(**a**, ξ) into another axiom of the same kind. The statement then follows from L18.

L20: A(F)$\supset$(Eξ) A(ξ) is strongly realizable.

Proof: Follows from L19 as L2 from L1.

L21: A(t)$\supset$(Ex) A(x) is strongly realizable.

Proof: The same reasoning which leads to L20 is used.

4.9. *The induction axiom*

L24: //A(**a**, 0)$\supset\cdot$(x) (A(**a**, x)$\supset$A(**a**, x'))$\supset$(z) A(**a**, z).

Proof: As before it is sufficient to prove the statement τ^{Φ}/A(**a**, 0)$\supset\cdot$(x) (A(**a**, x)$\supset$ $\supset$A(**a**, x'))$\supset$(z) A(**a**, z). The lemma then follows from the invariance of the form of the axiom against substitution. Assume //A(**a**, 0) and //(x) (A(**a**, x) $\supset$A(**a**, x')). The statement then follows if we can show τ^{Φ}/(z) A(**a**, z). Combining L1 and H9 we obtain //A(**a**, n) for all numerals n. In order to prove τ^{Φ}/(z) A(**a**, z) we have to find for every term t a continuity function σ of suitable type such that σ/A(**a**, t) holds. Let t be t(**a**, **b**) with **b** the list of those free choice variables which do not occur in the list **a**. With t there is associated a continuity function τ such that τ(**u**, **v**)$\neq$0 implies t(**a*****u**, **b*****v**) saturated. Assume τ(**u**, **v**)$\neq$0; put |t(**a*****u**, **b*****v**)| = m. Since //A(**a**, n) for all n we have in particular //A(**a**, m). From this we conclude that there is a $\hat{\tau}$ with $\hat{\tau}$/A(**a*****u**, m). According to H8 there is a τ' with τ'/A(**a*****u**, t(**a*****u**, **b*****v**)). Hence: if

$\tau(\mathbf{u}, \mathbf{v}) \neq 0$ then there exists a τ' with $\tau'/A(\mathbf{a}*\mathbf{u}, t(\mathbf{a}*\mathbf{u}, \mathbf{b}*\mathbf{v}))$. From H6 it follows that there is a σ with $\sigma/A(\mathbf{a}, t)$, what concludes the proof.

L23: The induction axiom is strongly realizable.

Proof: Follows from L22 in the same way as L2 from L1.

4.10 *The axiom of transfinite induction*

Let $t(a, b)=0$ be a prime formula containing the distinct free numbervariables a, b. Let us write $a \prec b$ in place of $t(a, b)=0$. Let $W(\prec)$ be an abbreviation for the following formula: $(\xi)(Ex)\neg\xi(x+1)\prec\xi(x)$. A formula is said to be an axiom of transfinite induction if it is of the following form: $W(\prec)\supset\cdot(y)\,((x)\,(x\prec y\supset A(x))\supset \supset A(y))\supset(z)\,A(z)$ (to be denoted by $TI(\prec, A)$). By definition every axiom of transfinite induction is an axiom of IA. In virtue of L12 a slightly more general form of transfinite induction is available in IA: namely formulas of the above form but with a quantifierfree formula $Q(a, b)$ in place of $t(a, b)=0$. In [5] it is shown that with the continuity axiom (which is available in IA) even the most general form of transfinite induction or bar induction can be reduced to our particular formulation above.

In order to show that each axiom $TI(\prec, A)$ is strongly realizable it is again sufficient to prove $//TI(\prec, A)$ for all formulas $\prec$, A which do not contain free number variables. Since substitution transforms $TI(\prec, A)$ into another such axiom, say $TI(\prec', A')$, it is sufficient to show that $\tau^{\Phi}/TI(\prec, A)$ holds for $\prec$, A not containing free number variables. Hence our aim is to prove

L24: Let $t(a,b)=0$ and $A(a)$ be a prime formula and an arbitrary formula respectively not containing free number variables. Then $\tau^{\Phi}/TI(\prec, A)$ holds.

From L24 we obtain according to our remarks immediately

L25: Every axiom $TI(\prec, A)$ is strongly realizable. Before coming to the proof of L24 we need a definition.

DEFINITION 5: Let $t(\alpha^{i_1}_{u_1}, \ldots, \alpha^{i_s}_{u_s}, a, b)$ be a term whose only free variables are those indicated (a, b number variables). A set D of natural numbers can be associated with the primeformula $t=0$ as follows: its elements are ordered pairs $\langle\langle v_1, \ldots, v_s\rangle, n\rangle$ (also written more briefly as $\langle v_1, \ldots, v_s/n\rangle$) such that $v_1, \ldots, v_s$ is an s-tupel of sequence numbers, all having the same length. A partial ordering $\sqsubset$ of D can be associated with $t=0$ as follows: $\langle v_1, \ldots, v_s/n\rangle \sqsubset \langle w_1, \ldots, w_s/m\rangle$ iff a) each v_i is a proper extension of w_i (that is $w_i \subset v_i$), b) $t(\alpha^{i_1}_{u_1} * v_1, \ldots, \alpha^{i_s}_{u_s} * v_s, n, m)$ is saturated and its value is 0. The main property of $\sqsubset$ is given by

L26: Let $t(\alpha^{i_1}_{u_1}, \ldots, \alpha^{i_s}_{u_s}, a, b)$ and $\sqsubset$ be as in definition 5. Let $a \prec b$ be short for $t(\alpha^{i_1}_{u_1}, \ldots, \alpha^{i_s}_{u_s}, a, b)=0$. If $W(\prec)$ is realizable then $\sqsubset$ is wellfounded. The easy proof of this lemma is omitted.

Proof of L24: For typographical reasons we treat a slightly simplified case, in which $t(a, b)$ contains precisely one free choice variable namely $\alpha^1_{\langle\,\rangle}$, and where the formula A in $TI(\prec, A)$ contains precisely two free choice variables, namely $\alpha^1_{\langle\,\rangle}$, $\alpha^2_{\langle\,\rangle}$. We indicate this by writing $t(\alpha^1_{\langle\,\rangle}, a, b)$ and $A(\alpha^1_{\langle\,\rangle}, \alpha^2_{\langle\,\rangle}, a)$ respectively. This particular case is typical in that the proof below can be generalized in a straight forward way to the case where an arbitrary set of free choice variables is present. For convenience we will also write the prime formula $t(\alpha^1_{\langle\,\rangle}, a, b) = 0$ briefly as $p(\alpha^1_{\langle\,\rangle}, a, b)$. We want to prove $\tau^{\Phi}/TI(\prec, A)$. This is done if we can show: if $//W(\prec)$ and $//(y)\ ((x)\ (x \prec y \supset A(x)) \supset A(y))$ then $\tau^{\Phi}/(x)\ A(x)$. Hence let us assume: I) $//W(\prec)$, II) $//(y)\ ((x)\ (x \prec y \supset A(x)) \supset A(y))$. In virtue of H10 we are through if we can prove for every numeral m: $//A(\alpha^1_{\langle\,\rangle}, \alpha^2_{\langle\,\rangle}, m)$. From assumption I) and L26 it follows that $\sqsubset$ is well-founded. In order to proceed by transfinite induction let $\langle u/m\rangle \in D$ be fixed and let us make the inductive assumption: III) if $\langle v/n\rangle \sqsubset \langle u/m\rangle$ then $//A(\alpha^1_v, \alpha^2_{\langle\,\rangle}, n)$. The transfinite induction is accomplished if we can show: if $F[\mathbf{a}]$ and $G[\mathbf{a}]$ are two functors such that $u \subseteq F[\mathbf{a}]$ holds then there is a τ such that $\tau/A(F[\mathbf{a}], G[\mathbf{a}], m)$ holds (with $\mathbf{a}$ the list of choice variables occurring either in F or in G). Hence let $F[\mathbf{a}]$ and $G[\mathbf{a}]$ with $u \subseteq F[\mathbf{a}]$ be given. From II), H1 and H10 we conclude: $\tau^{\Phi}/(x)\ (p(F[\mathbf{a}], x, m)\ \ A(F[\mathbf{a}], G[\mathbf{a}], x)) \supset A(F[\mathbf{a}], G[\mathbf{a}], m)$. The desired τ with $\tau/A(F[\mathbf{a}), G[\mathbf{a}), m)$ is found if on the basis of III) we can show IV): $//(x)\ (p(F[\mathbf{a}], x, m) \supset A(F[\mathbf{a}], G[\mathbf{a}], x))$. In virtue of H10 this is proved if we can show for every numeral n the statement V): $//p(F[\mathbf{a}], n, m) \supset A(F[\mathbf{a}], G[\mathbf{a}), n)$. The last statement finally is a consequence of the following statement VI): for every n, if $\mathbf{a} \subseteq \mathbf{H}[\mathbf{b}]$ then $\tau^{\Phi}/p(F[\mathbf{H}[\mathbf{b}]], n, m) \supset A(F[\mathbf{H}[\mathbf{b}]], G[\mathbf{H}[\mathbf{b}]], n)$. Hence let us concentrate on the proof of VI). To this end let n be an arbitrary numeral and $\mathbf{H}[\mathbf{b}]$ an arbitrary list of functors, only subject to the restriction $\mathbf{a} \subseteq \mathbf{H}[\mathbf{b}]$. In virtue of lemma 8 there is a continuity function σ having the following property: if $\sigma(\mathbf{w}) \neq 0$ then there is a sequence number v such that $u \subseteq v$, $v \subseteq F[\mathbf{H}[\mathbf{b} * \mathbf{w}]]$ and such that $t(\alpha^1_v, n, m)$ is saturated. We claim that the following statement VII) is true: if $\sigma(\mathbf{w}) \neq 0$ then $\tau^{\Phi}/p(F[\mathbf{H}[\mathbf{b} * \mathbf{w}]], n, m) \supset A(F[\mathbf{H}[\mathbf{b} * \mathbf{w}]], G[\mathbf{H}[\mathbf{b} * \mathbf{w}]], n)$ holds. Hence assume $//t(F[\mathbf{H}[\mathbf{b} * \mathbf{w}]], n, m) = 0$. Since $t(\alpha^1_v, n, m)$ is saturated and since $v \subseteq F[\mathbf{H}[\mathbf{b} * \mathbf{w}]]$ it follows that $t(F[\mathbf{H}[\mathbf{b} * \mathbf{w}]], n, m)$ is saturated and has the same value as $t(\alpha^1_v, n, m)$. From $//t(F[\mathbf{H}[\mathbf{b} * \mathbf{w}]], n, m) = 0$ it follows that this value must necessarily be 0. In combination with $u \subset v$ this implies $\langle v/n\rangle \sqsubset \langle u/m\rangle$. In virtue of our inductive assumption we have $//A(\alpha^1_v, \alpha^2_{\langle\,\rangle}, n)$ and hence $\tau/A(F[\mathbf{H}[\mathbf{b} * \mathbf{w}]], G[\mathbf{H}[\mathbf{b} * \mathbf{w}]], n)$ for some τ what proves statement VII). From VII), H4 and H6 it follows that $\tau^{\Phi}/p(F[\mathbf{H}[\mathbf{b}]], n, m) \supset A(F[\mathbf{H}[\mathbf{b}]], G[\mathbf{H}[\mathbf{b}]], n)$ is true. Since n, $\mathbf{H}[\mathbf{b}]$ where arbitrary (apart from $\mathbf{a} \subseteq \mathbf{H}[\mathbf{b}]$) it follows that statement VI) is true. From VI) we deduce IV) and from IV) and $\tau^{\Phi}/(x)\ (p(F[\mathbf{a}], x, m) \supset A(F[\mathbf{a}], G[\mathbf{a}], x) \supset A(F[\mathbf{a}], G[\mathbf{a}], m)$ we obtain the existence of a τ with $\tau/A(F[\mathbf{a}], G[\mathbf{a}], m)$. But $F[\mathbf{a}]$ and $G[\mathbf{a}]$ where only subject to the condition $u \subseteq F[\mathbf{a}]$ and otherwise arbitrary. Hence $//A(\alpha^1_u, \alpha^2_{\langle\,\rangle}, m)$ is true. The transfinite induction is done and the lemma proved.

4.11. *The axiom of choice*

As will be sketched in the appendix, we can derive the general axiom of choice from the full continuity axiom and the following special instance of the axiom of choice: (x) (Ex) Q(x, y) ⊃ (Eξ) (x) Q(x, ξ(x))$_{,}$ with Q a quantifierfree formula. On the other hand we know from L12 that every quantifierfree formula is provable equivalent to a certain prime formula. Therefore it is even sufficient to include among the axioms of IA only the following very particular instances of the axiom of choice: ACP) (x) (Ey) p(x, y) ⊃ (Eζ) (x) p(x, ζ (x)) where p(x, y) is a prime formula.

L27: (x) (Ey) p(x, y) ⊃ (Eζ) (x) p(x, ζ(x)) is strongly realizable (p prime).

Proof: As before it is sufficient to prove τ^{ϕ}/(x) (Ey) p(**a**, x, y) ⊃ (Eζ) (x) p(**a**, x, ζ (x)) where p(**a**, a_1, a_2) is any prime formula whose only free variables are the choice variables from the list **a** and the free number variables a_1, a_2. For simplicity we assume that the list **a** has the particular form $\alpha^1_{\langle\,\rangle}, \dots, \alpha^s_{\langle\,\rangle}$. Hence assume //(x) (Ey) p(**a**, x, y). This implies: for every n there is a τ_n such that τ_n/(Ey) p(**a**, n, y) holds. Therefore: if τ_n(u)$\neq$0 then //p(**a∗u**, n, t_n(**a∗u**)) for some term t_n which contains only free variables from the list **a∗u**. This in turn implies the existence of a continuity function σ^n_u having the property: if σ^n_u(v)$\neq$0 then p(**a∗u∗v**, n, t_n(**a∗u∗v**)) and t_n(**a∗u∗v**) both are saturated and p(**a∗u∗v**, n, t_n(**a∗u∗v**)) is true. In particular if |t_n(**a∗u∗v**)| = m then also p(**a∗u∗v**, n, m) is saturated and true (calling $q_1 = q_2$ saturated if q_1 and q_2 are saturated). Let μ be any continuity function of type (s, 1) having the following properties: 1) if μ(**w**, n)$\neq$0 then **w** admits a representation **w**=**u∗v** with τ_n(**u**)$\neq$0 and σ^n_u(**v**)$\neq$0, 2) if μ(**w**, n)$\neq$0, **w**=**u∗v**, τ_n(**u**)$\neq$0 and σ^n_u(**v**)$\neq$0 then μ(**w**, n) = |t_n(**a∗w**)| + 1. It is easy to prove the existence of such a μ. The language L contains by definition a functor constant $G^{\mu}_{s,0}$; we write F instead of $G^{\mu}_{s,0}$. By definition F[**a∗w**] (n) is saturated iff μ(**w**, n)$\neq$0 and in this case |F[**a∗w**] (n)| = |t_n(**a∗w**)| = μ(**w**, n)−1. Let μ_n be given by μ_n(**w**)=μ(**w**, n). We claim μ_n/p(**a**, n, F[**a** (n)). If μ_n(**w**)$\neq$0 then there is a splitting **w**=**u∗v** such that τ_n(**u**)$\neq$0 and σ^n_u(**v**)$\neq$0. But then p(**a∗w**, n, t_n(**a∗w**)) is saturated and true. Since t_n(**a∗w**) is also saturated and since |F [**a∗w**] (n)| = |t_n(**a∗w**)| by definition of μ it follows that p(**a∗w**, n, F [**a∗w**] (n)) is also saturated and true; hence μ_n/p(**a**, n, F [**a**](n)) for all n. According to H2 we have //p(**a**, n, F [**a**] (n)) for all n and this together with H10 yields //(x) p(**a**, x, F [**a**] (x)). From the definition of / and // one finally concludes that τ^{ϕ}/(Eζ) (x) p(**a**, x, ζ (x)) what proves the statement. Only minor modifications are needed in order to treat the case where the list a is of the more general type $\alpha^{i_1}_{u_1}, \dots, \alpha^{i_s}_{u_s}$.

4.12. *The continuity axiom*

In order to discuss the continuity axiom let φ_1, φ_2, φ_3 be two twoplace and a one place primitive recursive function respectively having the properties: a) $\varphi_1(u, v) = 0$ iff u, v are sequence numbers with $u \subseteq v$, b) $\varphi_2(a, b) = \frac{1}{2}((a+b)^2 + 3a + b)$, c) φ_3(n) is the length of n if n is a sequence number and 0 otherwise. These functions are represented in IA by certain functional constants c_1, c_2, c_3 of types (0,2), (0,2) and (0,1) respectively; IA contains suitable axioms with respect to c_1, c_2, c_3 which permit to derive the familiar properties of φ_1, φ_2, φ_3. For easy reading we adopt the following conventions: a) $c_1(p, q)=0$ is written as $p \subseteq q$, b) $c_2(p, q)$ is written as $\langle p, q\rangle$, c) if F is a functor we write $F(p, q)$ in place of $F(\langle p, q\rangle)$, d) $v \subseteq F$ is an abbreviation for $v \subseteq \bar{F}(c_3(v))$. Let $CT_1(\alpha)$ and $CT_2(\alpha)$ be the following formulas respectively: a) $(x, y)(x \subseteq y \wedge \alpha(x) \neq 0 \supset \alpha(y) \neq 0)$, b) $(\xi)(Ex)\,\alpha(\bar{\xi}(x)) \neq 0$. Let $CT(\alpha)$ be $CT_1(\alpha) \wedge \wedge\, CT_2(\alpha)$. A suitable form of the continuity axiom, which is equivalent to that one presented in [2] is

$$(\xi)(E\eta)\,A(\xi, \eta) \supset (E\sigma)\{(x)\,CT(\lambda y \sigma(y, x)) \wedge (\zeta, \mu)[(x, v)(v \subseteq \zeta \wedge \sigma(v, x) \neq 0 \supset \sigma(v, x) = \mu(x) + 1) \supset A(\zeta, \mu)]\}.$$

Denote this formula by CT(A). Our goal is to prove

L28: CT(A) is strongly realizable. As in earlier cases this is achieved if we can show

L29: //CT(A) for all A not containing free number variables. But CT(A) is clearly invariant against substitution and hence L29 follows from

L30: For A without free number variables τ^{Φ}/CT(A) holds. Before coming to the proof of this lemma, which will be given below we note

THEOREM 0: *If* $Z_3 \vdash A$ *then* A *is strongly realizable.*

Proof: According to the lemmas proved so far it follows that every axiom of Z_3 is strongly realizable and that if the premiss or the premisses of any inference are strongly realizable then the conclusion is strongly realizable.

COROLLARY 1 TO THEOREM 0: *Let* F, G *be two functors and* $A(\alpha)$ *a formula none containing free number variables. Then* $\tau^{\Phi}/(x)\,(F(x)=G(x)) \subseteq (A(F) \supset A(G))$ *holds.*

Proof: The formula on the righthandside of / (to be denoted by B) is obviously provable in Z_3; hence //B by theorem 0 and so τ^{Φ}/B.

COROLLARY 2 TO THEOREM 0: *Let* F, G, $A(\alpha)$ *be as in corollary 1. If for every* n *there is a* τ_n *with* $\tau_n/F(n)=G(n)$, *if* //A(F), *then* $\sigma/A(G)$ *for some* σ.

Proof: This is an immediate consequence of theorem 0, corollary 1, H2, H10 and the definition of $\tau^{\Phi}/U \supset V$.

Proof of L30: The proof is performed in several steps. In order to simplify the notation somewhat, we assume that A contains precisely the free variables α_u, β, γ and write therefore $A(\alpha_u, \beta, \gamma)$ and $(\xi)(E\eta)\,A(\alpha_u, \xi, \eta)$ respectively. This special case is representative in that the generalisation to the arbitrary case is straightforward. *Step 1:* We make the basic assumption $//(\xi)(E\eta)\,A(\alpha_u, \xi, \eta)$. This implies in particular $\tau^{\Phi}/(E\xi)\,A(\alpha_u, \beta, \xi)$ where β is a free choice variable associated with the empty sequent and different from α_u. By definition there is a functor $T[\alpha_u, \beta]$, whose only free variables are α_u, β such that the following statement I) holds: $//A(\alpha_u, \beta, T[\alpha_u, \beta])$. According to lemma 2 there is a continuity function σ_0 of type (2,1) such that $\sigma_0(v, w, n) \neq 0$ implies $T[\alpha_v, \beta_w](n)$ saturated with value $\sigma_0(v, w, n)-1$. Define a continuity function τ of type (1,1) as follows: 1) if m is not a sequence number then $\tau(\bar{f}(i), \langle m, k\rangle) = 1$, 2) if $i<j$ then $\tau(\bar{f}(i), \langle \bar{g}(j), k\rangle) = 1$, 3) if $j \leqslant i$ and if $\sigma_0(\bar{f}(j), \bar{g}(j), k) = 0$ then $\tau(\bar{f}(i), \langle \bar{g}(j), k\rangle) = 1$, 4) if $j \leqslant i$ and if $\sigma_0(\bar{f}(j), \bar{g}(j), k) = m+1$ then $\tau(\bar{f}(i), \langle \bar{g}(j), k\rangle) = m+2 = |T[\alpha_{\bar{f}(i)}, \beta_{\bar{g}(j)}](k)|+2$. The language L contains an operator symbol Γ of type (1,0) associated with τ. By definition $\Gamma[\alpha_v](\langle w, k\rangle)$ is saturated iff $\tau(v, \langle w, k\rangle) = m+1 \neq 0$ and its value in this case is m. *Step 2:* Let us abbreviate in the sequel the formula $(x, v)(v \subseteq \beta \wedge \alpha(v, x) \neq 0 \supset \alpha(v, x) = \gamma(x)+1)$ by $E(\alpha/\beta, \gamma)$. According to H10, H18 the lemma is proved if we can show: 1) for each n $//CT(\lambda y \Gamma[\alpha_u](y, n))$, 2) for all G, H and F such that $u \subseteq F$ we have $\tau^{\Phi}/E(\Gamma[F]/G, H) \supset A(F, G, H)$. We first concentrate on the verification of 2). As noted, we have $//A(F, G, T[F, G])$ for all F, G with $u \subseteq F$. Let $F[\mathbf{b}]$, $G[\mathbf{b}]$ and $H[\mathbf{b}]$ be fixed functors such that $u \subseteq F[\mathbf{b}]$ holds. Then we have II): $//A(F[\mathbf{b}], G[\mathbf{b}], T[F[\mathbf{b}], G[\mathbf{b}]])$. In virtue of corollary 2) to theorem 0 the verification of 2) is accomplished if we can show: if $E(\Gamma[F[\mathbf{b}]]/G[\mathbf{b}), H[\mathbf{b}])$ then there is for every n a τ^n such that $\tau^n/T[F[\mathbf{b}], G[\mathbf{b}]](n) = H[\mathbf{b}](n)$ holds. Hence let us assume III): $//E(\Gamma[F[\mathbf{b}]]/G[\mathbf{b}], H[\mathbf{b}])$. *Step 3:* From lemma 8 it follows that for every n there is a continuity function σ^n such that: if $\sigma^n(\mathbf{w}) \neq 0$ then there exist $\bar{f}(i)$ and $\bar{g}(j)$ such that $u * \bar{f}(i) \subseteq F[\mathbf{b} * \mathbf{w}]$, $\bar{g}(j) \subseteq G[\mathbf{b} * \mathbf{w}]$ and such that $\sigma_0(u * \bar{f}(i), \bar{g}(j), n) \neq 0$. Hint: 1) Put $\Delta(x) = 1$ if x is a sequence number with length$(u) <$ length(x) and $\Delta(x) = 0$ otherwise, 2) take as τ in lemma 1 the continuity function $\min(\Delta(x), \sigma_n(x, y, n))$, 3) observe that $u \subseteq F[\mathbf{b}]$ already holds. Then by definition of τ in step 1, $\tau(u * \bar{f}(i), \langle \bar{g}(j), n\rangle) = |T[\alpha_{u * \bar{f}(i)}, \beta_{\bar{g}(j)}](n)|+2$ and by definition of Γ the term $\Gamma[\alpha_{u * \bar{f}(i)}](\bar{g}(j), n)$ is saturated and its value $|T[\alpha_{u * \bar{f}(i)}, \beta_{g(j)}](n)|+1$. Now take a fixed n and a fixed $\mathbf{w}$ such that $\sigma^n(\mathbf{w}) \neq 0$ and let $\bar{f}(i)$ and $\bar{g}(j)$ be as in 1), 2) above. From assumption III) we infer $\tau^{\Phi}/E(\Gamma[F[\mathbf{b} * \mathbf{w}]]/G[\mathbf{b} * \mathbf{w}), H[\mathbf{b} * \mathbf{w}])$. This in turn implies $\tau^{\Phi}/\bar{g}(j) \subseteq G[\mathbf{b} * \mathbf{w}] \wedge \wedge \Gamma[F[\mathbf{b} * \mathbf{w}]](\bar{g}(j), n) \neq 0 \supset \Gamma[F[\mathbf{b} * \mathbf{w}]](\bar{g}(j), n) = H[\mathbf{b} * \mathbf{w}](n)+1$. Clearly, $\bar{g}(j) \subseteq \subseteq G[\mathbf{b} * \mathbf{w}]$ is saturated and true, hence $//\bar{g}(j) \subseteq G[\mathbf{b} * \mathbf{w}]$. Furthermore $\Gamma[F[\mathbf{b} * \mathbf{w}]](\bar{g}(j), n)$ is also saturated and its value $\neq 0$; hence $//\Gamma[F[\mathbf{b} * \mathbf{w}]](\bar{g}(j), n) \neq 0$ by H13. It follows from H7 that there exists a continuity function μ^n_w with the property IV): if $\mu^n_w(\mathbf{w}') \neq 0$ then $H[\mathbf{b} * \mathbf{w} * \mathbf{w}'](n)$ and $\Gamma[F[\mathbf{b} * \mathbf{w} * \mathbf{w}']](\bar{g}(j), n)$ are saturated and $|\Gamma[F[\mathbf{b} * \mathbf{w} * \mathbf{w}']](\bar{g}(j), n)| = |H[\mathbf{b} * \mathbf{w} * \mathbf{w}'](n)|+1$. But $\Gamma[F[\mathbf{b} * \mathbf{w}]](\bar{g}(j), n)$ and

$T[F[\mathbf{b}*\mathbf{w}], G[\mathbf{b}*\mathbf{w}]](n)$ are already saturated and the value of the first equals the value of the second plus one. Hence both $\Gamma[F[\mathbf{b}*\mathbf{w}*\mathbf{w}']](\bar{g}(j), n)$ and $T[F[\mathbf{b}*\mathbf{w}*\mathbf{w}'], G[\mathbf{b}*\mathbf{w}*\mathbf{w}']](n)$ are saturated and the value of the first equals the value of the second plus one. With the aid of property IV) we can sum up these considerations as follows: if $\sigma^n(\mathbf{w})\neq 0$ then there exist a μ_w^n such that $T[F[\mathbf{b}*\mathbf{w}*\mathbf{w}'], G[\mathbf{b}*\mathbf{w}*\mathbf{w}']](n)$ and $H[\mathbf{b}*\mathbf{w}*\mathbf{w}'](n)$ both are saturated and have the same value whenever $\mu_w(\mathbf{w}')\neq 0$ holds. But according to H6 this implies the existence of a τ^n with $\tau^n/T[F[\mathbf{b}], G[\mathbf{b}]](n) = H[\mathbf{b}](n)$. This concludes the proof of statement 2) mentioned at the beginning of step 2. *Step 4:* It remains to verify statement 1) mentioned at the beguinning of step 2. This amounts to prove for all n: a) $//(x, y)(x\subseteq y \wedge \Gamma[\alpha_u](x, n)\neq 0 \supset \Gamma[\alpha_u](y, n)\neq 0)$, b) $//(\xi)(Ex)\,\Gamma[\alpha_u](\xi(x), n)\neq 0$. We start with the verification of a). We have to prove for all sequence numbers v, w: $//w\subseteq v \wedge \Gamma[\alpha_u](w, n)\neq 0 \supset \Gamma[\alpha_u](v, n)\neq 0$. Hence let $F[\mathbf{b}]$ be a functor such that $u\subseteq F[\mathbf{b}]$ holds and assume I'): α) $//w\subseteq v$, β) $//[F[\mathbf{b}]](w, n)\neq 0$. I', α) implies that w and v are indeed sequence numbers and that v is a proper extension of w. From I', β) and lemma 8 we infer the existence of a continuity function σ such that $\sigma(\mathbf{r})\neq 0$ has the following consequences: 1) there is an $\bar{f}(i)$ such that $u*\bar{f}(i)\subseteq F[\mathbf{b}*\mathbf{r}]$ and length $(v)<$ length $(u*\bar{f}(i)$, 2) $\Gamma[\alpha_{u*\bar{f}(i)}](v, n)$ and $\Gamma[\alpha_{u*\bar{f}(i)}](w, n)$ both are saturated and the value of the second is $\neq 0$. Fix a list $\mathbf{r}$ such that $\sigma(\mathbf{r})\neq 0$. Since $v\subseteq w$, $u*\bar{f}(i)\subseteq F[\mathbf{b}*\mathbf{r}]$ and since $\Gamma[\alpha_{u*\bar{f}(i)}](w, n)$ is saturated with value $\neq 0$ we conclude from Lemma 4 that $\Gamma[F[\mathbf{b}*\mathbf{r}]](v, n)$ is saturated and has value $\neq 0$. To sum up: if $\sigma(\mathbf{r})\neq 0$ then $\Gamma[F[\mathbf{b}*\mathbf{r}]](v, n)$ is saturated and has value $\neq 0$. According to H13 this implies $\tau^{\phi}/\Gamma[F[\mathbf{b}]](v, n)\neq 0$ what concludes the verification of a). *Step 5:* It remains to verify b), that is $//(\xi)(Ex)\Gamma[\alpha_u](\xi(x), n)\neq 0$ for all n. Let n henceforth be fixed. Let β be a free choice variable, associated with the empty sequence and different from α_u. Combining H1, H12, H19 and H13 it is clear that b) is verified if we can find a term $t(\alpha_u, \beta)$ and a τ such that the following holds: if $\tau(v, w)\neq 0$ then $\Gamma[\alpha_{u*v}](\beta_w(t(\alpha_{u*v}, \beta_w)), n)$ is saturated and its value is different from 0. In step 1 we have introduced a continuity function σ_0 of type (2,1). Clearly there exists a continuity function μ of type (2,0) having the properties: if $\mu(\bar{f}(i), \bar{g}(i))\neq 0$ then 1) $k+1 = \mu(\bar{f}(i), \bar{g}(i))\leqslant i+1$, 2) $\sigma_0(\bar{f}(k), \bar{g}(k), n)\neq 0$. To μ there corresponds a functional symbol t of type (2,0) such that the following holds: 1) for $v=\bar{f}(i)$, $w=\bar{g}(i)$ $t(\alpha_v, \beta_w)$ is saturated iff $\mu(v, w)\neq 0$, 2) in this case $|t(\alpha_v, \beta_w)| = \mu(v, w)-1$. Obviously there exists a continuity function ν of type (2,0) having the following property: if $\nu(v, w)\neq 0$ then v allows a decomposition $v_1*v_2=v$ such that $\mu(u*v_1, w)\neq 0$. We claim: $\nu/\Gamma[\alpha_u](\bar{\beta}(t(\alpha_u, \beta)), n)\neq 0$. Assume $\nu(v, w)\neq 0$ and let $v=v_1*v_2$ be a decomposition such that $\mu(u*v_1, w)\neq 0$. Put $u*v_1=\bar{f}(i)$, $w=\bar{g}(i)$ and $k+1=\mu(u*v_1, w)$. According to the definition of μ we have $k\leqslant i$ and $\sigma_0(\bar{f}(k), \bar{g}(k), n)\neq 0$. Hence $\tau(\bar{f}(k), \langle\bar{g}(k), n\rangle)\leqslant 2$ and therefore $\Gamma[\alpha_{\bar{f}(k)}](\bar{g}(k), n)$ is saturated with value $\neq 0$. Since $u*v_1\subseteq u*v$ and $\bar{f}(k)\subseteq u*v_1$ it follows that $\Gamma[\alpha_{u*v}](\bar{g}(k), n)$ is saturated with value $\neq 0$. Since $k\leqslant i$ one easily verifies that $\bar{\beta}_w(k)$ is saturated with value $\bar{g}(k)$. Since $\mu(u*v_1, w)=k+1$ it follows that $t(\alpha_{u*v_1}, \beta_w)$ is saturated with value k. Hence $t(\alpha_{u*v}, \beta_w)$ is also saturated and its value is still k. Therefore $\bar{\beta}_w(t(\alpha_{u*v}, \beta_w))$ is saturated and its value is the same as that of $\bar{\beta}_w(k)$, namely $\bar{g}(k)$. Hence $\Gamma[\alpha_{u*v}](\beta_w(t(\alpha_{u*v}, \beta_w)), n)$ is saturated and its value is $\neq 0$. This concludes the proof.

Theorem 1: *If* A *is provable in the system* IA *of intuitionistic analysis gehn* A *is*

THEOREM 1: *If* A *is provable in the system* IA *of intuitionistic analysis then* A *is strongly realizable.*

Proof: According to the lemmas proved up to now it follows that all axioms of IA are strongly realizable. Furthermore, if the premiss or the premisses of an inference are strongly realizable then so is the conclusion according to lemmas L1 – L11. The theorem then follows by induction with respect to the length of proofs. Kleene-Vesleys system of intuitionistic analysis is contained in our system IA, as pointed out earlier, that is every formula provable in Kleene-Vesleys system is provable in IA (after eventually replacing some bound variables by others). Hence

COROLLARY TO THEOREM 1: *If a formula* A *is provable in the system of Kleene-Vesley then it is strongly realizable.*

5. A remark on Troelstras axiom

In [6] A. S. Troelstra introduces a certain system of intuitionistic analysis which contains the system of Kleene-Vesley as subsystem. An essential feature of this system is that it contains a new axiom, which for simplicity will be called Troelstras axiom. In order to state it let Rec (α) be a suitable formula expressing that α is a recursive function. Let $CT(\alpha)$ and $E(\alpha/\beta, \gamma)$ have the same meaning as in the proof of lemma L30. Let $A(\alpha)$ be a formula whose only free choice variable is α (associated with the empty sequence). Troelstras axiom looks as follows:

$$A(\alpha) \supset (E\sigma)\{Rec(\sigma) \wedge (x)\,CT(\lambda y\sigma(y, x)) \wedge (E\eta)\,E(\sigma/\eta, \alpha) \\ \wedge (\zeta, \zeta_1)(E(\sigma/\zeta, \zeta_1) \supset A(\zeta_1))\}.$$

Let us denote this formula by $Tr(A)$. If we drop $Rec(\alpha)$ in this formula then we obtain another formula to be denoted by $Tr*(A)$. While $Tr(A)$ is an essential new axiom this is not the case with $Tr*(A)$: it is easy to show that $Tr*(A)$ is provable in the system of Kleene-Vesley. Therefore, by corollary to theorem 1, $Tr*(A)$ is strongly realizable. In the model presented in the last chapter, $Tr(A)$ cannot be proved to be strongly realizable since this model has essentially set theoretic character: it contains no ingredients of recursive function theory. However, there is a constructive version of this model in which everything is codified in a suitable way by Goedel numbers; this constructive version will be presented in a subsequent paper. It can be shown that Tr(A) is strongly realizable in this constructive model. The proof of this splits up in two parts: an abstract, rather set theoretic part and a constructive part, which uses the fixed point theorem and whose main concern is to translate the abstract part in an appropriate way in the language of recursive function theory. Now it is quite usefull to have a direct verification of the strong realizability of Tr∗(A) at hand: it turns out that this direct verification essentially coincides with the abstract part of the above mentioned proof. Hence let us proceed to a direct verification of the strong realizability. One easily verifies on the basis of H5 that this is achieved if we can show

L31: If $A(\alpha)$ is a formula whose only free variable is the free choice variable associated with the empty sequence then //Tr∗(A).

Proof: We will not consider all the details but rather concentrate on the main points. Let $F[\mathbf{b}]$ be any functor without free number variables. Our aim is to show:

$$\tau^{\Phi}/A(F[\mathbf{b}]) \supset (E\sigma)\{(x)\,CT(\lambda y\sigma(y, x) \wedge (E\xi)\,E(\sigma/\xi, F[\mathbf{b}]) \\ \wedge (\zeta, \zeta_1)(E(\sigma/\zeta, \zeta_1) \supset A(\zeta_1))\}.$$

To this end assume $//A(F[\mathbf{b}])$. For simplicity we consider the case where $\mathbf{b}$ consists of two choice variables, α, β both associated with the empty sequence. The case where $\mathbf{b}$ is more general is treated in exactly the same way. We proceed by steps. *Step 1:* Clearly there are continuous operators F, G_1, G_2 of type (2,0), (1,0) and (1,0) respectively having the properties: 1) F maps the set of ordered pairs of numbertheoretic functions in a one one way onto the set of numbertheoretic functions, 2) $\lambda\xi(G_1[\xi], G_2[\xi])$ maps the set of numbertheoretic functions in a one one way onto the set of ordered pairs of numbertheoretic functions, 3) the mappings $\alpha \to (G_1[\alpha], G_2[\alpha])$ and $(\alpha, \beta) \to F[\alpha, \beta]$ are inverses of each other. It is clear that F, G_1, G_2 can be chosen in such a way that there are continuity functions τ, μ_1, μ_2 of suitable types such that $F = F_\tau$, $G_1 = F_{\mu_1}$ and $G_2 = F_{\mu_2}$; concrete examples can easily be found in connection with the pairing function $\frac{1}{2}((x+y)^2 + 3x + y)$. The language L contains by definition constants C, K_1 and K_2 which formally represent F, G_1, G_2 in IA. In addition there are sufficiently many axioms about C, K_1, K_2 in IA which permit us to derive all essential properties of F, G_1, G_2. In order to list them let $\alpha \sim \beta$ be an abbreviation for $(x)\,(\alpha(x) = \beta(x))$. Then we can prove in Z_3: 1) $C[\alpha, \beta] \sim \gamma \wedge C[\alpha', \beta'] \sim \sim \gamma \supset \alpha \sim \alpha' \wedge \beta \sim \beta'$, 2) $C[K_1[\gamma], K_2[\gamma]] \sim \gamma$, 3) $K_1[C[\alpha, \beta]] \sim \alpha$, 4) $K_2[C[\alpha, \beta]] \sim \beta$. *Step 2:* Concerning $E(\sigma/\alpha, \beta)$ we note that the following is provable in Z_3: $E(\sigma/\alpha, \beta) \supset \supset \cdot (E(\sigma/\alpha, \beta') \supset \beta \sim \beta')$. *Step 3:* Let for the moment being $G[\alpha]$ be a functor whose only free variable is α. According to lemma 2 there is a continuity function $\sigma_0(x, y)$ of type (1,1) with the property: if $\sigma_0(\nu, n) = m + 1$ then $G[\alpha_\nu](n)$ is saturated with value

m. Put $\mu(y, x)=\sigma_0(y, x)+1$. We can consider $\mu(y, x)$ as a continuity function of type (0,2). With μ there is associated a functional constant $e_\mu \in L$; denote e_μ by Δ. According to its definition $\Delta(v, n)$ is saturated for all v, n and its value is $\sigma_0(v, n)$. Now the following is true: $//E(\Delta/\alpha, G[\alpha])$. The verification of this statement, referred to as statement I), is entirely routine and is left to the reader. The particular case which we are interested in is where G is $F[K_1[\alpha], K_2[\alpha]]$; the Δ associated with this particular $G[\alpha]$ is now denoted by τ. $F[K_1[\alpha], K_2[\alpha]]$ will also be written as $F[\mathbf{K}[\alpha]]$. Hence we have II): $//E(\tau/\alpha, F[\mathbf{K}[\alpha]])$. *Step 4:* In virtue of section 4.4 we have accomplished our proof if we can show: a) $//(x)\, CT(\lambda y\tau(y, x))$, b) $//E(\tau/C[\alpha, \beta], F[\alpha, \beta])$, c) for all G, H without free number variables we have $\tau^{\Phi}/E(\tau/G, H)\supset A(H)$. We omit the verification of a) which is the same as in the proof of L31. In order to verify b) we use $Z_3 \vdash \alpha \sim K_1[C[\alpha, \beta]]$ and $Z_3 \vdash \beta \sim K_2[C[\alpha, \beta]]$. From this and $Z_3 \vdash \alpha \sim \beta \supset (A(\alpha) \supset \supset A(\beta))$ we derive in Z_3 the formula $E(\tau/C[\alpha, \beta], F[\mathbf{K}[C[\alpha, \beta]]]) \supset E(\tau/C[\alpha, \beta], F[\alpha, \beta]])$, to be denoted by U. Since $//E(\tau/\alpha, F[\mathbf{K}[\alpha]])$ holds by construction of τ, it follows that $//E(\tau/C[\alpha, \beta], F[\mathbf{K}[C[\alpha, \beta]]])$ holds. This, combined with U (theorem 0) and L1 yields $//E(\tau/C[\alpha, \beta], F[\alpha, \beta])$. But this, combined with H18 implies $//(E\xi)\, E(\tau/\xi, F[\alpha, \beta])$. *Step 5:* It remains to verify c). Hence assume $//E(\tau/G, H)$. According to the remarks made in step 2 it follows that the following formula V is provable in Z_3: $E(\tau/G, H) \supset \cdot E(\tau/G, F[\mathbf{K}[G]]) \supset (H \sim F[\mathbf{K}[G]])$. Hence //V. But $//(\tau/G, H)$ by assumption and $//E(\tau/G, F[\mathbf{K}[G]])$ by construction. Using L1 we obtain $//H \sim F[\mathbf{K}[G]]$. Now $\alpha \sim \beta \supset (A(\alpha) \supset A(\beta))$, (to be denoted by W in the sequel) is, as mentioned, provable in Z_3; hence //W. From this we obtain $//H \sim F[\mathbf{K}[G]] \supset \supset \cdot A(F[\mathbf{K}[G]]) \supset A(H)$. Now $//A(F[\mathbf{b}])$ is our basic assumption; therefore $//A(F[\mathbf{K}[G]])$. But $//H \sim F[\mathbf{K}[G]]$ has already been proved. Therefore, using //W, we obtain $//A(H)$ and hence in particular $\sigma/A(H)$ for some σ. This concludes the proof of the lemma.

Appendix. Axiom of choice and continuity

It remains to show that the general axiom of choice can be derived in IA. We proceed rather informally, but in such a way that it is evident that the reasoning given can be formalized in IA. Hence assume $(x)\,(E\xi)\,A(x, \xi)$. From this one easily deduces $(\eta)\,(E\xi)\,A(\eta(0), \xi)$. From the continuity axiom we infer the existence of a continuity function of type (1,1), say τ, satisfying I): $(\xi, \eta)\, E(\tau/\eta, \xi) \supset A(\eta(0)), \xi)$. Let $\varphi(z, y)$ be such that $\varphi(z, y)=z$ is an axiom of IA (there is of course such a constant φ). As consequence of I) we obtain II): $(z, \xi)\,(E(\tau/\lambda y\varphi(z, y), \xi) \supset A(z, \xi))$, where $\lambda y\varphi(z, y)\,(0)=z$ has been used. Clearly we can prove III): $(z, x)\,(Ev)\,(v \subseteq \lambda y\varphi(z, y) \wedge \wedge\, \tau(v, x) \neq 0)$. Put $\langle x, y\rangle = \frac{1}{2}((x+y)^2+3x+y)$ and let $p_1(x)$, $p_2(x)$ be those prim. rec. functions which satisfy $\langle p_1(z), p_2(z)\rangle = z$, $p_1(\langle x, y\rangle)=x$, $p_2(\langle x, y\rangle)=y$. With the aid of these we derive from III) the formula IV): $(u)\,(Ev)\,(v \subseteq \lambda y\varphi(p_1(u), y) \wedge \tau(v, p_2(u)) \neq 0)$.

Application of the axiom of choice for quantifierfree formulas yields the existence of a function ν such that V) holds: $(u)(\nu(u) \subseteq \lambda y \varphi(p_1(u), y) \wedge \tau(\nu(u), p_2(u)) \neq 0)$. By going back from u to pairs we get statement VI): $(z, x)(\nu(z, x) \subseteq \lambda y \varphi(z, y) \wedge \tau(\nu(z, x), x) \neq 0)$. From the continuity property one easily derives the following statement VII): if $\nu \subseteq \lambda y \varphi(z, y)$ and $\tau(v, x) \neq 0$ then $\tau(v, x) = \tau(\nu(z, x), x)$. Hence we obtain VIII): $(v, x)(v \subseteq \lambda y \varphi(z, y) \wedge \tau(v, x) \neq 0 \supset \tau(v, x) = \lambda y(\tau(\nu(z, y), y) - 1)(x) + 1)$. Therefore we can derive from the statement II) and VIII) $A(z, \lambda y(\tau(\nu(z, y), y) - 1))$. From this and a little bit of intuitionistic predicate calculus we finally obtain $(E\xi)(z) A(z, \lambda y \xi(z, y))$ what concludes the proof.

Conclusion

A. As noted in chapter V, the model presented in this paper is so to speak the abstract part of a more elaborate model (the "constructive" version), in which everything is codified by Goedel numbers; continuity functions in particular are then restricted to recursive ones. Actually, the author started with the investigation of this constructive version. However it quickly turned out that the main difficulties in proving the necessary lemmas were rather of an abstract, set theoretic nature. Once this abstract side of the problem was understood the application of the fixed point theorem became rather a question of routine. It was recognized that the sum of the abstract considerations formed a selfcontained totality which could be presented in closed form without any reference to recursiveness. This is one reason why we did present the abstract part of the full constructive model separately. The other reason is, that the part of the proof, which is concerned with coding everything by Goedel numbers is rather long and requires quite a number of applications of the fixed point theorem. The interest in fixed point techniques stems from the fact that it yields further interesting results. Among these we mention one: Kleene-Vesleys system is consistent with Churches thesis and Troelstras axiom. The full constructive model together with applications will be presented in another paper.

B. One might wonder, what kind of model we would get by working out definition D1 in the same way as definition D2 has been worked out in chapter IV. We will just mention the result. To this end let IA_0 be the system which differs from IA in the following points: 1) it contains only those axioms of continuity CT(A) for which A has prenex normal form, 2) it contains all formulas of the form $(x)\neg\neg A(x) \supset \supset \neg\neg(x) A(x)$ (x a bound individual variable). By working out D1 we get a model of IA_0. The technique of proof is quite the same as that one presented in this paper. There is also a constructive version of this model, in which everything is coded in a suitable way by Goedelnumbers of certain recursive functions. In this constructive version, Churches thesis is satisfied. Troelstras axiom Tr(A) holds, provided A has prenex normal form. The author does not know positively whether this model is really different from that one elaborated in this paper; however it seems highly probable that this is the case. This problem is under investigation.

C. Another question arises, namely whether the notion "provability" can be built into our model in order to obtain a notion "provable and realizable" and such that theorems like "if $\vdash A \vee B$ then $\vdash A$ or $\vdash B$" etc. can be reproved for IA (or rather a constructive variant of it). This is indeed the case. However a discussion of this point lies outside the scope of this work and will be postponed to the subsequent paper mentioned above.

D. From Dr. Troelstra the author learned that Joane Moschowakis has found another realizability notion in which Churches thesis and a new axiom, called "Vesleys principle", are satisfied. He also pointed out that his own axiom Tr(A) contradicts Vesleys principle. In virtue of our discussion (part A above) it seems that her model and ours differ in some essential point.

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