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Non-Simply-Connected Surgery and Some Results in Low Dimensional Topology

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§ 1. Introduction

In [27] we derived a Künneth formula for Wall's surgery obstruction groups $L_n(G)$. In particular, the inclusion induces an isomorphism $L_6(e) \to L_6(\mathbf{Z})$, $\mathbf{Z}$ the integers. It follows that non-framed surgery is possible in an orientable six-dimensional situation with fundamental group $\mathbf{Z}$. In § 6 of [27], [29], and [19] we derived some consequences of this fact for the classification of non-simply-connectred five-manifolds and related questions.

This paper continues the application of surgery to low dimensional manifolds. For example, we will prove a result somewhat more general than the following:

THEOREM 1.1. *Let M be a closed, connected, orientable, smooth 5-manifold with $\pi_1 M = \mathbf{Z}$. Suppose M is smoothly fibered over a circle with connected fibre. Let Q be a simply connected, closed, smooth 5-manifold. Then any closed smooth manifold of the homotopy type of $M \# Q$ is diffeomorphic to $M \# Q$.*

For the special case $M = N \times S^1$, this generalizes the result of Novikov [22] and Wall [35] that homotopy equivalent simply-connected closed smooth four manifolds are h-cobordant. In fact, we use an idea from Novikov's proof of this fact to help prove 1.1.

As a consequence of the proof of 1.1, we also have a splitting theorem for fibered 5-manifolds. Analogous theorems have been proven in higher dimensions by Farrell and W.-C. Hsiang [10], for more general fundamental groups.

THEOREM 1.2. *Let M be as 1.1, and let $N \subseteq M$ be a fibre. Let $h: K \to M$ be a homotopy equivalence of closed smooth manifolds. Then $\exists\, f$ homotopic to h, transverse to N, so that $f^{-1}(N)$ is diffeomorphic to N and $f: (K, f^{-1}N) \to (M, N)$ is a homotopy equivalence of pairs.*

Of course, in this theorem K is actually diffeomorphic to M. The analogous result is true in the piecewise linear case. (Theorem 1.1 holds in the P.L. case automatically because 5-dimensional P.L. manifolds are always smoothable).

Theorem 1.1 also sheds some light in the fibering problem. Let us say M, a closed 5-manifold with $\pi_1 M = \mathbf{Z}$, is *quasi-fibered over a circle* if there is a smooth (simply-connected) submanifold N of codimension one, N, so that the sequences

$$0 \to \pi_i N \to \pi_i M \xrightarrow{g_*} \pi_i S^1 \to 0$$

are exact where $g: M \to S^1$ represents any generator of $H^1(M; \mathbf{Z})$. The (quasi-)*fibering conjecture for 5-manifolds* asserts that any closed five manifold with $\pi_1 = \mathbf{Z}$, whose universal cover has the homotopy type of a finite complex, is quasi-fibred over a circle. The *smoothing conjecture for simply-connected finite Poincaré complexes* in dimension four asserts that if X is a closed Poincaré 4-complex, finite, ξ a vector bundle over X with spherical Thom class and $\bar{L}(\xi) = \text{index } X$, $\bar{L}(\xi) = L_1(p_1(\xi^{-1}))$ the dual Hirzebruch class, then $\exists$ a closed manifold N and a homotopy equivalence $h.$ $N \to X$ so that $h^*\xi$ is equivalent to the normal bundle of N.

THEOREM 1.3. *The quasi-fibering conjecture for five manifolds* (with $\pi_1 = \mathbf{Z}$) *is equivalent to the smoothing conjecture for simply connected Poincaré complexes in dimension four.*

Assuming the fibering conjecture, the smoothing conjecture follows by smoothing $X \times S^1$ up to homotopy and fibering. We use a stronger version of Theorem 1.1 to prove the converse, thus reducing the fibering problem to an abstract surgery problem rather than a problem in codimension one.

As a final application of surgery we will show the following (I = unit interval):

THEOREM 1.4. *Let* $(W, \partial_- W, \partial_+ W)$ *be an orientable h-cobordism with* dim $W = 5$. *Say* $\pi_1 W = \mathbf{Z}$. *Assume* $\exists$ *a retraction* $r: W \to \partial_- W$ *so that* $r \mid \partial_+ W: \partial_+ W \to \partial_- W$ is *a diffeomorphism. Then* $\exists$ *a diffeomorphism* $\varphi: W \to \partial_- W \times I$ *with*

$$\varphi \mid \partial W = (r \mid \partial_- W, 0) \cup (r \mid \partial_+ W, 1).$$

As a corollary, every diffeomorphism of a closed, orientable four-manifold with fundamental group $\mathbf{Z}$, homotopic to the identity, is psuedo-isotopic to the identity. This was already known for $S^1 \times S^3$ [14].

The simply-connected analogue of 1.4 is due to Barden. His proof has never appeared. As part of the proof of 1.4, we also prove the simply-connected analogue. The reader who is primarily interested in 1.4 can skip directly to the final section.

For larger fundamental groups, everything is much harder. For example, Theorem 1.1 is false for $S^3 \times T^2$ [27], and whether 1.2 or 1.3 hold for $S^3 \times T^2$ seems to be a very deep question.

§ 2. Browder-Novikov Theory

Let $(W, \partial W)$ be a smooth connected manifold with (possibly empty) boundary. We assume orientability of W, though for this section it is not necessary. By $hS(W, \partial W)$ we denote the equivalence classes of simple* homotopy equivalences

* In all cases of concern to us here, every homotopy equivalence is simple.

$h:(K, \partial K) \to (W, \partial W)$

where K is a smooth manifold and $h \mid \partial K: \partial K \to \partial W$ a diffeomorphism; h is equivalent to $h':(K', \partial K') \to (W, \partial W)$ if there is a diffeomorphism $\varphi:(K, \partial K) \to (K', \partial K')$ with $h' \circ \varphi$ homotopic to h relative ∂K.

Given h, let g be a homotopy inverse with $g \mid \partial W = (h \mid \partial K)^{-1}$. Let $\bar{g}:(W, \partial W) \to (K, \partial K) \times \mathbf{R}^m$, m large, be an embedding homotopic to $(g, 0)$ with $\bar{g} \mid \partial W = (g, 0)$. Let v be the normal bundle of $\bar{g}$; $\bar{g} \mid \partial W$ determines a natural trivialization of $v \mid \partial W$. By infinite repetition of the s-cobordism theorem, or by engulfing, there is a diffeomorphism

$c: E(v) \to K \times \mathbf{R}^m$

extending $\bar{g} \cup [(g \mid \partial W) \times id_{\mathbf{R}^m}]$. Here $E(v)$ denotes the total space of v. The composite $(h \times 1) \circ c$ is a fibre homotopy trivialization of the bundle v which is a linear bundle map of $v \mid \partial W$. Hence it defines a homotopy class $\eta(h)$ in $[W/\partial W; G/O]$ of maps of $W/\partial W$ into G/O, the classifying space for stable fibre homotopy trivializations of vector bundles. The invariant $\eta(h)$ is well-defined and depends only upon the class of h in $hS(W, \partial W)$. This is Sullivan's definition of the normal invariant [33].

We will need the original definition of Browder (see [3]) and Novikov [22] only in the case of tangential homotopy equivalences of closed manifolds. Let M^n be a closed, oriented manifold, and let ν^m be a high dimensional (i.e. $m > n+1$) normal bundle of M, i.e. the normal bundle of an embedding of M in S^{n+m}. For $m > n+1$, any two such embeddings are isotopic via an ambient isotopy that restricts to a bundle map of normal bundles. Let $D(\nu)$ $(S(\nu))$ be the associated disk (sphere) bundle, and let $T(\nu) = D(\nu)/S(\nu)$, the Thom space of ν. The orientation of M and of S^{m+n} determine an orientation of ν and so a Thom isomorphism $\psi: H_i(M) \to \tilde{H}_{i+m}(T(\nu))$. Let $[M]$ be the orientation class of M, and let $\bar{A}(M) = H^{-1}(\psi[M])$, $H: \pi_{m+n}(T(\nu)) \to \tilde{H}_{n+m}(T(\nu))$ the Hurewicz homomorphism.

An orientation preserving bundle map of ν to itself over the identity induces a map of $T(\nu)$ with itself which, by naturality of the Thom isomorphism, preserves the Thom class $\psi[M]$. Hence the equivalences of ν with itself, $\mathrm{Aut}(\nu)$, acts on $\bar{A}(M)$. Let $A(M)$ be the orbit set. By stability, $A(M)$ is really independent of the high dimension m of the fibre of ν.

Note also that since we are in the stable range, $\mathrm{Aut}(\nu) = [M; SO(m)] = [M; SO]$. Given $\xi \in \mathrm{Aut}(\nu)$, $\xi \oplus \mathrm{id}: \nu \oplus \nu^{-1} \to \nu \oplus \nu^{-1}$ is an automorphism of the trivial bundle and so determines an element of $[M; SO]$; this correspondence is bijective.

Given an embedding of M in S^{n+m}, the quotient map $D(\nu) \to T(\nu)$ extends to a map $S^{n+m} \to T(\nu)$ which carries $S^{n+m} - D(\nu)$ to a point; this defines a class $1_{n+m}(M)$ (or just 1_{n+m}) in $A(M)$. Suppose $h: K \to M$ is a tangential homotopy equivalence that preserves orientations, and let $b: \nu^m(K) \to \nu$ be a bundle map of normal bundles of fibre dimension m covering h. A map on Thom spaces, $T(b)$, is induced by b, and we define

$$\theta(h) = T(b)_* 1_{m+n}(K).$$

It is easy to see that $\theta(h) \in A(M)$ depends only upon the class in $hS(M)$ of h. (In the general definition of normal invariants one has to consider bundles over M other than the normal bundle.) So if $thS(M) \subset hS(M)$ denotes those classes which are tangential, then we have $\theta: thS(M) \to A(M)$.

Let $\bar{1}_{n+m}$ be a representative of 1_{n+m} in $\bar{A}(M)$. Then, according to Wall's refinement [39] of Spivak's uniqueness result for normal fibrations of Poincaré complexes, the map $\xi \to T(\xi)_* \bar{1}_{n+m}$ induces a bijection of isotopy classes of orientation-preserving fibre homotopy equivalences ξ of v with itself (over the identity) and $\bar{A}(M)$. (Basically this is just a consequence of the fact that $\sum^m M_+$ is the Spanier-Whitehead dual of $T(v^m)$.)

On the other hand, the isotopy classes of fibre homotopy equivalences of v with itself are in one-one correspondence with $[M; SG(m)] = [M; SG]$, $SG(m)$ the space of degree one maps of S^{m-1} to itself. The correspondence is exactly analogous to the one that gives $\mathrm{Aut}(v) = [M; SO]$, and so we get a map

$$\zeta: A(M) \to [M; G/O] = [M; SG/SO]$$

which is one-one and whose image is the elements that lift to $[M; G]$.

The following is part of a result that is well-known. It perhaps is buried in [38], but otherwise it does not seem to have appeared in the literature.

PROPOSITION 2.1. *The following diagram commutes:*

$$\begin{array}{ccc} thS(M) & \xrightarrow{\theta} & A(M) \\ \cap & & \downarrow \zeta \\ hS(M) & \xrightarrow{\eta} & [M; G/0]. \end{array}$$

Proof. Let $h: K \to M$ be a tangential homotopy equivalence. We will show that $\zeta^{-1}\eta(h) = \theta(h)$. Let $c: M \times \mathbf{R}^k \to K \times \mathbf{R}^k$ be a diffeomorphism with $c \mid M \times 0$ homotopic to $(g, 0)$, g a homotopy inverse to h, so that $\eta(h)$ is represented by $(h \times 1) \circ c$. We can also suppose $c(M \times \frac{1}{2}D^k) \supset K \times 0$.

Let $m > k + n + l$ and let $v^m = v^{m-k} \oplus \varepsilon^k$, v^{m-k} the normal bundle of M in S^{m+n-k} and ε^k trivial; v^m is the normal bundle of M in S^{m+n}. Then $\xi = (\mathrm{id}) \oplus (h \times 1) c: v^m \to v^m$ is a fibre-homotopy equivalence of v^m with itself, and $\zeta^{-1}\eta(h) = T(\xi)_* 1_{n+m}$.

Fix an embedding $M \subset S^{n+m}$ and let $f: S^{n+m} \to T(v^m)$ be the natural extension of the quotient map $D(v)$ to $T(v)$ by the trivial map on $S^{n+m} - D(v)$. Then $T(\xi) \circ f$ is transverse to $M \subset T(v^m)$, $(T(\xi) \circ f)^{-1} M = K$, and $T(\xi) \circ f \mid K$ is homotopic to h. It now follows as in [3, II.2.13] and from the definition that $\theta(h)$ is represented by $T(\xi) \circ f$.

Returning to Sullivan's definition, let $(W, \partial W)$ be a connected, orientable manifold pair with $n = \dim W \geqslant 5$. Then $\exists$ an action of the Wall surgery group $L_{n+1}(\pi_1 W)$ on $hS(W, \partial W)$ so that $\eta(x) = \eta(y)$ iff x and y are in the same orbit. This result is a simple consequence of the realization Theorem [38, 5.8 and 6.5] (also [27, 1.1]) for surgery obstructions and the definitions. There is also an exact sequence of pointed sets $hS(W, \partial W) \to [W/\partial W; G/O] \to L_n(\pi_1 W)$, but we will not use this here. These formulations are due basically to Sullivan. See [3], [33] and [38] for more details.

PROPOSITION 2.2 *Let $f: K \to P$ and $g: P \to M$ be (simple) homotopy equivalences of closed manifolds. Then in $[M; G/O]$,*

$$\eta(g \circ f) = \eta(g) + (g^{-1})^* \eta(f),$$

g^{-1} any homotopy inverse for g.

We omit the proof of this proposition. We use it only when $\eta(g) = 0$, a case in which the verification is quite simple.

PROPOSITION 2.3. *Let $f: K \to M$ be a (simple) homotopy equivalence of closed manifolds. Let $N \subseteq M$ be a submanifold of codimension one, let f be transverse to N, and suppose $h = f \mid f^{-1}N: f^{-1} \to N$ is also a (simple) homotopy equivalence. Let $i: N \subset M$ be the inclusion. Then $\eta(h) = i^* \eta(f)$.*

The proof of this proposition is straightforward using the definitions, and so omitted.

Suppose v^m and ξ^m are vector bundles over pointed spaces X and Y respectively. Then, by identifying fibres over a point, we have $v \vee \xi$ defined over the one-point union $X \vee Y$. The inclusions of the base points induce inclusion of S^m in $T(v)$ and $T(\xi)$, and the following is obvious.

PROPOSITION 2.4. *$T(v \vee \xi) = T(v) \cup_{S^m} T(\xi)$, the space obtained from the disjoint union of $T(v)$ and $T(\xi)$ by identifying the included copies of S^m.*

We conclude this section by defining, following Novikov [22], a twisted suspension. Let ξ^m be an oriented vector bundle over X. Let $\pi_n(X, \xi)$ be the subgroup of classes in $\pi_n(X)$ which induce from ξ the trivial bundle over S^n. Let $\kappa: S^m \to T(\xi)$ be induced by the inclusion of the basepoint. Note that $\kappa(S^m)$ is invariant under the action of $[M; SO(m)]$.

Let $\mu: S^n \to X$ represent an element of $\pi_n(X, \xi)$, and let $\hat{\mu}: \varepsilon^m \to \xi^m$ cover μ. Let ε^m be oriented so that $\hat{\mu}$ is orientation preserving. Then the standard orientation of S^n determines $x \in H_{n+m}(T(\varepsilon^m)) = \mathbf{Z}$, by the Thom isomorphism. Let $y \in \pi_{n+m}(T(\varepsilon^m))$ be any class whose Hurewicz image is x. This exists because $T(\varepsilon^m) = S^{n+m} \vee S^n$. The class $T(\hat{\mu})_* y$ in $\pi_{n+m}(T(\xi))/\kappa_*(\pi_{n+m}(S^m))$ depends only upon the class μ in $\pi_n(X, \xi)$. We denote this element by $S_\xi([\mu])$; this defines

$$S_\xi : \pi_n(X, \xi) \to \pi_{n+m}(T(\xi))/\text{Image}\,\kappa_* .$$

This map is natural with respect to bundle maps. If ξ is trivial, it coincides with the usual suspension; in particular S_ξ is a homomorphism.

Let M^n be an oriented, closed, connected manifold. Let ν^m be its normal bundle, m large. Assume $\pi_1 M$ acts trivially on $\pi_n M$; for example, let M be simply-connected. The orientation determines an isomorphism

$$H^n(M; \pi_n M) = \pi_n(M).$$

By $\pi_n^\nu M$ we denote those elements of $\pi_n M$ which pull back the normal bundle ν trivially and have zero Hurewicz image. Given $x \in \pi_n^\nu M$, there is a unique homotopy class of maps $g: M \to M$ so that g is the identity outside a given disk $D^n \subseteq M$ and so that the cohomology class of the difference co-cycle of f and the identity with respect to the identity homotopy on the $(n-1)$-skeleton is just x. (Recall that $M - D^n$ deforms to the $(n-1)$-skeleton.)

This defines a map $\omega: \pi_n^\nu(M) \to \pi^+(M, M)$, the last being homotopy classes of orientation preserving tangential homotopy equivalences of M with itself. There is a natural map of $\pi^+(M, M)$ into $hS(M)$. Moreover, $\pi^+(M, M)$ acts upon $A(M)$ by induced maps of covering maps; i.e. if $h: M \to M$ represents an element of $\pi^+(M, M)$, let $\hat{h}: \nu^m \to \nu^m$ cover h and let $[h] \cdot \alpha = T(\hat{h})_* \alpha$ in $A(M)$, where $[h]$ denotes the class of h.

Let $M^{(n-1)}$ be the $(n-1)$-skeleton of M. Then inclusion induces a monomorphism $\pi_{n+m}(T(\nu \mid M^{(n-1)})) \subseteq \pi_{n+m}(T(\nu))$, as $T(\nu)$ is, up to homotopy, $S^{n+m} \vee T(\nu \mid M^{(n-1)})$. If $\alpha \in \pi_{n+m}(T(\nu \mid M^{n-1}))$ and $\bar{1}_{n+m} \in \pi_{n+m}(T(\nu))$ is an element in the orbit $1_{n+m} \in A(M)$, then $\bar{1}_{n+m} + \alpha \in \bar{A}(M)$. We write $1_{n+m} + \alpha$ for the orbit of this element in $A(M)$.

PROPOSITION 2.5. (Novikov [22].) *Let* $\alpha \in \pi_{n+m}(T(\nu \mid M^{(n-1)}))$, *and let* $\gamma \in \pi_n^\nu(M)$. *Then*

$$(\omega(\gamma))(1_{n+m} + \alpha) \equiv 1_{n+m} + \alpha + S_\nu(\gamma), \text{ mod Image}\,\kappa_* .$$

COROLLARY 2.6. *Modulo the image of* κ_*,

$$\theta(\omega(\gamma)) = 1_{n+m} + S_\nu(\gamma).$$

If $g: M \to M'$ is a tangential homotopy equivalence, and $[h] \in \pi^+(M, M)$, let $[h]^g = [ghg^{-1}]$, g^{-1} any homotopy inverse g. The following is obvious:

PROPOSITION 2.7. $[\omega(\gamma)]^g = \omega(g_*\gamma)$, $\gamma \in \pi_n^\nu(M)$.

§ 3. Simply-Connected Four-Manifolds

In this section we study the Thom space of a simply-connected four-manifold N. Let ν^m be the normal bundle, m large. Since $[N; SO(m)]=0$ and $\pi_{m+4}(S^m)=0$, we have $A(N)=\bar{A}(N)$ and

$$S_\nu: \pi_4^\nu(N) \to \pi_{m+4}(T(\nu)).$$

Since $\pi_4(SO(m))=0$, $\pi_n^\nu(N)$ is just the homotopy classes of degree zero.

Let $g: N \to N$ be a homotopy equivalence. Then g is tangential, by the argument of [27, Theorem 6.1]. The map $T(g)_*: \pi_{m+4}(T(\nu)) \to \pi_{m+4}(T(\nu))$, the map induced by a bundle map covering g, is independent of the choice of covering map.

THEOREM 3.1. *Let $\alpha \in \pi_{m+4}(T(\nu \mid N^{(2)})) \subseteq \pi_{m+4}(T(\nu))$, $N^{(2)}$ the two-skeleton of N. Suppose $T(g)_*\alpha=\alpha$. Then there is a class $\gamma \in \pi_4^\nu(N)$ such that $S_\nu(\gamma)=\alpha$ and $g_*(\gamma)=\gamma$.*

For $g=id$, this result is stated in [22] and is not hard to prove. Together with Corollary 2.6, this case implies, since $T(\nu)=S^{m+4} \vee T(\nu \mid N^{(2)})$, the following:

COROLLARY 3.2. *Every element $\xi \in [N; G/O]$ that lifts to $[N; G]$ is the normal invariant of a homotopy equivalence of N with itself.*

Since $L_5(e)=0$, this implies, as observed by Novikov, that any four-manifold of the homotopy type of N is h-cobordant to N.

Proof of Theorem 3.1. We suppose first that $W^2(N)=0$; i.e. N is almost parallelizable. Up to homotopy, $N^{(2)}=S_1^{(2)} \vee \cdots \vee S_k^{(2)}$, $T(\nu \mid N^{(2)})=\Sigma^m N_+^{(2)}= =S^m \vee S_1^{m+2} \vee \cdots \vee S_k^{m+2}$, and $S_\nu \mid \pi_4^\nu(N)$ is just the suspension

$$\Sigma^m: \pi_4(S_1^2 \vee \cdots \vee S_k^2) \to \pi_{m+4}(S_1^{m+2} \vee \cdots \vee S_k^{m+2})$$

Let $\mu_i \in \pi_2(S_1^2 \vee \ldots \vee S_k^2)$. be the class carried by the i^{th} sphere S_i^2 with standard orientation. Let $\eta \in \pi_3(S^2)$ be the Hopf class and let $\Sigma\eta \in \pi_4(S^3)$ be its suspension. Let, for $l \leqslant i \leqslant k$, $\gamma_i=\mu_i \circ \eta \circ \Sigma\eta$; i.e. γ_i is the composite

$$S^4 \xrightarrow{\Sigma\eta} S^3 \xrightarrow{\eta} S^2 \xrightarrow{\mu_i} N^{(2)}.$$

LEMMA 3.3. *Σ^m carries the subgroup of $\pi_4(S_l^2 \vee \cdots \vee S_k^2)$ generated by the γ_i isomorphically onto $\pi_{m+4}(S_1^{m+2} \vee \cdots \vee S_k^{m+2})$.*

Proof. Each γ_i has order two. By homotopy excision [2], in the stable range, $\pi_{m+4}(S_1^{m+2} \vee \cdots \vee S_k^{m+2})=\sum_{i=1}^k \pi_{m+4}(S_i^{m+2})$. Moreover, the suspension $\pi_4(S^2) \to \pi_{m+4}(S^{m+2})=\mathbf{Z}_2$ is an isomorphism, and $\pi_4(S^2)$ is generated by $(\Sigma\eta)\circ\eta$. Hence Σ^m carries the subgroup in question onto the isomorphic group $\pi_{m+4}(S_l^{m+2} \vee \cdots \vee S_k^{m+2})$, and so is an isomorphism of these groups.

Now suppose $\alpha\in\pi_{m+4}(S_1^{m+2}\vee\cdots\vee S_k^{m+2})$, with $T(g)_*\alpha=\alpha$. Let $\beta=\sum_{i=1}^k\lambda_i\gamma_i$, with $\lambda_i=0$ or 1, be such that $\Sigma^m\beta=\alpha$. Then $\Sigma^m(g_*\beta)=\alpha$ also. We may suppose $g(N^{(2)})\subseteq N^{(2)}$, and we write g again for its restriction to $N^{(2)}$.

In general, $(\varrho+\sigma)\circ\eta=\varrho\circ\eta+\sigma\circ\eta+[\varrho,\sigma]$, where $[\varrho,\sigma]$ denotes the Whitehead product of ϱ and σ, since η has Hopf invariant one. On the other hand, composition with the suspension $\Sigma\eta$ is a homomorphism. So if

$$g_*\mu_i=\sum_j\xi_{ij}\mu_j,\ \text{then}$$

$$g_*\gamma_i=\sum_j\xi_{ij}\gamma_j=\sum_{j,k}\xi_{ij}\xi_{ik}[\mu_j,\mu_k]\circ\Sigma\eta,\ \text{and so}$$

$$g_*\beta=\sum_{i,j}\lambda_i\xi_{ij}\gamma_j+\sum_{i,j,k}\lambda_i\xi_{ij}\xi_{ik}([\mu_j,\mu_k]\circ\Sigma\eta).$$

Suspension annihilates Whitehead products, so

$$\Sigma^m(g_*\beta)=\sum_{i,j}\lambda_i\xi_{ij}S(\gamma_i)=\Sigma^m(\beta)=\sum_i\lambda_iS(\gamma_i).$$

Hence $\lambda_j\equiv\sum_i\lambda_i\xi_{ij}$ (mod 2). Thus if $\delta=\sum_i\lambda_i\mu_i$, then $g_*(\delta)=\delta+2\varepsilon$ some ε. Hence

$$g_*(\delta\circ\eta)=\delta\circ\eta+2(\varepsilon\circ\eta)+[\varepsilon,\varepsilon]+2[\varepsilon,\delta].$$

Since, $[\varepsilon,\varepsilon]$ is even, i.e. divisible by two, we can write

$$g_*(\delta\circ\eta)=\delta\circ\eta+2\varrho$$

Since composition with $\Sigma\eta$ is a homomorphism and since $\Sigma\eta$ has order two,

$$g_*(\delta\circ\eta\circ\Sigma\eta)=\delta\circ\eta\circ\Sigma\eta.$$

Also

$$\Sigma^m(\delta\circ\eta\circ\Sigma\eta)=(\Sigma^m\delta)\circ\Sigma^m(\eta\circ\Sigma\eta)=(\sum_i\lambda_i\Sigma^m\mu_i)\circ\Sigma^m(\eta\circ\Sigma\eta)$$
$$=\sum_i\lambda_i\Sigma^m(\mu_i\circ\eta\circ\Sigma\eta)=\sum_i\lambda_i\Sigma^m(\gamma_i)=\Sigma^m\beta=\gamma,$$

since composition with a suspension is a homomorphism. So if $\gamma=\delta\circ\eta\circ\Sigma\eta$, then $g_*(\gamma)=\gamma$ and $\Sigma^m(\gamma)=\alpha$.

Now suppose $W^2(N)\neq0$. The second Stiefel-Whitney class can be viewed as a homomorphism $W^2(N):H_2(N,\mathbf{Z})\to\mathbf{Z}_2$. $H_2(M;\mathbf{Z})$ has a basis over $\mathbf{Z}$ so that $W^2(N)$ vanishes except on the last basis element. So we may write, up to homotopy,

$$N^{(2)}=S_1^2\vee\cdots\vee S_k^2\vee S_{k+1}^2$$

with $\nu^m\mid S_i^2$ trivial for $1\leqslant i\leqslant k$ and non-trivial for $i=k+1$. It is easy to see that up to homotopy, $T(\nu\mid S_{k+1}^2)$ is just $X=S^m\cup_\varphi D^{m+2}$, $\varphi:S^{m+1}\to S^m$ non-trivial. For $T(\nu\mid S_{k+1}^2)$

has two cells, and if φ were trivial $\nu \mid S^2_{k+1}$ would be fibre-homotopy tirvial by Spivak [31], and so trivial because $\pi_1(o)=\pi_1(G)$. Hence by 2.4,

$$T(\nu \mid N^{(2)}) = S_1^{m+2} \vee \cdots \vee S_k^{m+2} \vee X.$$

LEMMA 3.4. $\pi_{m+4}(X)=0$. *Hence the inclusion induces an isomorphism of* $\pi_{m+4}(T(\nu \mid N^{(2)}))$ *with* $\pi_{m+4}(S_1^{m+2} \vee \cdots \vee S_k^{m+2})$.

Proof. By homotopy excision the stable range, the characteristic map $(D^{m+2}, S^{m+1}) \to (X, S^m)$ induces an isomorphism of $(m+4)$th homotopy groups. So we have an exact sequence

$$0 = \pi_{m+4}(S^m) \to \pi_{m+4}(X) \to \pi_{m+4}(D^{m+2}, S^{m+1}) \xrightarrow{\partial} \pi_{m+3}(S^m).$$

The domain of ∂ is $\mathbf{Z}_2$, and the image of a generator is represented by the composite $S^{m+3} \to S^{m+2} \to S^{m+1} \to S^m$ of the non-trivial maps. This is just the suspension of the Blakers-Massey element in $\pi_6(S^3)$ and so is non-trivial, by [26]. Hence $\pi_{m+4}(X)=0$.

The second statement follows from the first by homotopy excision.

As before, let $\mu_i \in \pi_2(N^{(2)})$ be the class carried by the i^{th} sphere, $1 \leqslant i \leqslant k+1$. Let $\gamma_i = \mu_i \circ \eta \circ \Sigma\eta$. The restriction of $S\nu$ to $\pi_4(S_1^2 \vee \cdots \vee S_k^2)$, a direct summand of $\pi_4(N^{(2)})$, is the suspension homomorphism Σ^m, hence by 3.2 it carries the subgroup generated by the γ_i, $1 \leqslant i \leqslant k$, isomorphically onto $\pi_{m+4}(T(\nu \mid N^{(2)}))$.

Again let $g_*(\mu_i) = \Sigma_j \xi_{ij} \mu_j$. Since $W^2(N) \circ g_* = W^2(N)$, and since $H_2(N) = H_2(N^{(2)}) = \pi_2(N^{(2)}) = \pi_2(N)$, we have $\xi_{i,k+1} \equiv 0 \pmod 2$, for $1 \leqslant i \leqslant k$.

Suppose $\alpha \in \pi_{m+4}(T(\nu \mid N^{(2)}))$, with $T(g)_* \alpha = \alpha$. Let $\beta = \sum_{i=1}^k \lambda_i \gamma_i$, $\lambda_i = 0$ or 1, be such that $\Sigma^m \beta = \alpha$. Again $S_\nu(g_* \beta) = \alpha$.

Let $\sigma_i = \sum_{j=1}^k \xi_{ij} \mu_j$. Then

$$g_* \gamma_i = (\sigma_i \circ \eta + (\xi_{i,k+1} \mu_{k+1}) \circ \eta + [\sigma_i, \xi_{i,k+1} \mu_{k+1}]) \circ \Sigma\eta.$$

The third term in parentheses is $\xi_{i,k+1}[\sigma_i, \mu_{k+1}]$; since $\xi_{i,k+1}$ is even, this vanishes under composition with $\Sigma\eta$. Similarly, $(\xi_{i,k+1}\mu_{k+1}) \circ \eta = \xi_{i,+1}(\mu_{k+1} \circ \eta) + \tau[\mu_{k+1}, \mu_{k+1}]$ also vanishes under composition with $\Sigma\eta$. So

$$g_* \gamma_i = \sigma_i \circ \eta \circ \Sigma\eta \quad \text{and}$$
$$g_* \beta = (\sigma_1 + \cdots + \sigma_k) \circ \eta \circ \Sigma\eta.$$

Now everything lives over $S_1^2 \vee \cdots \vee S_k^2$, where ν^m is trivial, and so the argument for the case $W^2(N)=0$ and the fact that $\xi_{i,k+1} \equiv 0$ (mod 2) imply that $g_*(\delta) = \delta + 2\varepsilon$, $\delta = \lambda_1 \mu_1 + \cdots + \lambda_k \mu_k$. Then it follows again that $g_*(\delta \circ \eta \circ \Sigma\eta) = \delta \circ \eta \circ \Sigma\eta$. By the same argument as for the case $W^2(N)=0$, we also have $S_\nu(\delta \circ \eta \circ \Sigma\eta) = \alpha$. So again, $\gamma = \delta \circ \eta \circ \Sigma\eta$ astisfies the conclusions of 3.1.

§ 4. Split Five-Manifolds With $\pi_1 = \mathbf{Z}$

Let W be a simply-connected 5-manifold with ∂W the disjoint union of closed, simply-connected four manifolds, N and N' say. Let $k: N \to N'$ be a homotopy equivalence with vanishing normal invariant (for example, a diffeomorphism) so that if $i: N \subset W$ and $i': N \subset W$ are inclusions, $i' \circ k \simeq i$. ("$\simeq$" means "is homotopic to.") Let $f: N \to N'$ be a diffeomorphism, and let M be obtained from W by identifying x with $f(x)$. We assume f preserves appropriate orientations, so that M is orientable. Let $\pi: W \to M$ be the quotient projection. Then we also require that $(\pi i)^*: H^2(M; \mathbf{Z}_2) \to H^2(N; \mathbf{Z}_2)$ be monic, or equivalently, that $H^2(W, \partial W; \mathbf{Z}_2) = 0$. It is easy to see that $\pi_1 M = \mathbf{Z}$. We say a manifold M obtained in this way is a *split five-manifold* (with simply-connected fibre). For example, if $M = N \times I$, $k =$ identity, and f is any orientation preserving diffeomorphism, we get the case of a manifold fibered over a circle.

The requirement that k have no normal invariant is somewhat unpleasant. This requirement will always be satisfied, however, if $\exists$ a retraction $r: W \to N$ with $r \mid N'$ a homotopy inverse to k and with $r^*(\tau N)$ stably equivalent to τW, the tangent bundle of W. So, for example, if W is an h-cobordism and $k: N \to N'$ is a homotopy equivalence with $i' \circ k \simeq i$, we will always have $\eta(k) = 0$.

Given any orientable 5-manifold M with $\pi_1 M = \mathbf{Z}$, one can always find a simply-connected four manifold $N \subset M$ with $H_2(N; \mathbf{Z}) \to H_2(M; \mathbf{Z})$ a surjection. Cutting M along N gives a simply-connected manifold W from which M can be recorvered by glueing up the two boundary components. However, it does not seem clear, in general, how to produce the homotopy equivalence k.

THEOREM 4.1. *Let M be a split 5-manifold with simply-connected fibre. Then any closed smooth manifold of the homotopy type of M is diffeomorphic to M.*

Assuming 4.1, let us derive 1.1. According to [27], $\eta: hS(M) \to [M; G/O]$ is monic with image consisting of those elements that lift to $[M; G]$. This is valid for any orientable five-manifold M with fundamental group $\mathbf{Z}$; it follows essentially from an analysis of the Wall groups $L_6(\mathbf{Z})$ and $L_5(\mathbf{Z})$.

Suppose P is a simply-connected closed 5-manifold. Let $h: L \to M \# P$ be a homotopy equivalence, and let $\xi = \eta(h)$. ($M \# P$ denotes the connected sum of M and P.) Then ξ lifts to $\lambda: M \# P \to G$. Since $\pi_4(G) = 0$, the restrictions of λ to M-(disk) and P-(disk) extend to $\lambda_1: M \to G$ and $\lambda_2: M \to G$, respectively. Let ξ_1 and ξ_2 be the images of λ_1 and λ_2, respectively, in $[M; G/O]$ and $[P; G/O]$. Then let $h_1: K \to M$ and $h_2: Q \to P$ be homotopy equivalences with $\eta(h_1) = \xi_1$ and $\eta(h_2) = \xi_2$. Then it is easy to verify that if $F = h_1 \# h_2: K \# Q \to M \# P$, then $\eta(F) = \xi = \eta(h)$. Moreover, by [1] or [22], Q is diffeomorphic to P. So this proves the following "splitting theorem".

PROPOSITION 4.2. *Let M be a closed connected orientable 5-manifold with*

$\pi_1 M = \mathbf{Z}$. Let $h: L \to M \# P$ be a homotopy equivalence, L a closed 5-manifold. Then $\exists$ a closed 5-manifold K, homotopy equivalences $h_1: K \to M$ and $h_2: P \to P$, and a diffeomorphism $\varphi: K \# P \to L$ so that $h \circ \varphi \simeq h_1 \# h_2$.

Clearly 1.1 follows from 4.2 and 4.1. In fact, if M^5 is any split manifold and P^5 is simply-connected, any manifold of the homotopy type of $M \# P$ is diffeomorphic to it.

We turn now to the proof of 4.1. Let M be a split 5-manifold with simply-connected fibre. Let W, N, π, i, i', k, and f be as in the definition of split manifolds. Let ν^m be the normal bundle of N. For any space X, let $CJ(X)$ be the image of $[X; G]$ in $[X; G/O]$.

LEMMA 4.4. *$(\pi i)^*: CJ(M) \to CJ(N)$ is a monomorphism.*

Proof. By [27], the map $\gamma_M: CJ(M) \to H^2(M; \mathbf{Z}_2)$ defined by $\gamma_M(\xi) = \xi^* \iota$, $\iota \in H^2(G/O; \mathbf{Z}_2)$ the non-zero class, is monic. Let $\gamma_N: CJ(B) \to H^2(M; \mathbf{Z}_2)$ be defined similarly. Then the following diagram commutes:

$$\begin{array}{ccc} CJ(M) & \xrightarrow{\gamma_M} & H^2(M; \mathbf{Z}_2) \\ \downarrow{\scriptstyle (\pi i)^*} & & \downarrow{\scriptstyle (\pi i)^*} \\ CJ(N) & \xrightarrow{\gamma_N} & H^2(N; \mathbf{Z}_2) \end{array}$$

Since $(\pi i)^*$ on the right is monic, so is $(\pi i)^*$ on the left.

Remark: This is the only place we use the requirement for a split manifold that $(\pi i)^*: H^2(M; \mathbf{Z}_2) \to H^2(N; \mathbf{Z}_2)$ be monic. It would suffice instead to require only that Lamma 4.4 hold.

Now let $\varrho \in CJ(M)$; write $\varrho: M \to G/O$ for a representative also. We are going to show that ϱ is the normal invariant of a homotopy equivalence of M with itself. Let $\xi = \varrho \pi i$ and $\xi' = \varrho \pi i'$. Then $\xi' k \simeq \xi$ and $\xi' f = \xi$. So if $g = f^{-1} k: N \to N$, then $\xi g = \xi' k \simeq \xi$; i.e. $g^* \xi = \xi$. Hence $(g^{-1})^* \xi = \xi$ in $[N; G/O]$.

By 2.2, $\eta(g) = f^* \eta(k) = 0$. By 3.2, $\xi = \eta(h)$ for some homotopy equivalence $h: N \to N$. So $(g^{-1})^* \xi = \eta(g \circ h) - \eta(g) = \eta(g \circ h)$, by 2.2 again. Thus $\eta(g \circ h) = \eta(h)$. So by 2.1, $\theta(g \circ h) = \theta(h)$. Thus if $\zeta^{-1}(\xi) = 1_{m+4} + \alpha$, we have

$$T(g)_* (1_{m+4} + \alpha) = 1_{m+4} + \alpha.$$

Since $\eta(g) = 0$, $T(g)_* 1_{m+4} = 1_{m+4}$. Hence

$$T(g)_* \alpha = \alpha.$$

Let $\gamma \in \pi_4^\nu(N)$ be such that $S_\nu(\gamma) = \alpha$ and $g_* \gamma = \gamma$; γ exists by Theorem 3.1. By 2.6,

$$\theta(\omega(\gamma)) = 1 + \alpha,$$

and so $\eta(\omega(\gamma))=\xi$. By 2.7, $[\omega(\gamma)]^g=\omega(\gamma)$. Also, $\omega(k_*\gamma)=[\omega(\gamma)]^k$; hence $\omega(k_*\gamma)= =[\omega(\gamma)]^f$.

Let $h:N\to N$ represent $\omega(\gamma)\in\pi^+(N, N)$, with h the identity outside a disk, and let $h'=fhf^{-1}$. Since $i'_*k_*\gamma=i_*\gamma$, $h\cup h'$ extends to $l: W\to W$ which is the identity outside a disk $D^5\subseteq W$ that meets ∂W in two subdisks, one in N and one in N'. We can easily also arrange things so that $l^{-1}(\partial W)=\partial W$ and so that for a given collar $c:\partial W\times[0, 1]\to W$, $l(c(x, t))=c(l(x), t)$.

Let $H:M\to M$ be the map induced by l. Then H is transverse to $(\pi i)(N)$ and $H^{-1}(\pi i(N))=\pi i(N)$. Moreover, $H\pi i=\pi ih$. It is not hard to see that H is a homotopy equivalence. By 2.3, $(\pi i)^*\eta(H)=\eta(h)=\xi=(\pi i)^*\varrho$. Hence by 4.4, $\eta(H)=\varrho$.

Thus every $\varrho\in CJ(M)$ is the normal invariant of a homotopy equivalence of M with itself. By [27, Theorem 6.6], $CJ(M)$ is the image of one-to-one map $\eta:hS(M)\to [M; G/O]$. Thus given any homotopy equivalence $G:K\to M$ of closed manifolds, $\exists H:M\to M$ and a diffeomorphism $\varphi:K\to M$ with $H\varphi\simeq G$. This proves Theorem 4.1.

This proof also proves Theorem 1.2. For if M is fibered, or even just split, with fibre N, then given $G:K\to M$, $H\varphi$ as in the preceding paragraph satisfies the requirements of 1.2.

For the special case $M=N\times S^1$, one can take $H=h\times 1$ in the proof of 4.1.

COROLLARY 4.5. *Every homotopy equivalence of a closed manifold K with $N\times S^1$ is homotopic to one of the form $(h\times 1)\circ\varphi$, where $\varphi:K\to N\times S^1$ is a diffeomorphism and $h:N\to N$ is a homotopy equivalence that is the identity outside a disk.*

We will use this corollary in the next section. Another corollary is the following (compare [36]).

COROLLARY 4.6. *Let N be a simply connected closed four-manifold. Then every automorphism of the bilinear form Q on $H_2(N; \mathbf{Z})$ given by intersection numbers is induced by a diffeomorphism of $S^1\times N$ with itself.*

Proof. Let $H:N\to N$ be a homotopy equivalence of N with itself inducing a given automorphism of the form Q. Then $(H\times 1)\simeq(h\times 1)\circ\varphi$ as in Corollary 4.5. Clearly φ is the required diffeomorphism.

Barden's theorem, the simply-connected version of 1.4, can also be derived from our arguments in this section. However, we will give a much simpler proof in § 6.

§ 5. Fibering Five-Manifolds And Smoothing 4-Complexes

Let M be a closed, connected orientable five-manifold with fundamental group $\mathbf{Z}$. We say M is quasi-fibered over S^1 (with connected fibre) if $\exists$ a closed submanifold $N\subset M$ of codimension one so that the following sequence is exact for $f:M\to S^1$ any map representing a generator of $H^1(M, \mathbf{Z})$:

$$0 \to \pi_i(N) \to \pi_i(M) \xrightarrow{f_*} \pi_i(S^1) \to 0: \quad i \geqslant 0.$$

Clearly N has the homotopy type of $\hat{M}$, the universal cover of M. Fibered manifolds are quasi-fibered and quasi-fibered manifolds are split. Quasi-fibered manifolds will be fibered if the h-cobordism theorem is true in dimension five.

Let M^5 be a given closed orientable manifold with $\pi_1 M = \mathbf{Z}$. Suppose $\hat{M}$ has the homotopy type of a finite complex; equivalently, by [40], suppose the homotopy groups of M are finitely generated. For manifolds of higher dimension with fundamental group $\mathbf{Z}$, this requirement on the universal covering space is sufficient to insure that the manifold fibres over a circle [7]. In particular, the universal cover has the homotopy type of a closed manifold of one less dimension.

In our situation, let v be the normal bundle of M and $p: \hat{M} \to M$ the universal covering map. Then $\xi = p^* v$ is reducible; equivalently the Thom class of ξ is spherical [31]. Furthermore, $\hat{M}$ is a Poincaré complex of formal dimension four and $L_1(\xi^{-1}) = \frac{1}{3} p_1(\xi^{-1}) [\hat{M}]$ a generator of $H_4(\hat{M}, \mathbf{Z})$. All this follows, for example, by fibering $M \times S^4$ and $M \times CP^2$ over a circle and applying Poincaré duality and the Hirzebruch formula [20] to the fibres. In the analogous situation in dimensions $4k$, $k > 1$, we could conclude that $\hat{M}$ had the homotopy type of a smooth manifold [5]. In dimension four, however, it is not known whether this is the case.

THEOREM 5.1. *Let M be a closed, connected, orientable five-manifold with $\pi_1 M = \mathbf{Z}$. Let $\hat{M}$ be the universal covering space of M. Suppose $\hat{M}$ has the homotopy type of a smooth closed four-manifold. Then M is quasi-fibered over a circle.*

Proof. The basic idea of the proof of 5.1 is illustrated by the following special case: suppose $\pi_1 M$ acts trivially on $\pi_i M$ for all i. Then it is not hard to see that M has the homotopy type of $\hat{M} \times S^1$. Hence M also has the homotopy type of $N \times S^1$, N a simply-connected four-manifold. By Theorem 1.1, M is diffeomorphic to $N \times S^1$, and in particular, M is quasi-fibered, indeed fibered, over S^1.

In the general case, let $t: \hat{M} \to \hat{M}$ be a generator of the group of covering transformations of $\hat{M}$. Let $T(t)$ be the *mapping torus* of t; i.e. $T(t)$ is obtained from $\hat{M} \times I$ by identifying $(x, 0)$ with $(tx, 1)$. The composite of the natural projection of $\hat{M} \times I$ to $\hat{M}$ and the covering projection induces a map of $T(t)$ to M. Using the fact that $T(t)$ is a fibre space over S^1 with fibre $\hat{M}$, it is not hard to see that this map induces isomorphisms of homotopy groups. Hence $T(t)$ and M have the same homotopy type.

Let $\lambda: \hat{M} \to N$ be a homotopy equivalence, where N is a simply-connected closed four-manifold. Let $h: N \to N$ be a homotopy equivalence, simplicial with respect to a fixed smooth triangulation of N, so that $h\lambda$ is homotopic to λt. Then

$$\lambda \cup (h\lambda t^{-1}) : \hat{M} \times \partial I \to N \times \partial I$$

extends to a map of $\hat{M} \times I$ to $N \times I$ which agrees with identifications and so induces $H : T(t) \to T(h)$. The universal cover of $T(h)$, for example, consists of infinitely many copies of $N \times I$ laid end to end and glued together by h. Using this description, it is not hard to verify that H induces isomorphisms of homology groups of universal covering spaces, as well as an isomorphism of fundamental groups. Hence H induces isomorphisms of homotopy groups, and so is a homotopy equivalence.

Thus M has the homotopy type of the mapping torus $T(h)$. Suppose h had no normal invariant; i.e. $\eta(h)=0$ in $[N; G/O]$. Then if g is a homotopy inverse to h, $\eta(g)=0$ also, by Proposition 2.2. Hence g is normally cobordant [3] to the indentity; and since $L_5(e)=0$, it follows that $\exists$ an h-cobordism W with $\partial W = N$ and a map $\varphi : W \to N \times I$ with $\varphi(x) = (gx, 0)$ for $x \in \partial_- W$ and $\varphi \mid \partial_+ W : \partial_+ W \to N \times 1$ a diffeomorphism. The map $(h \times 1)(g \times 1)\varphi : \partial_+ W \to N \times 1$ is homotopic to $\varphi \mid \partial_+ W$. Hence by the homotopy extension property, we may find $\psi : W \to N \times I$ with $\psi(x) = (gx, 0)$ for $x \in \partial_- W$ and $\psi(x) = (hg \times 1)\varphi(x)$ for $x \in \partial_+ W$.

Let K be obtained from W by identifying $x \in \partial_- W = N$ with $\varphi^{-1}(x, 1) \in \partial_+ W$. Then ψ agrees with the identifications and so induces a map $G : K \to T(h)$. Again, G is a homotopy equivalence because it induces isomorphisms of fundamental groups and homology groups of universal covering spaces. Thus K and M have the same homotopy type. But K is quasi-fibered. Hence, by Theorem 4.1, K and M are diffeomorphic, and so M is also quasi-fibered.

Thus we would like to show that $\eta(h)=0$. This unfortunately seems in general not to be the case. However, let $\varrho : M \times S^1 \to T(h) \times S^1 = T(h \times id_{S^1})$ be a homotopy equivalence. The inclusion $N = N \times 1 \subseteq N \times I$ induces an inclusion of N in the Poincaré complex $T(h)$. By the splitting theorem of [10], we may suppose $\varrho^{-1}(N \times S^1) = Q$ is a submanifold and

$$\varrho : (M \times S^1, Q) \to (T(h) \times S^1, N \times S^1)$$

is a homotopy equivalence of pairs. The splitting theorem applies because

$$Wh(\mathbf{Z} \oplus \mathbf{Z}) = 0 .$$

By the s-cobordism theorem [15], $M \times S^1$ is diffeomorphic to the mapping torus of a diffeomorphism f of Q with itself. The map ϱ splits to a map

$$\bar{\varrho} : Q \times I \to N \times S^1 \times I \quad \text{with}$$
$$\bar{\varrho}(fx, 1) = ((h \times 1)\pi\varrho(x), 1),$$

$\pi : N \times S^1 \times I \to N \times S^1$ the natural projection. Thus the restriction of $\bar{\varrho}$ to $Q \times 0$ gives a homotopy equivalence $\beta : Q \to N \times S^1$ with $(h \times 1)\beta$ homotopic to βf. (Here $1 = id_{S^1}$.)

By Corollary 4.5, $\beta \simeq (j \times 1)\,\sigma$, where $j: N \to N$ is a homotopy equivalence and $\sigma: Q \to N \times S^1$ is a diffeomorphism. Let k be a homotopy inverse to j; we assume k and j are both simplicial with respect to the triangulation of N. Let $g = khj$. Then $(g \times 1) \simeq \mu = \sigma f \sigma^{-1}$, a diffeomorphism. Hence $\eta(g \times 1) = 0$. The map $[N; G/O] \to [N \times S^1; G/O]$ induced by the natural projection is monic and carries $\eta(g)$ to $\eta(g \times 1)$. Hence $\eta(g) = 0$.

There is a map of $N \times S^1 \times I$ to itself that sends $(x, 0)$ to $(x, 0)$ and $(x, 1)$ to $((g \times 1)\,\mu^{-1}(x), 1)$, $x \in N \times S^1$. This map induces a map $T(\mu) \to T(g \times 1) = T(g) \times S^1$ which is again easily seen to induce isomorphisms of fundamental groups and homology groups of universal covering spaces. Also, $(k \times 1)$ is homotopic to $(h \times 1)(k \times 1)\,\mu^{-1}$, and the homotopy induces a map $T(\mu) \to T(h) \times S^1$ which is also a homotopy equivalence. Thus as $T(h)$ has the homotopy type of M, we get a homotopy equivalence $F: T(g) \times S^1 \to M \times S^1$.

Let α generate $\pi_1(T(g))$, let δ generate $\pi_1 M$, and let γ generate $\pi_1 S^1$. Then $\pi_1(T(g) \times S^1) = \pi_1(T(g)) \times \pi_1(S^1)$, for example, and it follows from the construction that $F_*(0, \gamma) = (0, \pm\gamma)$. After composition with $(id) \times (-1)$ if necessary, we may assume $F_*(0, \gamma) = (0, \gamma)$. We must have $F_*(\alpha, 0) = (\pm\delta, m\gamma)$, $m \in \mathbf{Z}$, since F_* is an isomorphism.

Let $\bar{l}: M \times S^1 \to S^1$ be such that $\bar{l}_*(\delta, 0) = -m\gamma$ and $\bar{l}_*(0, \gamma) = \gamma$. Let $l: M \times S^1 \to M \times S^1$ be $l(x, y) = (x, \bar{l}(x, y))$. Then $(l \circ F)_*(\alpha, 0) = l(\pm\delta, m\gamma) = (\pm\delta, 0)$. Hence $l \circ F$ is a homotopy equivalence that lifts to a homotopy equivalence of the infinite cyclic cover $T(g) \times \mathbf{R}$ of $T(g) \times S^1$ with the cover $M \times \mathbf{R}$ of $M \times S^1$. Thus M has the homotopy type of $T(g)$, $g: N \to N$ a homotopy equivalence with vanishing normal invariant. Now the above argument that we wanted to apply for h goes through for g and proves Theorem 5.1.

Theorem 5.1 shows that the *smoothing conjecture* implies the (quasi-)*fibering conjecture*. To prove the remainder of Theorem 1.3, let us assume the (quasi-)*fibering conjecture* to be valid. Let X be a Poincaré complex, finite, simply-connected, of formal dimension four. Let ξ^k, k large, be a vector bundle over X with spherical Thom class and with $L_1(p_1(\xi^{-1})) = \frac{1}{3}p_1(\xi^{-1}) = \text{Index } X$. Then let $f: S^{k+4} \to T(\xi)$ be a map representing the Thom class in $H_{k+4}(T(\xi); \mathbf{Z})$, transverse regular to the zero-section $X \subset T(\xi)$. Let $N = f^{-1}X$ and let $g = f \mid N: N \to X$. Then g has degree one with respect to suitable orientations, and g is covered by a bundle map of the normal bundle $\nu(N)$ to ξ; equivalently, there is a stable framing F of $\tau(N) \oplus g^*\xi$. The surgery obstruction for (N, g, F) is just

$$s(N, g, F) = \tfrac{1}{8}(\text{Index}\, X - \text{Index}\, M) = 0\,.$$

Unfortunately, this does not allow us to perform surgery to get a homotopy equivalence.

The periodicity theorem [33] for simply-connected surgery obstructions asserts that $s((N, g, F) \times CP^2) = 0$. Indeed, this is clear from the facts that the index of a product is the product of the indices and complex projective space CP^2 has index 1. Hence $s((M, g. F)\times S^1\times CP^2)=0$, and so by the general periodicity theorem for (non-simply-connected) surgery obstructions of [38], $(N, g, F)\times S^1$ is cobordant to (K, h, G) with $h:K\to X\times S^1$ a homotopy equivalence. The *fibering conjecture* asserts that there is a simply-connected closed four-manifold $N\subset K$ with $\pi_iN=\pi_iK$ for $i\geqslant 2$. The composite

$$N\subset K\overset{g}{\to}X\times S^1\overset{\pi}{\to}X,$$

π the natural projection, is a homotopy equivalence which clearly pulls back $\xi^k\oplus\varepsilon^1$ to the $(k+1)$-dimensional normal bundle of N.

§ 6. H-cobordisms of Four-manifolds with $\pi_1=\mathbf{Z}$

In this section we prove Theorem 1.4. To make the basic idea clearer, we first prove Barden's theorem, the simply-connected analogue of 1.4. No proof of Barden's theorem has ever appeared.

THEOREM 6.1. *Let (W, V, V_1) be a simply-connected h-cobordism with* dim $W=5$. *Let $r: W\to V$ be a retraction such that $r \mid V_1: V_1\to V$ is a diffeomorphism. Then there is a diffeomorphism $\varphi: V\times I\to W$ so that $\varphi(x, 0)=x$* and *$\varphi(x, 1)=(r \mid V_1)^{-1}x$* for *$x\in V$.*

Proof. We may as well assume W is connected. Let $\mu:(W, V, V_1)\to(I, 0, 1)$ be a Morse function [21]. Then $(r, \mu): W\to V\times I$ is a homotopy equivalence that is a diffeomorphism of boundaries. Hence (r, μ) represents an element of $hS\,(V\times I, V\times\partial I)$. Clearly, it suffices to prove this element is trivial.

We have a map $\eta:hS\,(V\times I, V\times\partial I)\to[\Sigma V_+; G/O]$ and an action of $L_6(e)$ on $hS\,(V\times I, V\times\partial I)$ so that the inverse images via η of points are empty or orbtis of this action. Here $\{e\}$ is the trivial group and $\Sigma V_+ = V\times I/V\times\partial I$ is the reduced suspension of the union of V with a disjoint point.

There is a map $\varphi: S^3\times S^3\to S^6$ of degree one and a stable framing F of $M=S^3\times S^3$ so that the surgery obstruction $s(M, \varphi, F)\in\mathbf{Z}_2=L_6(e)$ is non-trivial. See [22], for example. Given $h:(K, \partial K)\to(V\times I, V\times\partial I)$, let ξ be a bundle so that $h^*\xi$ is equivalent to the normal bundle of K, and choose a framing G of $\tau(K)\oplus h^*\xi$. Then take the connected sum in the interior, $(K\times I, h\times I, G\times I)\ \#\ (M, \varphi, F)$. The surgery obstruction of the result is still non-trivial, by additivity of surgery obstructions [3]. This shows that the non-zero element of $L_6(e)$ acts trivially.

On the other hand ΣV_+ has a cell-decomposition with one one-cell, some three-cells, and a five-cell. But $\pi_1(G/O)=\pi_3(G/O)=\pi_5(G/O)=0$. Indeed, PL/O is 6-connected as $\Gamma_i=0$ for $i\leqslant 6$ (see [8], [21], and [18]) and the odd homotopy groups of

G/PL are all trivial by [33]. Hence $[\Sigma V_+; G/O]=0$. So $hS(V\times I, V\times \partial I)$ has one element. This proves the theorem.

Now suppose (W, V, V_1) is an h-cobordism, $\dim W=5$, but $\pi_1 W=\mathbf{Z}$. Assume that W is orientable and connected. Let $r: W\to V$ be a retraction with $r \mid V_1: V_1\to V$ a diffeomorphism. Let μ be a Morse function on W. Then $(r,\mu): W\to V\times I$ again represents an element of $hs(V\times I, V\times \partial I)$, and it would suffice to show this element to be trivial to prove Theorem 1.4.

Since $L_6(e)\to L_6(\mathbf{Z})$ induced by inclusion is an isomorphism, by [27, Theorem 5.1] and the periodicity $L_6=L_{10}$, the same proof as in the simply-connected case shows that $L_6(\mathbf{Z})$ acts trivially upon $hS(V\times I, V\times \partial I)$. Hence

$$\eta: hS(V\times I, V\times \partial I)\to [\Sigma V_+; G/O]$$

is a monomorphism. Unfortunately, $[\Sigma V_+, G/O]$ is not trivial.

Let $h:(M, \partial M)\to(V\times I, V\times \partial I)$ be a homotopy equivalence that restricts to a diffeomorphism of boundaries. Let K be obtained from M by identifying $h^{-1}(x, 0)$ with $h^{-1}(x, 1)$ for $x\in V$. Then h induces a homotopy equivalence $f:K\to V\times S^1$. It follows easily from the definitions that $\eta(f)$ is the image in $[V\times S^1; G/O]$ of $\eta(h)$ under the natural quatient projection of $V\times S^1$ on $\Sigma V_+=V\times S^1/V\times pt$.

Let $\iota\in H^2(G/O; \mathbf{Z}_2)$ be the non-zero class. Let $\xi=\eta(h)$. By the results of [27, § 6], $\xi=0$ if and only if $\xi^*\iota=0$. In fact, ξ comes from $[V\times S^1; G]$ and the evaluation on ι is a monomorphism on such elements.

The map $[\Sigma V_+; G/O]\to[V\times S^1; G/O]$ is a monomorphism. For its kernel is isomorphic to the cokernel of the map $[\Sigma (V\times S^1); G/O]\to[\Sigma V; G/O]$ induced by the inclusion $V=V\times pt\subseteq V\times S^1$. This map is clearly onto; the suspension of the natural projection of $V\times S^1$ on V induces a right inverse.

The evaluation on ι is natural with respect to induced maps. Hence it defines a map

$$\gamma: [\Sigma V_+: G/O]\to H^2(\Sigma V_+; \mathbf{Z}_2)=\mathbf{Z}_2$$

with the property that for $h:(M, \partial M)\to(V\times I, V\times \partial I)$ a homotopy equivalence that is a diffeomorphism of boundaries, $\gamma(\eta(h))=0$ if and only if $\eta(h)=0$.

Thus to prove Theorem 1.4 it suffices to find a homotopy equivalence $h: V\times I\to V\times I$, with $h \mid V\times \partial I$ the identity, and $\eta(h)\neq 0$ in $[\Sigma V_+; G/O]$.

LEMMA 6.2. *A generator of* $H_3(V)=\mathbf{Z}$ *is spherical.*

Assuming this lemma, let us complete the proof of Theorem 1.4. Let $D^5\subset V\times(\frac{1}{4},\frac{3}{4})$ be a disk. Let $\alpha_0\in\pi_5(V\times I)$ be represented by the composite

$$S^5 \xrightarrow{p} S^3 \xrightarrow{\beta} V\times(\tfrac{1}{4},\tfrac{3}{4}),$$

the first map being the non-trivial map and the second representing a generator of

$H_3(V)$. Then let $h: V\times I\to V\times I$ be the identity outside D and so that the obstruction to homotopy of h to the identity relative $(V\times I-\mathring{D})$ is $\alpha_0\in H^5(V\times I, (V\times I-\mathring{D}); \pi_5 V) =\pi_5 V$.

Glueing $V\times 0$ to $V\times 1$ by the identity, we get a map $f: V\times S^1\to V\times S^1$ induced by h. If $\eta(h)$ were equal to zero, h would represent the trivial element of $hS(V\times I, V\times\partial I)$, i.e. the class of the identity. Certainly in this case f would be homotopic to a diffeomorphism. So it suffices to show that f is not homotopic to a diffeomorphism.

Let ν^k be the normal bundle of $V\times S^1$. Then by 2.6, $\theta(f)=1_{5+k}+S_\nu(\alpha_0)$, and we could proceed by showing that $1_{5+k}+S_\nu(\alpha_0)\neq 1_{5+k}$ in $A(V\times S^1)$. Instead we give a direct argument suggested by J. Morgan.

Let $Q\subset V\times S^1$ be a closed submanifold, $\dim Q=2$, representing a class in $H_2(V\times S^1; \mathbf{Z}_2)$ dual to the mod 2 reduction of a generator of $H_3(V)\subset H_3(V\times S^1)$. Q exists by Steenrod representability [34] and the Whitney embedding theorem. (Actually, one can use a spectral sequence argument to represent the dual class by a sphere.) We may assume $Q\cap D=\emptyset$.

We may assume $\beta: S^3\to V\times S^1$ is transverse regular to Q. Then $\beta^{-1}Q$ will be an odd number of points, $q_1, \ldots, q_s$ say, s odd. We can assume $p: S^5\to S^3$ has $q_1, \ldots, q_s$ as regular values. Then $p^{-1}(q_i)=U_i$ is a submanifold of S^5, and if F_i is the framing of U_i induced via p, the Kervaire invariant $c(U_i, F_i)$ is not zero. This is because the Kervaire invariant and the Thom construction give an isomorphism of $\pi_5(S^3)$ with $\mathbf{Z}_2$ [24]. For example, we could arrange to have $U_i=S^1\times S^1$ with F_i the "wrong framing."

Thus we have a representative $\alpha: S^5\to V\times S^1$ of α_0, transverse to Q, with $\alpha^{-1}(Q)= =U_1\cup\cdots\cup U_s$. Without changing this, we may alter α so that on a small disk $D_0\subset S^5$ so that $\alpha\mid D_0: D_0\to D$ is a diffeomorphism. Let us identify D with the complementary disk $S^5-\operatorname{Int} D_0$. Then we may choose f so that $(f\mid D)\cup(\alpha\mid D_0)=\alpha$. So f will be transverse to Q, and $f^{-1}(Q)=Q\cup U_1\cup\cdots\cup U_s=W$. Let $\varphi=f\mid W: W\to Q$, a map of degree one on Q and degree zero on each U_i. Let ξ be the stable normal bundle of Q; $\xi=\nu(Q, V\times S^1)\oplus\nu\mid Q$, where the first summand is the normal bundle of Q in $V\times S^1$. Then from transversality we have a stable bundle map from the normal bundle of W to ξ covering φ, and so a stable framing of $\tau W\oplus\varphi^*\xi$. Clearly the Kervaire (surgery) obstruction $c(W, \varphi, F)=\sum c(V_i, F_i)$ does not vanish.

Suppose h were homotopic to a diffeomorphism. Then, making the homotopy transverse to Q, we get a cobordism of (W, φ, F) to (P, ψ, G), $\psi: P\to Q$ a homotopy equivalence, indeed a diffeomorphism. But the Kervaire obstruction is a cobordism invariant (see [3], [6]), so this is impossible. Hence h is not homotopic to a diffeomorphism. This proves Theorem 1.4, assuming Lemma 6.2.

Proof of 6.2. Recall V is a closed orientable manifold with $\pi_1 V=\mathbf{Z}$. There is a space X of the homotopy type of V and a Serre fibration $p: X\to S^1$ with simply-connected fiber F. (See, e.g. [30].) F has the (weak) homotopy type of the universal covering space of V. Let (E_r) be the cohomology spectral sequence of the Serre fibration p. Then $E_2^{p,q}=H^p(S^1; \mathfrak{H}^q(F))$, cohomology with local coefficients. We also have a filtration $H^3(X)=F^{0,3}\supseteq F^{1,2}\supseteq F^{2,1}\supseteq F^{3,0}\supseteq 0$. As $E_2^{2,1}=E_2^{3,0}=0$, $F^{2,1}=0$ and $F^{1,2}=E_\infty^{1,2}=E_2^{1,2}$.

Suppose $y\in F^{1,2}$. Let $x\in H^1(X)=E_2^{1,0}=E_\infty^{1,0}=F^{1,0}$. Then $x\cup y\in F^{2,2}$. But $E_2^{2,2}=E_2^{3,1}=E_2^{4,0}=0$; hence $F^{2,2}=0$. So $x\cup y=0$. By Poincaré duality, this implies $y=0$. Thus $E_\infty^{1,2}=0$, and the map

$$H^3(X)\to F^{0,3}\to E_\infty^{0,3}\subseteq E_2^{0,3}\subset H^3(F),$$

which is just the map induced by inclusion, is monic. The image consists of those elements fixed under the action of $\pi_1 S^1$.

Now $H_2(F)=H_2(\hat{V})$, $\hat{V}$ the universal cover of V. Let $\Lambda=\mathbf{Z}[\pi_1 V]$, the integral group ring of $\pi_1 V=\mathbf{Z}$. Let C_* be the chains of $\hat{V}$ with respect to a cell-decomposition induced from one of V by the covering map. C_* is a Λ-module. By Poincaré duality (see [39]), $H_2(V; \Lambda)=H_2(\hat{V})=H_2(C_*; \Lambda)=H^2(\mathrm{Hom}(C_*; \Lambda))=H^2(V; \Lambda)$. So by universal coefficients over Λ (see [30]),

$$H_2(F)=\mathrm{Hom}_\Lambda(H_2(\hat{V}); \Lambda)\oplus \mathrm{Ext}_\Lambda(H_1(\hat{V}); \Lambda).$$

But $H_1(\hat{V})=0$, and, as $\mathbf{Z}$-modules,

$$\mathrm{Hom}_\Lambda(H_2(\hat{V}); \Lambda)\subseteq \mathrm{Hom}_{\mathbf{Z}}(H_2(\hat{V}); \mathbf{Z}).$$

Hence $H_2(F)$ is free. By universal coefficients, this means $H^3(F)=\mathrm{Hom}(H_3(F); \mathbf{Z})$.

The subgroup of elements of $H^3(F)$ fixed under the action of the fundamental group is a direct summand because it is the kernel of a homomorphism of the free module $H^3(F)$ onto a submodule and hence onto a free module. Hence $\exists w\in H_3(F)$ with $\langle i^*x, w\rangle=0$, where $i:F\to X$ is inclusion, $x\in H^3(X)$ is a generator, and $\langle,\rangle$ denotes Kronecker product. Hence i_*w is a generator of $H_3(X)=H_3(V)$. Since F is simply-connected, w is spherical, by the Hurewicz theorem. Hence i_*w is spherical. This proves the lemma.

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