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## A Note on Bases in Ordered Locally Convex Spaces

by J. T. MARTI and D. R. SHERBERT

Let  $E$  be a real Fréchet space ordered by a cone  $K$  and let the dual space  $E'$  be ordered by the dual cone  $K' = \{f \in E' : f(x) \geq 0, x \in K\}$ . Besides the topologies on  $E$  defined by the dual system  $\langle E, E' \rangle$ , e.g., the Mackey topology  $\tau(E, E')$  determined by the metric on  $E$ , and the weak topology  $\sigma(E, E')$ , there are also topologies defined in terms of the order structure. One of these is  $o(E, E')$ , the topology of uniform convergence on the order bounded subsets of  $E'$ .  $o(E, E')$  is defined when each  $x$  in  $E$  regarded as a linear functional on  $E'$  is an order bounded linear functional on  $E'$ , that is, whenever  $\{f(x) : f \in S\}$  is bounded for each order bounded set  $S$  in  $E'$ . This condition is satisfied when  $K$  is generating. If  $E$  is a barreled space, in particular, if  $E$  is a Fréchet space, with a generating cone, then  $o(E, E')$  is consistent with  $\langle E, E' \rangle$  as is shown in proposition 1 below. Also, if  $E$  is a locally convex lattice, then  $o(E, E')$  is always consistent and is the coarsest topology finer than the weak topology for which the lattice operations are continuous. See [6, 7].

In this note, we give an answer to the question of whether each  $o(E, E')$ -basis for  $E$  is a  $o(E, E')$ -Schauder basis for  $E$ . A  $\mathfrak{T}$ -basis for a topological vector space  $E(\mathfrak{T})$  is a sequence  $\{x_i\}$  in  $E$  such that for each  $x$  in  $E$ , there is a unique sequence  $\{\alpha_i\}$  of scalars such that  $x = \sum_{i=1}^{\infty} \alpha_i x_i$ , where the convergence of the series is with respect to the topology  $\mathfrak{T}$  [4]. The uniqueness implies that each  $\alpha_i$  may be regarded as a linear functional on  $E$ . If each  $\alpha_i$  is  $\mathfrak{T}$ -continuous, then  $\{x_i\}$  is called a  $\mathfrak{T}$ -Schauder basis for  $E$ . The weak basis theorem [2, 3, 5] for Fréchet spaces states that each  $\sigma(E, E')$ -basis is a  $\sigma(E, E')$ -Schauder basis for  $E$ . As a consequence of this fundamental theorem one gets the result that each  $\sigma(E, E')$ -basis for a Fréchet space  $E$  is a  $\tau(E, E')$ -Schauder basis for  $E$ . We show that if  $E$  is a Fréchet space ordered by a generating cone, then the analogous weak basis theorem for  $o(E, E')$  is valid. Moreover, it is also shown that each lattice theoretically absolutely convergent  $o(E, E')$ -basis for a complete metrizable locally convex lattice  $E$  is an unconditional  $\tau(E, E)$ -Schauder basis for  $E$ .

The following proposition is related to Corollary 2.4 of [7, p. 130]. It shows that if  $E$  is barreled, then the hypotheses that the cone in  $E$  is closed and  $E'$  is a full subspace of the algebraic dual  $E^*$  or of the order dual  $E^+$  can be dispensed with.

**PROPOSITION 1.** *If  $E$  is a barreled space ordered by a generating cone, then  $o(E, E')$  is consistent with the dual system  $\langle E, E' \rangle$ .*

*Proof.* Let  $S$  be the class of all order intervals in  $E'$  and let  $\bar{S}$  be the saturated hull of  $S$  (i.e., the class of all scalar multiples of the  $\sigma(E', E)$ -closed, convex circled hulls

of finite families of  $S$ ). Then the topology of uniform convergence on the sets in  $\bar{S}$  coincides with the topology  $o(E, E')$ . Since the cone  $K$  in  $E$  is generating,  $K'$  is normal for the topology  $\sigma(E', E)$  in  $E'$  [7, p. 74]. From this it follows that each order interval in  $E'$ , and hence each member of  $\bar{S}$ , is  $\sigma(E', E)$  bounded. Since  $E$  is barreled, these sets are therefore equicontinuous. Thus, the sets in  $\bar{S}$  are  $\sigma(E', E)$ -compact and the proposition is a consequence of the Mackey–Arens theorem.

**PROPOSITION 2.** *If  $\{x_i\}$  is a  $o(E, E')$ -basis for a Fréchet space  $E$  ordered by a generating cone  $K$ , then  $\{x_i\}$  is a  $o(E, E')$ -Schauder basis for  $E$ .*

*Proof.* The result follows immediately from proposition 1 and a generalization of the weak basis theorem that states that if  $\mathfrak{T}$  is a topology consistent with  $\langle E, E' \rangle$ , then every  $\mathfrak{T}$ -basis for  $E$  is a  $\mathfrak{T}$ -Schauder basis for  $E$  [1, p. 508]. However, the result can also be proved directly by adapting the techniques used for proving the weak basis theorem [2, 3]. In this case, particular use must be made of the fact that convergence for  $o(E, E')$  implies convergence for  $\sigma(E, E')$ , and that the order intervals in  $E'$  are equicontinuous sets.

**PROPOSITION 3.** *If  $E(\mathfrak{T})$  is a Fréchet space ordered by a generating cone  $K$ , then every  $o(E, E')$ -basis for  $E$  is a  $\mathfrak{T}$ -Schauder basis for  $E$ .*

*Proof.* Each  $x$  in  $E$  has the weak series expansion  $\sum_{i=1}^{\infty} \alpha_i x_i$ , but in order to conclude that  $\{x_i\}$  is actually a weak basis, we must show that the unique sequence  $\{\alpha_i\}$  corresponding to  $x$  in the  $o(E, E')$  expansion of  $x$  is also unique for the weak series expansion of  $x$ . To do this, we use the fact that each  $\alpha_i$  belongs to  $E'$ . Then if  $\sum_{i=1}^{\infty} \beta_i x_i = \theta$ , where we assume the series converges weakly, we have  $\sum_{i=1}^{\infty} \beta_i \alpha_j(x_i) = \theta$  for  $j=1, 2, \dots$ . Since  $\{x_i\}$  is a  $o(E, E')$ -basis for  $E$ ,  $\alpha_j(x_i) = \delta_{ij}$ . Hence  $\beta_j = 0$  for each  $j$  and we conclude that the coefficients in the weak series expansion of  $x$  are unique. Thus,  $\{x_i\}$  is a weak basis for  $E$  and the weak basis theorem implies that  $\{x_i\}$  is a weak Schauder basis for  $E$ . From this we obtain that  $\{x_i\}$  is a  $\mathfrak{T}$ -Schauder basis for  $E$ .

**COROLLARY 4.** *Every  $o(E, E')$ -basis for a complete metrizable locally convex lattice  $E(\mathfrak{T})$  is a  $\mathfrak{T}$ -Schauder basis for  $E$ .*

*Proof.* Since  $E$  is a locally convex lattice, the cone in  $E$  is obviously generating so that the corollary follows from proposition 3.

If  $E$  is a locally convex lattice, then a  $\theta$ -neighborhood basis for  $o(E, E')$  is given by polars of order intervals in  $E'$  of the form  $[-f, f]$  where  $f$  is in  $K'$ . Thus  $o(E, E')$  is generated by the family  $\{P_f: f \in K'\}$  of seminorms defined by

$$P_f(x) = \sup \{|g(x)| : -f \leq g \leq f\}$$

and these seminorms have the simple form

$$P_f(x) = f(|x|), \quad f \in K', \quad x \in E$$

where  $|x|$  is the lattice theoretic absolute value of  $x$  in  $E$ . To see this, first note that the inequality  $P_f(x) \leq f(|x|)$  is evident since  $-f \leq g \leq f$  implies that  $g(x) \leq f(|x|)$ . To obtain the reverse inequality, we may use the fact that the canonical mapping  $\varphi: E \rightarrow E''$  is a lattice isomorphism of  $E$  onto a sublattice of  $E''$ . Then for  $f$  in  $K'$  we have

$$f(|x|) = |\varphi(x)|(f) = \sup \{ \varphi(x)(g) : |g| \leq f \} = \sup \{ g(x) : -f \leq g \leq f \}$$

See [9, p. 212]. From this we have  $f(|x|) \leq P_f(x)$  and hence  $P_f(x) = f(|x|)$  for  $f$  in  $K'$  and  $x$  in  $E$ .

Using the concept of lattice theoretical absolute convergence of a series introduced by Pietsch [8], one gets a new type of  $o(E, E')$ -basis. A  $o(E, E')$ -basis  $\{x_i\}$  for  $E$  is called *lattice theoretically absolutely convergent* if for each  $x$  in  $E$  the sequence  $\{\sum_{i=1}^n |\alpha_i x_i|\}$  is majorized in  $E$ , where  $\sum_{i=1}^{\infty} \alpha_i x_i$  is the basis expansion of  $x$ . Since  $E'$  is a lattice ideal in the order dual  $E^+$  of  $E$  and since  $E^+$  coincides with the order bound dual  $E^b$  [7, 9], it follows that  $\{x_i\}$  is a lattice theoretically absolutely convergent basis for  $E$  if and only if it is a  $o(E, E')$ -basis such that  $\sum_{i=1}^{\infty} P_f(\alpha_i x_i)$  is finite for  $x$  in  $E$ ,  $f$  in  $K'$  [8, p. 17] (i.e., a  $o(E, E')$ -absolutely convergent basis for  $E$ ).

**PROPOSITION 5.** *If  $\{x_i\}$  is a lattice theoretically absolutely convergent  $o(E, E')$ -basis for a complete metrizable locally convex lattice  $E(\mathfrak{T})$ , then  $\{x_i\}$  is an unconditional  $\mathfrak{T}$ -basis for  $E$ .*

*Proof.* Let  $\Sigma$  denote the collection of all finite subsets of the set of positive integers. For each  $\sigma \in \Sigma$ , define the continuous linear transformation  $T_\sigma: E \rightarrow E$  by  $T_\sigma x = \sum_{i \in \sigma} \alpha_i x_i$  where  $\{\alpha_i\}$  is the sequence of coefficients corresponding to  $x$ . We have for all  $\sigma \in \Sigma$  and  $f \in E'$  that

$$|f(T_\sigma x)| \leq \sum_{i \in \sigma} |f(\alpha_i x_i)| \leq \sum_{i \in \sigma} |f|(|\alpha_i x_i|) \leq \sum_{i=1}^{\infty} P_{|f|}(\alpha_i x_i)$$

which is finite. Thus, for each  $x$  in  $E$ , the family  $\{T_\sigma x: \sigma \in \Sigma\}$  is weakly bounded, and hence bounded in  $E(\mathfrak{T})$ . It then follows that the family  $\{T_\sigma: \sigma \in \Sigma\}$  is equicontinuous. Let  $U$  be any  $\theta$ -neighborhood in  $E$ . Then there is a circled  $\theta$ -neighborhood  $V$  such that  $V + V \subset U$ . Choose a  $\theta$ -neighborhood  $W$  in  $E$  such that  $T_\sigma(W) \subset V$  for all  $\sigma$ . Since  $\{x_i\}$  is a  $\mathfrak{T}$ -basis for  $E$  by proposition 3, there exists an integer  $n$  such that  $x - T_{\sigma_n} x$  is in  $V \cap W$  where  $\sigma_n = \{1, 2, 3, \dots, n\}$ . Then  $T_\sigma(x - T_{\sigma_n} x)$  is in  $V$  for all  $\sigma \in \Sigma$ . Hence, for all  $\sigma \supset \sigma_n$  we have

$$x - T_\sigma x = x - T_{\sigma_n} x - T_\sigma(x - T_{\sigma_n} x) \in V + V \subset U.$$

This shows that  $\sum_{i=1}^{\infty} \alpha_i x_i$  is unconditionally convergent to  $x$  and the proof is complete.

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