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Groups of Self Homotopy Equivalences of Induced Spaces

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Introduction

Methods have been given for effectively calculating, at least up to extension, the group of homotopy classes $\mathscr{E}(X)$ of self homotopy equivalences of a topological space X in case X has a very simple structure: for example when X has a two stage Postnikov system (3.3 of [10]), when X^{n+1} is obtained from X^{n-1} by attaching one cell of appropriate dimension (6.1 of [2]) or when X is a pseudo-projective plane (see [11]). The purpose of this article is to give effective methods for calculating, up to extension, $\mathscr{E}(X)$ and its important subgroups when X is an induced fibre or cofibre space in terms of information available from the fibre or cofibre sequence. These methods can for example be applied inductively to a Postnikov decomposition to *effectively* compute $\mathscr{E}(X)$ when the penultimate stage of the Postnikov decomposition is an H-space.

Let $\Omega K \to P_h \to B \xrightarrow{h} K$ be an induced fibre sequence. Then there is an action $\Omega K \times P_h \to P_h$ making $P_h \to B$ into a principal morphism in the homotopy category. In § 2 I show that with certain restrictions on the range of dimensions of non-zero homotopy groups of ΩK and B, the group $\mathscr{E}(P_h)$ may be calculated up to extensions in terms of $\mathscr{E}(B)$ and aut ΩK: this particular extension theorem is a corollary to the results of [13].

In § 3 an alternative extension sequence is obtained which specializes in case $P_h \to B$ is part of a Postnikov system to the sequence

$$0 \to H^s(B; G)/h_1 \pi_1 \pi_1^B(B; 1) \to \mathscr{E}(P_h) \to \mathscr{E}(h) \to 0.$$

As an application $\mathscr{E}(X)$ is determined in § 4 for the three stage Postnikov system with second k-invariant the cup product $\bigcup: K(\pi, m) \times K(G, n) \to K(\pi \otimes G, m+n)$. Finally § 5 contains an extension sequence for the group of *fibre* homotopy classes of fibre homotopy equivalences.

The dual results are discussed concurrently and are marked with an asterisk.

1. Preliminaries

Let $\mathscr{H}$ be the category of based spaces having the homotopy type of CW complexes and base point preserving homotopy classes of maps[1]): the zero object is a one

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[1]) The same symbol is used for a map and its homotopy class.

point space and zero morphisms are the classes of constant maps: products and sums are simply ordinary topological products $X\times Y$ and wedges $X\vee Y$ (unions with base points identified): the set of morphisms $X\rightarrow Y$ is denoted (X, Y). The group objects in this category are H-spaces, that is spaces, G, with a multiplication $G\times G\rightarrow G$ having the constant function as homotopy identity and having a homotopy inverse; and the cogroup objects are H'-spaces. The class of $f:X\rightarrow Y$ is an equivalence if f is a homotopy equivalence; this is true if and only if f is bijective between sets of path components, induces an isomorphism of the fundamental groups, and either isomorphisms of all homotopy groups π_r $(r\geqslant 2)$ or, equivalently, of all homology groups H_r $(r\geqslant 2)$ of the universal covering spaces.

Given a space K with base point $*$, the space PK is the set of maps $\{l:I\rightarrow K:l(0)=*\}$ with the compact open topology where $I=[0, 1]$ is the closed interval. The loop space ΩK is the subspace of PB of maps with $l(0)=*=l(1)$, and is a group object in $\mathscr{H}$. Let $h:B\rightarrow K$ be a *map* and define the induced fibre map $p:P_h\rightarrow B$ by

$$P_h=\{(b, l)\in B\times PK: h(b)=l(1)\} \quad \text{and} \quad p(b, l)=b\,.$$

The (reduced) cone on K is $CK=K\times I/K\times\{0\}\bigcup *\times I$; the (reduced) suspension of K is the quotient space $SK=K\times I/K\times\{0\}\bigcup K\times\{1\}\bigcup *\times I$ which is a cogroup object in $\mathscr{H}$; and the (reduced) cylinder is $K\divideontimes I=K\times I/*\times I$. Let $h:K\rightarrow B$ be a map then the mapping cone, or induced cofibre space, is $C_f=B\cup_f CK$, the quotient space obtained from the topological sum $B+CK$ by identifying $(k, 1)$ and $f(k)$ for each k in K. Then C, S and C_f are dual to P, Ω and P_f.

The group of self equivalences of X in $\mathscr{H}$ is denoted $\mathscr{E}(X)$; and, if $f:X\rightarrow Y$, the group of self equivalences of X retracting to self equivalences of Y is denoted $\mathscr{R}_f(X)$, the group of self equivalences of Y lifting to self equivalences of X is denoted $\mathscr{L}_f(Y)$; thus the diagram is commutative in $\mathscr{H}$ — or commutative up to homotopy in $\mathscr{T}\!op$.

$$\begin{array}{ccc} X & \rightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \rightarrow & Y \end{array}$$

Also $\mathscr{R}_f^1(X)$ and $\mathscr{L}_f^1(Y)$ denote respectively those equivalences retracting or lifting to the identity.

2. The Exact Sequences

Let $P_h\rightarrow B$ be the fibre space induced by $h:B\rightarrow K$, then an ΩK-action on P_h is given by taking $\kappa:\Omega K\times P_h\rightarrow P_h$ to be the class of the map $(l, (b, m))\rightarrow(b, l+m)$, and $p:P_h\rightarrow B$ is then a κ-principal morphism (definition in § 5 of [13], and full details in § 2 of [17]): there is in general no difference morphism (§ 7 of [13]) for p in the category $\mathscr{H}$. Let $i:\Omega K\rightarrow P_h$ be the inclusion[2]) and define $i_\flat:(P_h, \Omega K)\rightarrow(\Omega K, \Omega K)$ and $\kappa_\#:(P_h, \Omega K)\rightarrow$

[2]) Clearly the inclusion is in the same homotopy class as $\kappa(1, *):\Omega K\rightarrow\Omega K\times P_h\rightarrow P_h$.

$\to(P_h, P_h)$, as in § 3 of [13], by $i_\flat(\xi)=\xi i+1$ and $\kappa_\#(\xi)=\kappa(\xi, 1)$. As in § 3 of [13] binary structures $\times$ and $\oplus$ are defined on $(P_h, \Omega K)$ by $\xi\times\zeta=\xi i\zeta+\zeta+\xi$ and $\xi\oplus\zeta=\xi\kappa(\zeta, 1)+\zeta$.

$$\begin{array}{ccc}(P_h, \Omega K) & \xrightarrow{\kappa\#} & (P_h, P_h) \\ \downarrow{\scriptstyle i\flat} & & \\ (\Omega K, \Omega K) & & \end{array}$$

The properties of $i_\flat$ and $\kappa_\#$ *vis à vis* the structures $\times$ and $\oplus$ are given in § 3 of [13]. The next lemma is an elementary consequence of theorem 6.1 of [4].

LEMMA 2.1. *$p: P_h \to B$ is a κ-principal $\mathscr{E}$-morphism.*[3,4]

It is possible to apply the exact sequence theorem 5.5 of [13] to this situation with certain restrictions on homotopy groups and $\mathscr{R}_p(P_h)$ can thus be calculated, up to extension, provided the kernel $\kappa_\#^{-1}(1)$ is known. Let h_p be the composite homomorphism

$$h_p: \pi_1^{P_h}(B; p) \xrightarrow{h_*} \pi_1^{P_h}(K, hp) \xrightarrow{(hp)\flat} \pi_1^{P_h}(K; *) = (P_h, \Omega K)$$

where $(hp)_\flat$ is the isomorphism[5] given by a nulhomotopy of hp. Of course $(hp)_\flat$ is not unique, but any two such isomorphisms differ by an inner automorphism of $(P_h, \Omega K)$; thus h_p is unique in case K is an H-space and in this case $(hp)_\flat$ is the isomorphism described in § 1.2 of [12] (see ii) and iii) of 1.2.2 of [12]). If further B is an H-space, h_p may, for subsequent purposes, be replaced by the homomorphism

$$\Delta(h, p): (P_h, \Omega B) \xrightarrow{p\flat^{-1}} \pi_1^{P_h}(B, p) \xrightarrow{h_*} \pi_1^{P_h}(K, hp) \xrightarrow{(hp)\flat} (P_h, \Omega K)$$

whose properties are given in § 1 and § 2 of [12]; and which is calculated explicitly in theorem 2.4.1 of [12] when K is a product of Eilenberg–MacLane spaces.

Now by theorem 3.5 of [13] $\kappa_\#: \kappa_\#^{-1}\ \mathscr{E}(P_h) \to \mathscr{E}(P_h)$ is a homomorphism with respect to the $\oplus$-structure on the first group. Thus $\kappa_\#^{-1}(1)$ is a group; clearly $\oplus$ and the operation $+$ of the group $(P_h, \Omega K)$ give the same structure on it. The next lemma is now elementary from remarks in § 1.3 of [12].

LEMMA 2.2 *The function $h_p: \pi_1^{P_h}(B, p) \to \kappa_\#^{-1}(1)$ is an epimorphism with respect to the structure $\oplus$ on the group $\kappa_\#^{-1}(1)$.*

Now let $t: P_h \cup C\Omega K \to S\Omega K$ be the collapsing map, and $\pi: P_h \cup C\Omega K \to B$ the

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[3]) Defined in § 5 of [13]: all morphisms $\Omega K \to \Omega K$ lifting elements of $\mathscr{R}_p^1(P_h)$ must be equivalences.

[4]) Rather more than this is true if the base is simply connected. Using the five lemma and the usual arguments on weak homotopy equivalence on the homotopy sequence of the fibration, it is elementary that $\mathscr{R}_p^1(P_h) = \kappa_\# i_\flat^{-1} \mathscr{E}(\Omega K)$ (cf. lemma 5.4 of [13]).

[5]) The isomorphisms $(hp)_\flat$ and later $p_\flat$ are not to be confused with the function $i_\flat$ defined above.

extension of $p: P_h \to B$ which is constant on $C\Omega K$.

$$\begin{array}{ccccc} \Omega K & \xrightarrow{i} & P_h & \xrightarrow{p} & B \\ & \searrow & \uparrow \pi & & \\ & & P_h \cup C\Omega K & \xrightarrow{t} & S\Omega K \end{array}$$

PRINCIPAL FIBRE SPACE THEOREM 2.3. *Let $h: B \to K$ induce the principal fibre map $p: P_h \to B$ for which P_h is $(m-1)$ connected and ΩK has non vanishing homotopy groups*[6]) *$\pi_r(\Omega K)$ only in a range $n \leqslant r \leqslant m+n-1$. Then $h_p\, \pi_1^{P_h}(B; p) \subset i_\flat^{-1}$ aut ΩK as a subgroup, and there is an exact sequence of homomorphisms*

$$0 \to i_\flat^{-1} \operatorname{aut} \Omega K / h_p \pi_1^{P_h}(B, p) \xrightarrow{\kappa\#} \mathscr{R}_p(P_h) \to \mathscr{L}_p(B)/\mathscr{L}_p^1(B) \to 0$$

with respect to the $\times$ structure on the first group. If also[7]) *B has homotopy groups only in the range $m \leqslant r \leqslant n-1\,(m \geqslant 2)$,*

$$\text{then} \quad \mathscr{E}(P_h) = \mathscr{R}_p(P_h),\ \mathscr{L}_p^1(B) = 1 \quad \textit{and} \quad \operatorname{aut}(\Omega K) = \mathscr{E}(\Omega K);$$

and $i_\flat^{-1}$ aut $\Omega K / h_p\, \pi_1^{P_h}(B; p)$ is a group extension[8])

$$0 \to (B, \Omega K)/h_1 \pi_1^B(B, 1) \cup \pi^{*-1} t^*(S\Omega K, \Omega K) \to i_\flat^{-1} \operatorname{aut} \Omega K / h_p \pi_1^{P_h}(B; p)$$
$$\to (L+1) \cap \operatorname{aut} \Omega K \to 0$$

where $L = ker((S\Omega K, K) \to (P_h \cup C\Omega K, K))$.

Proof. By lemma 7.2 $(P_h, \Omega K) = \mathscr{T}_\kappa(P_h, \Omega K)$[9]) and thus the $\oplus$ and $\times$ structures are the same by proposition 3.7 of [13]. The first part of this theorem is now immediate from lemma 5.3 of [13] and the exact sequence theorem 5.5(i) of [13]. Also $\mathscr{E}(P_h) = \mathscr{R}_p(P_h)$, $\mathscr{L}_p^1(B) = 1$ and aut $\Omega K = \mathscr{E}(\Omega K)$ are lemmas 7.4, 7.5 and 7.1. In the following diagram π^* is an isomorphism by an obstruction theory[10]) argument based on 1.1 of [5] and a Postnikov decomposition for ΩK; also the horizontal sequences are

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[6]) Including $r = 0$.

[7]) Of course (for $m \leqslant n$) K is an H-space; and, with the further restriction that $\pi_r(\Omega K)$ is non zero only in a range $n \leqslant r \leqslant m+n-2$ $(n \geqslant 2)$, *any* fibre space over B with fibre ΩK is equivalent to an induced one (see for example theorem 3 of [8]): this latter result is extended in theorem 2.4 and corollary 2.5 of [6].

[8]) The group $(B, \Omega K)$ is abelian (see lemma 3.3 of [13]) since K is an H-space.

[9]) $\mathscr{T}_\kappa(P_h, \Omega K)$ is the set of κ-twisted morphisms $\xi: P_h \to \Omega K$: that is classes satisfying $\xi\kappa = \xi i p_1 + p_1 + \xi p_2 - p_1$ (see § 3 of [13]).

[10]) For lemmas on obstruction theory see § 8.

exact (see theorem 1 of [14] in case of the lower sequence).

$$\begin{array}{ccccccccc}
(S\Omega K, \Omega K) & \xrightarrow{t^*} & (P_h \cup C\Omega K, \Omega K) & \rightarrow & (P_h, \Omega K) & \rightarrow & (\Omega K, \Omega K) & \rightarrow & (P_h \cup C\Omega K, K) \\
 & & \uparrow{\scriptstyle \pi^*} \quad \nearrow & & \uparrow & & \uparrow & & \\
 & & (B, \Omega K) & & {\scriptstyle h_p} & & 0 & & \\
 & & \uparrow{\scriptstyle h_1} & & | & & \| & & \\
 & & \pi_1^B(B; 1) & \rightarrow & \pi_1^{P_h}(B; p) & \rightarrow & \pi_1^{\Omega K}(B; *) & &
\end{array}$$

With the dimensional restrictions indicated K is an H-space, hence h_1 and h_p are unique and the diagram is commutative by 1.2.2 of [12]: an elementary argument now gives the second sequence of the theorem. This completes the proof of theorem 2.3.

The group $\mathscr{L}_p(B)/\mathscr{L}_p^1(B)$ may, in the exact sequence, be replaced by $p^* \mathscr{E}(B) \cap \ker h_*$ where $h_*: (P_h, B) \rightarrow (P_h, K)$.

As mentioned previously, in case B is an H-space $h_p \pi_1^{P_h}(B, p)$ can be replaced by $\Delta(h, p)(P_h, \Omega B)$, and thus the theorem can be used to calculate the group of homotopy equivalences of a simply connected space with a finite Postnikov decomposition whose penultimate stage is an H-space.

In the dual case, which is considered next, the full force of the theorem applies only when there is at least one homology dimension missing between the cobase and the cofibre; so that, theoretically at least, the dual is not quite so useful.

Dually then let $A \rightarrow C_h$ be induced by $h: K \rightarrow A$, then an SK-coaction on C_h is given by taking $\lambda: C_h \rightarrow SK \vee C_h$ to be the class of the map which collapses the slice $(K, \frac{1}{2})$ of the cone to the base point, and $i: A \rightarrow C_h$ is then a λ-coprincipal morphism. Let $p: C_h \rightarrow SK$ be the projection onto the cofibre and define $p^\flat: (SK, C_h) \rightarrow (SK, SK)$ and $\lambda^\#: (SK, C_h) \rightarrow (C_h, C_h)$, as in § 3 of [13], by $p^\flat(\xi) = p\xi + 1$ and $\lambda^\#(\xi) = (\xi, 1)\lambda$. As in § 3 of [13] binary structures $\times$ and $\oplus$ are defined on (SK, C_h) by $\xi \times \zeta = \xi p \zeta + \xi + \zeta$ and $\xi \oplus \zeta = (\xi, 1)\lambda\zeta + \xi$.

$$\begin{array}{ccc}
(SK, C_h) & \xrightarrow{\lambda^\#} & (C_h, C_h) \\
{\scriptstyle p^\flat}\downarrow & & \\
(SK, SK) & &
\end{array}$$

The proof of the next lemma is elementary using the usual arguments on weak homotopy equivalences.

LEMMA 2.1* *Let SK be simply connected, then $i:A\to C_h$ is a λ-coprincipal $\mathscr{E}$-morphism.*[11])

Let h^i be the composite homomorphism

$$h^i:\pi_1^A(C_h;\, i)\xrightarrow{h^*}\pi_1^K(C_h;\, ih)\xrightarrow{(ih)\flat}\pi_1^K(C_h;\, *)=(SK,\, C_h)$$

where $(ih)_\flat$ is the isomorphism given by a nulhomotopy of ih; as above $(ih)_\flat$ is not necessarily unique, but is unique in case K is an H'-space, and in this case is the isomorphism described in § 1.2 of [12] (see ii) and iii) of 1.2.2 of [12]). If further A is an H'-space, h^i may, for subsequent purposes, be replaced by the homomorphism[12])

$$\Gamma(i,\, h):(SA,\, C_h)\xrightarrow{i\flat^{-1}}\pi_1^A(C_h;\, i)\xrightarrow{h^*}\pi_1^K(C_h;\, ih)\xrightarrow{(ih)\flat}(SK,\, C_h)$$

whose properties are given in § 3 of [12]; and which is calculated explicitly in theorem 3.4.3 of [12] in the case A and K are both suspensions.

By theorem 3.5* of [13] $\lambda^\#:\lambda^{\#-1}\,\mathscr{E}(C_h)\to\mathscr{E}(C_h)$ is a homomorphism with respect to the $\oplus$-structure on the first group. Thus $\lambda^{\#-1}(1)$ is a group; clearly $\oplus$ and the operation $+$ of the group $(SK,\, C_h)$ give opposite [13]) structures on it.

LEMMA 2.2* *The function $h^i:\pi_1^A(C_h;\, i)\to\lambda^{\#-1}(1)$ is an anti-epimorphism with respect to the structure $\oplus$ on the group $\lambda^{\#-1}(1)$.*

Now, for the cofibration $A\xrightarrow{i}Ch\xrightarrow{p}SK$, the space P_p, defined as in § 1, is the fibre of p. Let $\imath:A\to P_p$ be the lifting of i given by $\imath(a)=(i(a)),\, *)$.

$$\begin{array}{ccccc} & & A & \xrightarrow{i} & C_h \xrightarrow{p} SK \\ & & {\scriptstyle \imath}\downarrow & \nearrow_{v} & \\ \Omega SK & \xrightarrow{u} & P_p & & \end{array}$$

COPRINCIPAL COFIBRE SPACE THEOREM 2.3.*[14]) *Let $h:K\to A$ induce the principal cofibre map $i:A\to C_h$ for which C_h and SK are respectively $(m-1)$ and $(n-1)$ connected $(n\geqslant 2)$, and let SK have non zero integral homology groups $H_r(SK)$ only for $r\leqslant m+n-2$ with* $\operatorname{ext}(H_{m+n-2}(SK),\, \pi_n(SK)\otimes\pi_m(C_h))=0$. *Then $h^i\ \pi_1^A\ (C_h;\, i)$ $\subset p^{\flat-1}$* aut*SK as a subgroup, and there is an exact sequence of homomorphisms*

$$0\to p^{\flat-1}\,\operatorname{aut}SK/h^i\pi_1^A(C_h;\, i)\xrightarrow{\lambda\#}\mathscr{L}_i(C_h)\to\mathscr{R}_i(A)/\mathscr{R}_i^1(A)\to 0$$

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11) Defined in § 5 of [13]; all morphisms $SK\to SK$ retracting elements of $\mathscr{L}^1{}_i\,(C_h)$ must be equivalences.

12) Again $i_\flat$ here should not be confused with the function $i_\flat:(P_h,\, \Omega K)\to(\Omega K,\, \Omega K)$ defined previously.

13) In the dual case the structures are equal.

14) In this theorem hypotheses are made demanding that certain top dimensional homology groups are free. This is done to avoid complicated statements: in fact all that is necessary is that these homology groups make certain ext functors vanish: the details are clear from the proof and the lemmas of § 7 and § 8.

with respect to the $\times$ *structure on the first group. If also A has homology groups only in the range* $m \leqslant r \leqslant n-2$ $(m \geqslant 2)$ *and* $H_{n-2}(A)$ *is free and SK has homology groups*[15] *only in the range* $n \leqslant r \leqslant m+n-3$ *and* $H_{m+n-3}(SK)$ *is free,*[16] *then* $\mathscr{E}(C_h)=\mathscr{L}_i(C_h)$, $\mathscr{R}_i^1(A)=1$, *and* $\operatorname{aut} SK=\mathscr{E}(SK)$; *and* $p^{b-1} \operatorname{aut} SK/h^i \pi_1^A(C_h; i)$ *is a group extension*

$$0 \to (SK, A)/h^1\pi_1^A(A; 1) \cup \iota_*^{-1} u_*(SK, \Omega SK) \to p^{b-1} \operatorname{aut} SK/h^i\pi_1^A(C_h; i) \to (L+1) \cap \operatorname{aut} SK \to 0$$

where $L=\ker((K, \Omega SK) \to (K, P_p))$.

Proof. The proof of the first part is dual to that of theorem 2.3 using lemma 7.2*. Also $\mathscr{E}(C_h)=\mathscr{L}_i(C_h)$, $\mathscr{R}_i^1(A)=1$ and $\operatorname{aut} SK=\mathscr{E}(SK)$ are lemmas 7.4*, 7.5* and 7.1*. In the following diagram ι_* is an isomorphism by an obstruction theory argument based on lemma 3.1 of [5]; also the horizontal sequences are exact as in the dual case

$$\begin{array}{ccccccccc}
(SK, \Omega SK) & \xrightarrow{u_*} & (SK, P_p) & \to & (SK, C_h) & \to & (SK, SK) & \to & (K, P_p) \\
 & & \uparrow{\scriptstyle \iota_*} & \nearrow & \uparrow{\scriptstyle h^i} & & & & \\
 & & (SK, A) & & & & 0 & & \\
 & & \uparrow{\scriptstyle h^1} & & & & \| & & \\
 & & \pi_1^A(A; 1) & \to & \pi_1^A(C_h; i) & \to & \pi_1^A(SK; *) & &
\end{array}$$

With the dimensional restrictions indicated K is a suspension, hence h^1 and h^i are unique and the diagram is commutative by 1.2.2 of [12]. The theorem now follows as before.

3. An Alternative Sequence

Alternative extension sequences are now considered which in some applications are more useful than previous ones.

DEFINITION. An *equivalence* of $h: B \to K$ is a pair of equivalences $c: B \to B$ and $\bar{a}: K \to K$ satisfying $\bar{a}h = hc$. The group of equivalences of h, with composition $(\bar{a}, c) \times (\bar{a}', c') = (\bar{a}\bar{a}', cc')$, is denoted $\mathscr{E}(h)$ and is regarded as a subgroup of $\mathscr{L}_h \times \mathscr{R}_h$.

Clearly $\mathscr{E}(h)$ is the isotropy group of h under the obvious action of $\mathscr{E}(K) \times \mathscr{E}(B)$ on (B, K).

Now with the restrictions on homotopy groups given in the hypotheses of the following theorem, each element b of $\mathscr{R}_p(P_h)$ determines unique classes of equivalences

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[15]) Of course (for $m \leqslant n-1$) K is a suspension.

[16]) If this is extended to the range $n \leqslant r \leqslant m+n-2$ with $H_{m+n-2}(SK)$ free, then the remainder of the theorem is valid except that, in the exact sequence, the function $(SK, A)/h^1 \pi_1^A(A; 1) \cup \iota_*^{-1} u_*(SK, \Omega SK) \to p^{b-1} \operatorname{aut} SK/h^i \pi_1^A(C_h; i)$ is not mono.

$\bar{a}:K \to K$ and $c:B \to B$, making the following diagram commutative:

$$\begin{array}{ccccccc} \Omega K & \to & P_h & \to & B & \xrightarrow{h} & K \\ \downarrow{\scriptstyle \Omega\bar{a}} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \bar{a}} \\ \Omega K & \to & P_h & \to & B & \xrightarrow{h} & K \end{array}$$

Full details and proof are given in § 6. Thus there is a homomorphism $\varrho: \mathscr{R}_p(P_h) \to \to \mathscr{L}_h(K) \times \mathscr{R}_h(B)$ and using standard constructions the image of this homomorphism is seen to be $\mathscr{E}(h)$. The following theorem is proved in § 6.

PRINCIPAL FIBRE SPACE THEOREM 3.1. *Let B have non zero homotopy groups $\pi_r(B)$ only in the rage $m \leqslant r \leqslant n$ $(m \geqslant 2)$, let K be simply connected and let ΩK have non zero homotopy groups only in the range $n \leqslant r \leqslant m+n-1$. Then the induced fibre sequence $\Omega K \xrightarrow{i} P_h \xrightarrow{p} B \xrightarrow{h} K$ gives rise to the exact sequence of homomorphisms*[17])

$$0 \to p^*(B, \Omega K)/h_p \pi_1^{P_h}(B; p) \xrightarrow{\kappa\#} \mathscr{R}_p(P_h) \to \mathscr{E}(h) \to 0.$$

where the first set has the standard group structure induced by ΩK. Furthermore if B has non zero homotopy groups only in the range $m \leqslant r \leqslant n-1$ then $\mathscr{R}_p(P_h) = \mathscr{E}(P_h)$.

This theorem contains as special cases theorems 2.1 and 3.2 of [10].

Now the function p^* induces an isomorphism

$$(B, \Omega K)/h_1 \pi_1^B(B; 1) \cup \pi^{*-1} t^*(S\Omega K, \Omega K) \to p^*(B, \Omega K)/h_p \pi_1^{P_h}(B; p)$$

where $t: P_h \cup C\Omega K \to S\Omega K$ is the quotient map and $\pi: P_h \cup C\Omega K \to B$ is the extension of p which is trivial on $C\Omega K$: details are given in § 6. Thus in case $(S\Omega K, \Omega K) = 0$, the exact sequence of the theorem may be rewritten in a basically simpler form where the first and last groups do not involve P_h. This is the situation that exists in the various stages of a Postnikov system for example. The following corollary is now proved.

COROLLARY 3.2.[18]) *Let B have non zero homotopy groups $\pi_r(B)$ only in a range $m \leqslant r \leqslant n-1$ $(m \geqslant 2)$, and let $K = K(G, s+1)$, $s \geqslant n$, be an Eilenberg-MacLane space. Then there is an exact sequence of homomorphisms:*

$$0 \longrightarrow H^s(B; G)/h_1 \pi_1^B(B; 1) \xrightarrow{\kappa\# p^*} \mathscr{E}(P_h) \longrightarrow \mathscr{E}(h) \longrightarrow 0$$

Other exact sequences relating the equivalences between various stages of a Postnikov system have been considered by Kahn [9], Arkowitz and Curjel [1] and Shih [16].

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[17]) In case B is an H-space, $h_p \pi^{P_h}{}_1 (B; p)$ may again be replaced by $\Delta(h,p)(P_h, \Omega B)$.

[18]) In case B is also an Eilenberg-MacLane space, then $\pi^B{}_1 (B; 1) = 0$ as in § 1.2 of [12], and this sequence is corollary 3.3 of [10] and is a correction to corollary 2 of [16].

The following corollary is a simple application of either of the principal fibre space theorems.

COROLLARY 3.3. *Let X have non zero homotopy groups only in a range $2 \leqslant r \leqslant N$, and let these groups be finite, then $\mathscr{E}(X)$ is finite.*

Dually with the restrictions on homology groups given below, each element b of $\mathscr{E}(C_h)$ determines unique classes of equivalences $\bar{c}: K \to K$ and $a: B \to B$ making the following diagram commutative

$$\begin{array}{ccccccc} K & \xrightarrow{h} & A & \to & C_h & \to & SK \\ \bar{c}\downarrow & & a\downarrow & & b\downarrow & & \downarrow Sc \\ K & \xrightarrow[h]{} & A & \to & C_h & \to & SK \end{array}$$

Thus there is an epimorphism $\sigma: \mathscr{E}(C_h) \to \mathscr{E}(h)$.

The following theorem is proved in § 6.

COPRINCIPAL COFIBRE SPACE THEOREM 3.1*. *Let A and K be simply connected and let A have non zero homology groups only in the range $m \leqslant r \leqslant n-2$ and SK have non zero homology groups only in the range*[19] *$n \leqslant r \leqslant m+n-2$ with $H_{n-2}(A)$ and $H_{m+n-2}(SK)$ free. Then the induced cofibre sequence $K \xrightarrow{h} A \xrightarrow{i} C_h \xrightarrow{p} SK$ gives rise to the exact sequence of homomorphisms*

$$0 \to i_*(SK, A)/h^i\pi_1^A(C_h; i) \xrightarrow{\lambda\#} \mathscr{E}(C_h) \xrightarrow{\sigma} \mathscr{E}(h) \to 0$$

where the first set has the standard group structure induced by SK.

This theorem contains theorem 6.1 of [2] as a special case.[20]

Also i_* induces an isomorphism

$$(SK, A)/h^1\pi_1^A(A; 1) \cup \iota_*^{-1}u_*(SK, \Omega SK) \to i_*(SK, A)/h^i\pi_1^A(C_h; i)$$

where $u: \Omega SK \to P_p$ is the inclusion of the fibre and $\iota: A \to P_p$: note that in case $n \leqslant r \leqslant m+n-3$, ι_* is an isomorphism. This equivalence can undoubtedly be used to simplify computation particularly if A is an abelian suspension in which case $h^1\pi_1^A(A; 1) = \Gamma(1, h)\,[SA, A]$ can be calculated by theorem 3.4.3 of [12]. There is, however, no dual to corollary 3.2 since, for a Moore space K, $(SK, \Omega SK)$ is not generally zero.

---

19) Because of the missing homology dimension emodied in these hypotheses, this theorem, as theorem 2.3*, is potentially less useful than its dual in the general theory.

20) A related exact sequence has also been obtained by Y. Kudo and K. Tsuchida (see theorems 2.2 and 2.8 of [18]).

4. Some Calculations

EXAMPLE 4.1. Let $X=K(\pi, m)\times K(G, n)$ with $m<n$. Then by corollary 3.2, there is an exact sequence

$$0\rightarrow H^n(K(\pi, m); G)\rightarrow \mathscr{E}(X)\rightarrow \operatorname{aut}\pi \times \operatorname{aut} G\rightarrow 0.$$

By direct considerations this sequence is a semi direct product and $\mathscr{E}(X)$ may be faithfully represented as the group of upper triangular matrices $\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}$ $c\in\operatorname{aut}\pi$, $a\in \operatorname{aut} G$ and $b\in H^n(K(\pi, m); G)$ with the usual operator structures and the usual matrix multiplication:

$$\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c'\end{pmatrix}=\begin{pmatrix} aa' & ab'+bc' \\ 0 & cc'\end{pmatrix}$$

and inverse

$$\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}^{-1}=\begin{pmatrix} a^{-1} & -a^{-1}bc^{-1} \\ 0 & c^{-1}\end{pmatrix}$$

EXAMPLE 4.2. Let X be the space induced by a cup product map $\bigcup : K(\pi, m)\times \times K(G, n)\rightarrow K(\pi\otimes G, m+n)$. (see § 2.2 of [12]). Assume $m<n$, then X has a three stage Postnikov system with first k invariant $k_1=*$. Let $p: P_{\cup}\rightarrow K(\pi, m)\times K(G, n)$ be the fibration induced by $\bigcup$ with fibre $K(\pi\otimes G, m+n-1)$. Denote by T the subgroup of triangular matrices $\begin{pmatrix} a & b \\ 0 & c\end{pmatrix}$ considered in example 4.1 for which $c\cup b=0$ in $H^{m+n}(K(\pi, m), \pi\otimes G)$: note that

$$cc'\cup(ab'+bc')=(c\otimes a)(c'\cup b')+(c\cup b)\,c'$$

and that

$$c^{-1}\cup(-a^{-1}bc^{-1})=-(c^{-1}\otimes a^{-1})(c\cup b)\,c^{-1}.$$

Clearly $\mathscr{E}(\bigcup)$ is canonically isomorphic to T. According to 2.3.2 and 2.5.1 of [12],

$$\Delta(\bigcup, 1): (K(\pi, m)\times K(G, n), K(\pi, m-1)\times K(G, n-1)) \rightarrow (K(\pi, m)\times K(G, n), K(\pi\otimes G, m+n-1))$$

is given by $\Delta(\bigcup, 1)\,x=(-1)^m\,\iota\cup x$ for[21] $x\in H^{n-1}(K(\pi, m), G)$, $\iota\in H^m(K(\pi, m), \pi)$ the fundamental class, and $\iota\cup x\in H^{m+n-1}(K(\pi, m), \pi\otimes G)$. On applying corollary

[21]) Using the usual representability of cohomology: see 2.2 and 2.5 of [12].

3.2, the following exact sequence is obtained

$$0 \to H^{m+n-1}(K(\pi, m), \pi \otimes G)/\iota \cup H^{n-1}(K(\pi, m), G) \oplus H^{m+n-1}(K(G, n), \pi \otimes G) \to \mathscr{E}(P_{\cup}) \to T \to 0.$$

Since T is fully described above, this determines $\mathscr{E}(P_{\cup})$ up to extension.

As an elementary example let $\bigcup : K(Z, 2) \times K(G, n) \to K(Z \otimes G, n+2)$, then, since the cohomology ring $H^*(K(Z, 2))$ is the polynomial ring generated by ι, the exact sequence is

$$0 \to \text{ext}(G, G) \to \mathscr{E}(P_{\cup}) \to T \to 0$$

where $T = Z_2 \times (\text{aut}\, G)$ in case $n \geqslant 3$ is odd, and in case $n \geqslant 4$ is even T is the semi direct product

$$0 \to G \to T \to Z_2 \times (\text{aut}\, G) \to 0$$

with structure $Z_2 \times \text{aut}\, G \to \text{aut}\, G$ given by $(\alpha, \beta) \to \beta\alpha$ where Z_2 acts on G in the non trivial way.

5. Fibre Homotopy Equivalence

The principal fibre space theorem 2.3 determines immediately the homotopy classes of fibre homotopy equivalences as an extension. With only minor changes in proof, the fibre homotopy classes can be similarly determined.

According to theorem 6.1 of [4] any homotopy equivalence $P_h \to P_h$ over $1 : B \to B$ is necessarily a fibre homotopy equivalence. Moreover it is elementary that the principal action $\Omega K \times P_h \to P_h$ gives an action of $(P_h, \Omega K)$ on the set of fibre homotopy classes $P_h \to P_h$ which is both effective and transitive thus determining a bijection. Let $\overline{\mathscr{R}}^1_p(P_h)$ denote the fibre homotopy classes of fibre homotopy equivalences with the group structure given by composition. Then, making minor amendments to the proof of theorem 2.3, the following theorem is obtained.

THEOREM 5.1. *Let $h : B \to K$ induce the principal fibre map $p : P_h \to B$ for which B has non vanishing homotopy groups only in the range $m \leqslant r \leqslant n-1$ and ΩK has non vanishing homotopy groups only in the range $n \leqslant r \leqslant m+n-1$, then there is an exact sequence of homomorphisms*

$$0 \to (B, \Omega K)/\pi^{*-1}\iota^*(S\Omega K, \Omega K) \to \overline{\mathscr{R}}^1_p(P_h) \xrightarrow{ib} (L+1) \cap \text{aut}\, \Omega K \to 0$$

where $L = \ker((S\Omega K, K) \to (P_h \cup C\Omega K, K))$.

For example if K is an Eilenberg-MacLane space then the set of fibre homotopy classes of equivalences which preserve the fibre of $P_h \to B$ is isomorphic to $(B, \Omega K)$ (cf. theorem 1 of [7]).

Dually let $\overline{\mathscr{L}}_i^1(C_h)$ denote the cofibre homotopy classes for a cofibre sequence $A \to C_h \to SK$. With the dimensional restrictions of the following theorem, any homotopy equivalence $C_h \to C_h$ extending $1: A \to A$ is necessarily a cofibre homotopy equivalence by using the standard cellular argument for example.

THEOREM 5.1.* *Let $h: K \to A$ induce the principal cofibre map $i: A \to C_h$ for which A and K are simply connected A has homology only in the range $m \leqslant r \leqslant n-2$ and SK has homology only in the range $n \leqslant r \leqslant m+n-3$ with $H_{n-2}(A)$ and $H_{m+n-3}(SK)$ free. Then there is an exact sequence of homomorphisms*

$$0 \to (SK, A)/\iota_*^{-1} u_*(SK, \Omega SK) \to \overline{\mathscr{L}}_i^1(C_h) \xrightarrow{pb} (L+1) \cap \operatorname{aut} SK \to 0$$

where $L = \ker((K, \Omega SK) \to (K, P_p))$.

6. Proofs of § 3

Some of the secondary lemmas needed in these proofs are delayed to § 7 and § 8. In particular § 8 contains several lemmas on obstruction theory which are used extensively here and elsewhere, often without explicit reference.

Proof of theorem 3.1. The sequence $\Omega K \xrightarrow{i} P_h \xrightarrow{p} B \xrightarrow{h} K$ with the stated conditions on homotopy groups gives rise to the following commutative diagram, where all the sequences are short exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{R}_p^1 \cap \mathscr{L}_i^1 & \to & \mathscr{L}_i^1 & \to & \mathscr{R}_h^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{R}_p^1 & \to & \mathscr{R}_p & \to & \mathscr{L}_p = \mathscr{R}_h & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}_h^1 & \to & \mathscr{L}_h & \to & \mathscr{L}_h/\mathscr{L}_h^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

To see this observe first of all that $\mathscr{L}_p^1 = 1$ by lemma 7.5. Also $\mathscr{R}_h \subset \mathscr{L}_p$ by the standard construction. Conversely each lifting equivalence of p lifts to a fibre homotopy equivalence by 6.1 of [4] whose restriction to ΩK is therefore an equivalence (not uniquely determined) which is the loop of an equivalence of K since $(K, K) \to (\Omega K, \Omega K)$ is bijective by lemma 7.3; it is now immediate from theorem 4.3 of [15] that this is a retraction of the original equivalence of B, which proves $\mathscr{L}_p = \mathscr{R}_h$. Thus the second and third horizontal sequences and the third vertical sequence are exact by the $\mathscr{L} - \mathscr{R}$ duality theorem 2.2 of [13]. Consider now the second vertical sequence; since

$(\Omega K, \Omega B)=0$ by obstruction theory, the obvious homomorphism $\mathscr{R}_p \to \mathscr{R}_i \cap \mathscr{L}_{\Omega h}$ is well defined by 2.8′ of [17] applied to the fibre sequence of $P_p \to P_h$. De-looping the image gives, since $(K, K) \to (\Omega K, \Omega K)$ is bijective, an element of $\mathscr{L}_h$ by theorem 4.3 of [15]. It is now clear that the first and second vertical sequences are exact. According to theorems 4.1 and 4.3 of [15] an element of $\mathscr{L}_i^1$ determines a unique element of $\mathscr{R}_h^1$ and clearly the first horizontal sequence is exact. The diagram is commutative. In the usual way then the sequence[22])

$$0 \to \mathscr{R}_p^1 \cap \mathscr{L}_i^1 \to \mathscr{R}_p \to \mathscr{L}_h \times \mathscr{R}_h \xrightarrow{(h^*,\ -h_*)} \mathscr{L}_h / \mathscr{L}_h^1 \to 0$$

is exact in the category of sets and all functions except $(h^*, -h_*)$ are linear. The image of $\mathscr{R}_p \to \mathscr{L}_h \times \mathscr{R}_h$ is $\mathscr{E}(h)$, which proves the exactness of the sequence of homomorphisms

$$0 \to \mathscr{R}_p^1 \cap \mathscr{L}_i^1 \to \mathscr{R}_p \to \mathscr{E}(h) \to 0 .$$

Consider now the following commutative diagram of track group homomorphisms:

$$\begin{array}{ccccc}
\pi_1^{P_h \cup C\Omega K}(B;\pi) & \xrightarrow{j^*} & \pi_1^{P_h}(B;p) & \xrightarrow{i^*} & \pi_1^{\Omega K}(B;pi) \\
\uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle 1} & & \\
\pi_1^B(B;1) & \xrightarrow{p^*} & \pi_1^{P_h}(B;p) & & \\
\downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} & \searrow{\scriptstyle h_p} & \\
\pi_1^B(K;h) & \xrightarrow{p^*} & \pi_1^{P_h}(K;hp) & \xrightarrow{(hp)_\#} & (P_h, \Omega K)
\end{array}$$

The top sequence is exact by theorem 1 of [14] and thus, since $\pi_1^{\Omega K}(B;pi) \approx (\Omega K, \Omega B)=0$, j^* is an epimorphism. By theorem 1.1 of [5] the function $\pi: P_h \cup C\Omega K \to B$ is $(m+n)$ connected and hence, by lemma 8.2, π^* is an isomorphism. It is now clear that $h_p \pi_1^{P_h}(B;p) \subset (hp)_\# \, p^* \, \pi_1^B(K;h)$. In the next commutative diagram the first and fourth sequences are exact (see theorem 1 of [14] for the first), and each of the functions π^* is an isomorphism as above.

$$\begin{array}{ccccccc}
\pi_1^{P_h \cup C\Omega K}(K;h\pi) & \xrightarrow{j^*} & \pi_1^{P_h}(K;hp) & \xrightarrow{i^*} & \pi_1^{\Omega K}(K;hpi) \\
\uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} \\
\pi_1^B(K;h) & \xrightarrow{p^*} & \pi_1^{P_h}(K;hp) & \to & \pi_1^{\Omega K}(K;hpi) \\
\downarrow{\scriptstyle h\flat} & & \downarrow{\scriptstyle (hp)_\#} & & \downarrow{\scriptstyle (hpi)_\#} \\
(B, \Omega K) & \to & (P_h, \Omega K) & \to & (\Omega K, \Omega K) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
(S\Omega K, \Omega K) \xrightarrow{t^*} (P_h \cup C\Omega K, \Omega K) & \xrightarrow{j^*} & (P_h, \Omega K) & \to & (\Omega K, \Omega K)
\end{array}$$

---

22) There is also the exact sequence

$0 \to \mathscr{R}_p^1 \cup \mathscr{L}_i^1 \to \mathscr{L}_i^1 \times \mathscr{R}_p^1 \to \mathscr{R}_p \to \mathscr{L}_h / \mathscr{L}_h^1 \to 0$

which is clearly related to the first principal fibre space theorem 2.3.

Now $(hp)_{\#}\, p^* \pi_1^B(K; h) = p^*(B, \Omega K)$ from the exactness of the second and third sequences, and $i_\flat^{-1}(1) = i^{*-1}(0) = p^*(B, \Omega K)$. It is clear that $p^*: (B, \Omega K) \to (P_h, \Omega K)$ is anti-linear with respect to the $\times$ structure on $(P_h, \Omega K)$, and thus is linear when K is an H-space; and also that $\kappa_{\#}: p^*(B, \Omega K) \to \mathscr{R}_p^1 \cap \mathscr{L}_i^1$ is an epimorphism with kernel $h_p\, \pi_1^{P_h}(B; p)$. This completes the proof.

Proof of corollary 3.2. In the particular case that K is an Eilenberg–MacLane space, $(S\Omega K, \Omega K) = 0$ and thus $p^*: (B, \Omega K) \to (P_h, \Omega K)$ is a monomorphism by the exactness of the lower sequence. This proves that $\kappa_{\#}\, p^*: (B, \Omega K) \to \mathscr{R}_p^1 \cap \mathscr{L}_i^1$ is an epimorphism with kernel $h_1\, \pi_1^B(B; 1)$.

Proof of theorem 3.1*. The sequence $K \xrightarrow{h} A \xrightarrow{i} C_h \xrightarrow{p} SK$, with the stated homology conditions, gives rise to the following diagram where all the sequences are short exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}_i^1 \cap \mathscr{R}_p^1 & \to & \mathscr{R}_p^1 & \to & \mathscr{L}_h^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}_i^1 & \to & \mathscr{L}_i & \to & \mathscr{R}_i = \mathscr{L}_h & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{R}_h^1 & \to & \mathscr{R}_h & \to & \mathscr{R}_h/\mathscr{R}_h^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The proof of this basically dual to that of theorem 3.1, using the usual arguments on weak homotopy equivalence in place of Dold's result; the duals of the results referred to in the proof of theorem 3.1 are to be found in the same places as their counterparts. Thus the exact sequence of homomorphisms is obtained:

$$0 \to \mathscr{L}_i^1 \cap \mathscr{R}_p^1 \to \mathscr{L}_i \to \mathscr{E}(h) \to 0.$$

Now by lemma 3.1 of [5] $\iota: A \to P_p$ is $m+n-2$ connected. In each of the following two commutative diagrams of homomorphisms the homomorphisms ι_* are epi by lemma 8.3*. In the first diagram the top sequence and in the second diagram all horizontal sequences are exact.

$$\begin{array}{ccccc}
\pi_1^A(P_p; \iota) & \xrightarrow{v_*} & \pi_1^A(C_h; i) & \xrightarrow{p_*} & \pi_1^A(SK; pi) \\
\iota_* \uparrow & & \uparrow 1 & & \\
\pi_1^A(A; 1) & \xrightarrow{i_*} & \pi_1^A(C_h; i) & & \\
h^* \downarrow & & \downarrow h^* & \searrow^{h^i} & \\
\pi_1^K(A; h) & \xrightarrow{i_*} & \pi_1^K(C_h; ih) & \xrightarrow[(ih)^{\#}]{} & (SK, C_h)
\end{array}$$

$$\begin{array}{ccccc}
 & \pi_1^K(P_p; ih) & \xrightarrow{v_*} \pi_1^K(C_h; ih) & \xrightarrow{p_*} \pi_1^K(SK; pih) \\
 & {\scriptstyle i_*}\uparrow & \quad 1\uparrow & \quad 1\uparrow \\
 & \pi_1^K(A; h) & \xrightarrow{i_*} \pi_1^K(C_h; ih) & \rightarrow \pi_1^K(SK; pih) \\
 & {\scriptstyle h\flat}\downarrow & \quad {\scriptstyle (ih)_\#}\downarrow & \quad {\scriptstyle (pih)_\#}\downarrow \\
 & (SK, A) & \xrightarrow{i_*} (SK, C_h) & \rightarrow (SK, SK) \\
 & {\scriptstyle i_*}\downarrow & \quad 1\downarrow & \quad 1\downarrow \\
(SK, \Omega SK) \xrightarrow{u_*} & (SK, P_p) & \xrightarrow{v_*} (SK, C_h) & \rightarrow (SK, SK)
\end{array}$$

Thus as before $h^i\ \pi_1^A(C_h;\ i) \subset (ih)_\#\ i_*\ \pi_1^K(A; h)$ since $\pi_1^A(SK; pi) \approx (SA, SK) = 0$; also $(ih)_\# i_* \pi_1^K(A; h) = i_*(SK, A) = p^{b-1}(1)$. In this case $i_*: (SK, A) \to (SK, C_h)$ is linear with respect to the $\times$-structure on (SK, C_h); and $\lambda^\#: i_*(SK, A) \to \mathscr{L}_i^1 \cap \mathscr{R}_p^1$ is an epimorphism with kernel $h^i\ \pi_1^A(C_h; i)$. The theorem now follows from lemma 7.4*.

7. Several Lemmas

The proofs of the lemmas in this section rely on obstruction theory (see § 8).

LEMMA 7.1. *Let ΩK be path connected and have non vanishing homotopy groups*[23]) $\pi_r(\Omega K)$ *only in a range*[24]) $n \leqslant r \leqslant 2n-1$, *then* $\mathscr{E}(\Omega K) = \mathrm{aut}(\Omega K)$.

Proof. A proof is given which can be roughly dualized. Consider the short exact sequence of homomorphisms

$$0 \to (S(\Omega K \circledast \Omega K), K) \xrightarrow{(S\pi)^*} (S(\Omega K \times \Omega K), K) \xrightarrow{(S\tau)^*} (S(\Omega K \vee \Omega K), K) \to 0$$
$$\| $$
$$(S\Omega K, K) \times (S\Omega K, K)$$

Let $j_r: \Omega K \vee \Omega K \to \Omega K \to \Omega K \times \Omega K$ $r = 1, 2$ be given by $j_1(p, q) = (p, *)$ and $j_2(p, q) = (*, q)$, then $(Sj_1)^* + (Sj_2)^*$ is a (non linear) splitting for the sequence in the sense that $(S\tau)^*\ \{(Sj_1)^* + (Sj_2)^*\} = 1$. Given $g: \Omega K \to \Omega K$, let $\bar{g}: S\Omega K \to K$ be its adjoint and consider the element $\bar{g}(Sm_{\Omega K}) = \bar{g}\bar{m}_{\Omega K}: S(\Omega K \times \Omega K) \to S\Omega K \to K$. Clearly $\bar{g}(Sm_{\Omega K})\ (S\tau) = \bar{g}(Sc)$ where $c: \Omega K \vee \Omega K \to \Omega K$ is the folding map. Now $w = \bar{g}(Sm_{\Omega K}) - \bar{g}(Sc)\ (Sj_1 + Sj_2)$ is the obstruction to the linearity of g; g is linear if and only if $w = 0$. Also $(S\tau)^* w = 0$ and therefore $w \in (S\pi)^*\ (S(\Omega K \circledast \Omega K), K)$; this latter group is zero under the conditions indicated on the homotopy groups by the usual obstruction theory argument based on a Postnikov system for ΩK. The lemma is immediate.

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[23]) Including $r = 0$.

[24]) The proof in case $n = 1$ is elementary.

LEMMA 7.2. *Let P_h be $(m-1)$ connected and let ΩK have non vanishing homotopy groups $\pi_r(\Omega K)$ only in a range $n \leqslant r \leqslant m+n-1$ then $(P_h, \Omega K) = \mathscr{T}_\kappa(P_h, \Omega K)$.*

Proof. Again a proof is given which can be dualized. Consider the short exact sequence of homomorphisms

$$0 \to (S(\Omega K \circledast P_h), K) \xrightarrow{(S\pi)^*} (S(\Omega K \times P_h), K) \xrightarrow{(S\tau)^*} (S(\Omega K \vee P_h), K) \to 0$$
$$\| $$
$$(S\Omega K, K) \times (SP_h, K)$$

With the given connectivity conditions the left hand group is zero. Thus, as in the proof of 7.1, for $\xi : P_h \to \Omega K$

$$\begin{aligned} \overline{\xi\kappa} &= \bar{\xi}(S\kappa)(S\tau)(Sj_1 + Sj_2) \\ &= \overline{\xi i p_1} + \overline{\xi p_2} \\ &= \overline{\xi i p_1 + p_1 + \xi p_2 - p_1} \end{aligned}$$

LEMMA 7.3. *The loop map $(K, K) \to (\Omega K, \Omega K) \approx (S\Omega K, K)$ is bijective for ΩK having non zero homotopy groups only in a range $n \leqslant r \leqslant 2n$ $(n \geqslant 2)$.*

Proof. According to 3.2 of [3], the fibre of the evaluation $S\Omega K \to K$ is $S(\Omega K \circledast \Omega K)$ which is $2n$ connected. Applications of the Whitehead theorem, the universal coefficient theorem, and obstruction theory on a Postnikov system for K, now yield the result.

LEMMA 7.4. (Corollary 1 of [15]) *Let B have homotopy groups only in the range $2 \leqslant r \leqslant n-1$, and let ΩK be $(n-1)$ connected, then $\mathscr{E}(P_h) = \mathscr{R}_p(P_h)$.*

LEMMA 7.5. *Let B have homotopy groups only in the range $2 \leqslant r \leqslant n$ and let ΩK be $(n-1)$ connected then $\mathscr{L}_p^1(B) = 1$.*

Proof. The function $(B, B) \to (P_h, B)$ is injective by obstruction theory based on a Postnikov decomposition for B: the result is immediate.

LEMMA 7.1*. *Let SK be $(n-1)$ connected and have non zero integral homology $H_r(SK)$ only for $r \leqslant 2n-3$, or for $r \leqslant 2n-2$ if also*

$$\operatorname{ext}(H_{2n-2}(SK), \pi_n(SK) \otimes \pi_n(SK)) = 0, \quad \textit{then} \quad \mathscr{E}(SK) = \operatorname{aut} SK.$$

Proof. Consider the short exact sequence of homomorphisms

$$0 \to (K, \Omega(SK \flat SK)) \to (K, \Omega(SK \vee SK)) \to (K, \Omega(SK \times SK)) \to 0$$
$$\|$$
$$(K, \Omega SK) \times (K, \Omega SK)$$

Now $(SK \flat SK)$ has the weak homotopy type of $S(\Omega SK \circledast \Omega SK)$ (see § 2 of [5]), which is $2n-2$ connected; and by the usual arguments in obstruction theory using the universal coefficient and Kunneth theorems, the left hand group is zero. The remainder of the proof is dual to that of lemma 7.1.

LEMMA 7.2*. *Let* C_h *and* SK *be respectively* $(m-1)$ *and* $(n-1)$ *connected and let* SK *have non zero integral homology groups* $H_r(SK)$ *only for* $r \leqslant m+n-3$, *or for* $r \leqslant m+n-2$ *if also* $\operatorname{ext}(H_{m+n-2}(SK), \pi_n(SK) \otimes \pi_m(C_h))=0$, *then* $(SK, C_h)=\mathscr{T}_\lambda(SK, C_h)$.

Proof. Consider the short exact sequence of homomorphisms

$$0 \to (K, \Omega(SK \flat C_h)) \to (K, \Omega(SK \vee C_h)) \to (K, \Omega(SK \times C_h)) \to 0$$
$$\| $$
$$(K, \Omega SK) \times (K, \Omega C_h).$$

As above $(SK \flat C_h)$ has the weak homotopy type of $S(\Omega SK \circledast \Omega C_h)$ and the first group in the sequence is zero. The proof is now dual to that of lemma 7.2.

LEMMA 7.3*. *The suspension map* $(K, K) \to (SK, SK) \approx (K, \Omega SK)$ *is bijective for* K *simply connected and* SK *having non zero homology groups only in a range* $n \leqslant r \leqslant 2n-4$ *with* $H_{2n-4}(SK)$ *free.*

Proof. According to the generalized *EHP* sequence (see 5.4 of [5]) $\pi_{3n-5}(K) \to \to \pi_{3n-5}(\Omega SK) \to \pi_{3n-5}(K \circledast K) \to \cdots$, the map $K \to \Omega SK$ is $(2n-3)$-connected. The result follows by obstruction theory.

LEMMA 7.4*. (Corollary 1* of [15]). *Let* A *be simply connected and have homology groups only in the range* $2 \leqslant r \leqslant n-1$ *with* $H_{n-1}(A)$ *free and let* SK *be* $(n-1)$-*connected then* $\mathscr{E}(C_h)=\mathscr{L}_i(C_h)$.

LEMMA 7.5*[25]. *Let* $A \to X$ *be m-connected and let* A *be simply connected and have non zero homology groups only in the range* $r \leqslant m-1$ *with* $H_{m-1}(A)$ *free,*[26] *then* $\mathscr{R}_i^1(A)=1$.

Proof. The function $(A, A) \to (A, X)$ is bijextive by obstruction theory.

8. Obstruction Theory

The present section contains several lemmas on obstruction theory which are used extensively in the previous proofs. The first four are standard: their proofs are elementary and are omitted.

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[25]) This lemma is weaker than its dual; in particular it does not allow consecutive homology dimensions for A and X/A.

[26]) This situation occurs for example when $(m+1)$ cells are attached to a complex of dimension $(m-1)$.

LEMMA 8.1. *Let P be m connected ($m \geqslant 1$), and let Q have non vanishing homotopy groups only in the range $1 \leqslant r \leqslant m$, then $(P, Q) = 0$.*

LEMMA 8.1*. *Let Q be m connected ($m \geqslant 1$) and P simply connected with non zero homology only in the range $1 \leqslant r \leqslant m$ with $\operatorname{ext}(H_m(P), \pi_{m+1}(Q)) = 0$, then $(P, Q) = 0$.*

LEMMA 8.2. *Let $f: A \to B$ be m-connected ($m \geqslant 1$), and let X have non zero homotopy groups only in the range $1 \leqslant r \leqslant n$, then $f^*: (B, X) \to (A, X)$ is bijective for $n \leqslant m-1$ and injective for $n \leqslant m$.*

LEMMA 8.2*. *Let $f: A \to B$ be n connected ($n \geqslant 1$), and let X be simply connected with non zero homology only in the range $2 \leqslant r \leqslant n$ with $H_n(X)$ free, then $(X, A) \to (X, B)$ is surjective. If further X has non zero homology only in the range $2 \leqslant r \leqslant n-1$ and $\operatorname{ext}(H_{n-1}(B), k) = 0$ where k is the kernel of the epimorphism $\pi_n(A) \to \pi_n(B)$, then $(X, A) \to (X, B)$ is bijective.*

LEMMA 8.3. *Let $f: A \to B$ be n-connected ($n \geqslant 1$), let X have non zero homotopy groups only the range $2 \leqslant r \leqslant m$, and let $g: B \to X$; then $f^*: \pi_s^B(X; g) \to \pi_s^A(X; gf)$ ($s \geqslant 1$) is mono for $m-s \leqslant n$ and iso for $m-s+1 \leqslant n$.*

Proof. Let $\beta_r: X_r \to K_{r+2}(\pi_{r+1}(X))$ be the k-invariant for a Postnikov decomposition of X, and let $g_r: B \xrightarrow{g} X \to X_r$. Consider the following commutative diagram.

$$
\begin{array}{ccc}
\pi_s^B(X_r; g_r) & \xrightarrow{\quad f^* \quad} & \pi_s^A(X_r; g_r f) \\
\downarrow & & \downarrow \\
\pi_s^B(K_{r+2}; \beta_r g_r) & \longrightarrow & \pi_s^A(K_{r+2}; \beta_r g_r f) \\
\Big| \quad {\scriptstyle (\beta_r g_r)^\flat} \searrow \approx & & \approx \swarrow {\scriptstyle (\beta_r g_r f)^\flat} \quad \Big| \\
\downarrow & H^{r-s+2}(B; \pi_{r+1}(X)) \to H^{r-s+2}(A; \pi_{r+1}(X)) & \downarrow \\
\pi_{s-1}^B(X_{r+1}; g_{r+1}) & \longrightarrow & \pi_{s-1}^A(X_{r+1}; g_{r+1} f) \\
\downarrow & & \downarrow \\
\pi_{s-1}^B(X_r; g_r) & \longrightarrow & \pi_{s-1}^A(X_r; g_r f) \\
\downarrow & & \downarrow \\
\pi_{s-1}^B(K_{r+2}; \beta_r g_r) & \longrightarrow & \pi_{s-1}^A(K_{r+2}; \beta_r g_r f) \\
{\scriptstyle (\beta_r g_r)^\flat} \searrow \approx & & \approx \swarrow {\scriptstyle (\beta_r g_r f)^\flat} \\
 & H^{r-s+3}(B; \pi_{r+1}(X)) \to H^{r-s+3}(A; \pi_{r+1}(X)) &
\end{array}
$$

The vertical sequences are exact (see theorem 1 of [14]) and the diagram is commutative by lemma 1.2.2 iv) of [12]. Also $f^*: H^t(B; \pi_{r+1}) \to H^t(A; \pi_{r+1})$ is iso for $t \leqslant n-1$ and mono for $t = n$; thus by an induction on r using the five lemma, the function $f^*: \pi_{s-1}^B(X_{r+1}; g_{r+1}) \to \pi_{s-1}^A(X_{r+1}; g_{r+1} f)$ is epi for $r-s+3 \leqslant n$, and by a further induction on r is mono for $r-s+2 \leqslant n$.

LEMMA 8.3*. *Let $f: A \to B$ be n-connected* ($n \geqslant 1$), *let X be a simply connected m dimensional* CW *complex, and let $g: X \to A$; then $f_*: \pi_s^X(A; g) \to \pi_s^X(B; fg)$* ($s \geqslant 1$) *is iso for $m+s \leqslant n-1$ and epi for $m+s \leqslant n$.*

Proof. Let $\alpha_r: \vee S^r \to X^r$ be the attaching map and let $g_r = f \mid X^r$. Consider the following commutative diagram.

$$
\begin{array}{ccccc}
\pi_s^{X^r}(A; g_r) & & \xrightarrow{\quad f_* \quad} & & \pi_s^{X^r}(B; fg_r) \\
\downarrow{\scriptstyle \alpha^*_r} & & & & \downarrow \\
\pi_s^{\vee S^r}(A; g_r\alpha_r) & & \longrightarrow & & \pi_s^{\vee S^r}(B; fg_r\alpha_r) \\
\Big\downarrow & {\scriptstyle (g_r\alpha_r)^\flat} \searrow \approx & & \approx \swarrow {\scriptstyle (fg_r\alpha_r)^\flat} & \Big\downarrow \\
 & & \Pi\pi_{r+s}(A) \to \Pi\pi_{r+s}(B) & & \\
\pi_{s-1}^{X^{r+1}}(A; g_{r+1}) & & \longrightarrow & & \pi_{s-1}^{X^{r+1}}(B; fg_{r+1}) \\
\downarrow & & & & \downarrow \\
\pi_{s-1}^{X^r}(A; g_r) & & \longrightarrow & & \pi_{s-1}^{X^r}(B; fg_r) \\
\downarrow & & & & \downarrow \\
\pi_{s-1}^{\vee S^r}(A; g_r\alpha_r) & & \longrightarrow & & \pi_{s-1}^{\vee S^r}(B; fg_r\alpha_r) \\
 & {\scriptstyle (g_r\alpha_r)^\flat} \searrow \approx & & \approx \swarrow {\scriptstyle (fg_r\alpha_r)^\flat} & \\
 & & \Pi\pi_{r+s-1}(A) \to \Pi\pi_{r+s-1}(B) & &
\end{array}
$$

The vertical sequences are exact (see theorem 1* of [14]) and the diagram is commutative by lemma 1.2.2iv) of [12]. Also $f_*: \pi_{r+s}(A) \to \pi_{r+s}(B)$ is iso for $r+s \leqslant n-1$ and epi for $r+s=n$. An induction on r, using the five lemma, proves that

$$f_*: \pi_{s-1}^{X^{r+1}}(A; g_{r+1}) \to \pi_{s-1}^{X^{r+1}}(B; fg_{r+1})$$

is epi for $r+s \leqslant n$, and another induction on r then proves that the same function is mono for $r+s \leqslant n-1$. The result now follows.

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