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Some Results on the Fourier-Stieltjes Algebra of a Locally Compact Group

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1. Introduction

This paper is concerned with the study of the set $B(G)$ of all finite complex linear combinations of continuous positive definite functions over an arbitrary locally compact group G . We prove (corollary of theorem 6) that on the boundary of the unit sphere of $B(G)$ ([5]), the compact-open topology and the weak topology $\sigma(B(G), L^1(G))$ coincide (we consider $L^1(G)$ with left Haar measure dx). This extends the following theorem due to D. A. Raikov ([2] p. 260–261): the above topologies coincide on the set of all continuous positive definite functions with value 1 at the unit element e of G . We need, for this purpose, some general results on C^* -algebras which are developed in section 2.

As an application we give a new proof in section 4 of a result of H. Leptin ([6]) concerning the existence of an approximate identity in the Fourier algebra $A(G)$ of G ([5]). More precisely, we obtain directly the existence of an approximate identity of the following type: for every $u \in A(G)$ and $\varepsilon > 0$ there exists $v \in A(G)$ with $\|v\| = 1$ such that $\|u - uv\| < \varepsilon$ implies that the group G has property (R). We recall that a locally compact group G is said to have property (R) if the constant 1 on G can be approximated uniformly on compact sets by functions of the form $x \mapsto \int_G k(xy) \overline{k(y)} dy$, where k is a continuous complex-valued function on G with compact support.

Finally in section 5 we extend the following multiplier theorem to every locally compact group G , satisfying property (R): if $u \in C^G$ and if $uf \in B(G)$ for every $f \in A(G)$, then $u \in B(G)$. Moreover, Prof. P. Eymard told me that, under the same assumptions, we have

$$\|u\| = \text{Sup} \{ \|uf\| \mid f \in A(G), \|f\| \leq 1 \}.$$

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2. Some Remarks on the Polar Decomposition of Linear Continuous Functionals on a C^* -Algebra

Let A be a C^* -algebra, A' the set of all linear continuous functionals on A . For

$f \in A'$ define

$$\|f\| = \sup \{|f(x)| \mid \|x\| \leq 1, x \in A\}.$$

By ([2] p. 243) there is only one positive linear functional $|f|$ on A , such that $\|f\| = \||f|\|$, and $|f(x)|^2 \leq \|f\| \cdot |f|(xx^*)$. Consider on A' the locally convex topology τ defined by the following systems of sets:

$$U_\tau(f_0; \varepsilon, x_1, \dots, x_n) = \{f \in A' \mid |f(x_j) - f_0(x_j)| < \varepsilon \quad 1 \leq j \leq n, \|\|f\| - \|f_0\|\| < \varepsilon\}$$

where $f_0 \in A'$, $\varepsilon > 0$ and $\{x_1, \dots, x_n\} \subset A$.

PROPOSITION 1. *The map $f \mapsto |f|$ is a continuous map of (A', τ) in $(A', \sigma(A', A))$.*

This proposition was proved in ([3] Lemma 3.5).

Remark. The result remains valid if for the definition of the topology τ one takes the $\{x_1, \dots, x_n\}$ in an arbitrary dense subspace of A .

PROPOSITION 2. *To every $f \in A'$ there exists a $*$ -representation of A (π, H) with cyclic vector ξ such that $|f|(x) = (\pi(x)\xi, \xi)$. There also exists a unique $\zeta \in H$ with $f(x) = (\pi(x)\zeta, \xi)$. We have*

$$\|f\|^{1/2} = \|\zeta\| = \|\xi\|.$$

Proof. On the dense subspace $\pi(A)\xi$ consider the linear functional defined by

$$F(\pi(x^*)\xi) = \overline{f(x)}.$$

The inequality

$$|f(x)|^2 \leq \|f\| \|\pi(x^*)\xi\|^2$$

implies the continuity of F and therefore the existence and unicity of $\zeta \in H$ with

$$\overline{f(x)} = (\pi(x^*)\xi, \zeta)$$

i.e.

$$f(x) = (\pi(x)\zeta, \xi).$$

From

$$\sup_{\|\pi(x^*)\xi\| \leq 1} |(\pi(x^*)\xi, \zeta)| = \|\zeta\|$$

there follows $\|\zeta\| \leq \|f\|^{1/2}$. On the other hand, the inequalities

$$|f(x)| \leq \|\pi(x^*)\xi\| \|\zeta\| \leq \|x\| \|\xi\| \|\zeta\|$$

imply

$$|f(x)| \leq \|x\| \|f\|^{1/2} \|\zeta\|,$$

that is $\|f\|^{1/2} \leq \|\zeta\|$, and finally $\|f\|^{1/2} = \|\zeta\|$.

Remark. Suppose that (π', H') is a $*$ -representation of A with cyclic vector ξ' , such that $|f|(x) = (\pi'(x)\xi', \xi')$. Let be ζ' the corresponding vector of H' with $f(x) = (\pi'(x)\zeta', \xi')$. It is well known that there exists only one linear isometry ψ of H onto H' with $\psi\pi(x)\xi = \pi'(x)\xi'$. It is easy to verify that $\psi(\zeta) = \zeta'$.

PROPOSITION 3. For every $f \in A'$, and $x, y \in A$, we have

$$|f(x) - f(y)|^2 \leq \|f\| [\|f\| (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f|(yx^*)].$$

Proof. Consider π, H, ξ and ζ as in proposition 2. We have

$$f(x) = (\zeta, \pi(x^*)\xi);$$

that is

$$|f(x - y)|^2 = |(\zeta, \pi(x^* - y^*)\xi)|^2 \leq \|\zeta\|^2 \|\pi(x^* - y^*)\xi\|^2.$$

From

$$\|\pi(x^* - y^*)\xi\|^2 = \|\pi(x^*)\xi\|^2 + \|\pi(y^*)\xi\|^2 - 2 \operatorname{Re} (\pi(x^*)\xi, \pi(y^*)\xi)$$

and

$$(\pi(x^*)\xi, \pi(y^*)\xi) = |f|(yx^*)$$

we obtain

$$\|\pi(x^* - y^*)\xi\|^2 \leq \|\xi\|^2 (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f|(yx^*)$$

and therefore

$$|f(x - y)|^2 \leq \|f\| [\|f\| (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f|(yx^*)].$$

Remark. To every $f \in A'$ there corresponds a unique positive linear functional $p(f)$ on A such that

$$\|f\| = \|p(f)\| \quad \text{and} \quad |f(x)|^2 \leq \|f\| p(f)(x^*x).$$

We have $p(f) = |f^*|$; for $p(f)$ proposition 1 remains valid. Proposition 2 has to be modified in the following way: Let (π_p, H_p) a $*$ -representation of A with cyclic vector ξ_p such that $p(f)(x) = (\pi_p(x)\xi_p, \xi_p)$. Then there exists a unique $\zeta_p \in H_p$ with $f(x) = (\pi_p(x)\xi_p, \zeta_p)$ etc. ... The inequality of proposition 3 becomes

$$|f(x) - f(y)|^2 \leq \|f\| [\|f\| (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} p(f)(y^*x)].$$

3. Topologies on the Fourier-Stieltjes Algebra of a Locally Compact Group

In the sequel, G is an arbitrary locally compact group, $C_{00}(G)$ the set of all complex-valued continuous functions on G with compact support, $C^b(G)$ the set of all continuous complex-valued bounded functions on G and $P(G)$ the set of all continuous positive definite functions. For $f \in C^b(G)$ let \tilde{f} denote the function $\tilde{f}(x) = \overline{f(x^{-1})}$. Let $C^*(G)$ be the C^* -algebra of the group G . The isomorphism of $B(G)$ and $C^*(G)'$ permits us (see [5] p. 192–193) to define on $B(G)$ a norm $u \mapsto \|u\|$ and an "absolute value" $u \mapsto |u|$.

PROPOSITION 4. *To every $u \in B(G)$ there corresponds a unitary continuous representation (π, H) with cyclic vector ξ , such that $|u|(x) = (\pi(x)\xi, \xi)$ and with a unique vector $\zeta \in H$ such that $u(x) = (\pi(x)\zeta, \xi)$. We have, moreover,*

$$\|\zeta\| = \|u\|^{1/2} = \|\xi\| = |u|(e)^{1/2}.$$

This proposition is a direct consequence of prop. 2. A similar result is proved directly in ([5] p. 195).

PROPOSITION 5. *Every $u \in B(G)$ satisfies the inequality*

$$|u(x) - u(y)|^2 \leq 2 \|u\| [|u|(e) - \operatorname{Re} |u|(yx^{-1})],$$

for all $x, y \in G$.

Proof. The proof is similar to that of prop. 3. It is enough to choose π, H, ξ and ζ as in prop. 4. From

$$u(x) = (\zeta, \pi(x^{-1})\xi)$$

follows

$$|u(x) - u(y)|^2 \leq \|\zeta\|^2 \|\pi(x^{-1})\xi - \pi(y^{-1})\xi\|^2.$$

On the other hand,

$$\|\pi(x^{-1})\xi - \pi(y^{-1})\xi\|^2 = 2\|\xi\|^2 - 2\operatorname{Re}(\pi(x^{-1})\xi, \pi(y^{-1})\xi)$$

and

$$\|\xi\|^2 = |u|(e) = \|\zeta\|^2 = \|u\|$$

imply the required inequality.

Remarks

- (1) The same inequality is well known for elements of $P(G)$.
- (2) It is possible to consider proposition 5 as a special case of proposition 3.

Consider the topology (denoted τ) on $B(G)$ defined by the following systems of sets:

$$U_\tau(u_0; \varepsilon, f_1, \dots, f_n) = \{u \in B(G) \mid |\int f_j(x) (u(x) - u_0(x)) dx| < \varepsilon \quad 1 \leq j \leq n$$

and

$$\{ \|u\| - \|u_0\| < \varepsilon \}$$

where $u_0 \in B(G)$, $\varepsilon > 0$ and $\{f_1, \dots, f_n\} \subset L^1(G)$.

THEOREM 6. *The topology τ on $B(G)$ is stronger than the compact-open topology.*

Proof. Given $u_0 \in B(G)$, $\varepsilon > 0$ and $K \in \mathfrak{K}$ (\mathfrak{K} denotes the set of all compact subsets of G), we have to show the existence of $\lambda \in \mathbb{R}^+$ and $\{h_1, \dots, h_k\} \subset L^1(G)$ such that

$$\begin{aligned} U_\tau(u_0; \lambda, h_1, \dots, h_k) &\subset U(u_0; \varepsilon, K) \\ &= \{u \in B(G) \mid |u(x) - u_0(x)| < \varepsilon \text{ for every } x \in K\}. \end{aligned}$$

Consider a compact neighbourhood V of e , such that $x \in V$ implies

$$||u_0|(x) - |u_0|(e)| < \frac{\varepsilon^2}{54(1 + \|u_0\|)}.$$

Let Φ_V denote the function on G defined as 1 on V and 0 outside. For every $x \in G$ and $u \in B(G)$ we have

$$m(V)^{-1} \Phi_V * u(x) - u(x) = m(V)^{-1} \int \Phi_V(t) (u(t^{-1}x) - u(x)) dt$$

(where $m(V)$ is $\int \Phi_V(t) dt$). From prop. 5 follows

$$|m(V)^{-1} \Phi_V * u(x) - u(x)| \leq 2^{1/2} m(V)^{-1} \int_V \|u\|^{1/2} (|u|(e) - \operatorname{Re} |u|(t))^{1/2} dt$$

and using the Hölder's inequality we obtain

$$|m(V)^{-1} \Phi_V * u(x) - u(x)| \leq 2^{1/2} m(V)^{-1/2} \|u\|^{1/2} \left| \int_V (|u|(e) - |u|(t)) dt \right|^{1/2}.$$

From the remark of prop. 1 follows the possibility of finding $\{f_1, \dots, f_n\} \subset L^1(G)$ and $\eta > 0$ such that $u \in U_\tau(u_0; \eta, f_1, \dots, f_n)$ implies

$$|u| \in U \left(|u_0|, \frac{\varepsilon^2}{54(1 + \|u_0\|)}, \frac{\Phi_V}{m(V)} \right).$$

We can assume that

$$\eta \leq \min \left(\frac{\varepsilon^2}{54(1 + \|u_0\|)}, 1 \right).$$

Using the inequality

$$\begin{aligned} \left| \int_V (\|u\| - |u|(t)) dt \right| &\leq \left| \int_V (\|u\| - \|u_0\|) dt \right| + \left| \int_V (\|u_0\| - |u_0|(t)) dt \right| \\ &\quad + \left| \int_V \Phi_V(t) (|u_0|(t) - |u|(t)) dt \right|, \end{aligned}$$

we get for

$$u \in U_\tau(u_0; \eta, f_1, \dots, f_n) / \left| \int_V (\|u\| - |u|(t)) dt \right| < \frac{\varepsilon^2}{18(1 + \|u_0\|)} m(V)$$

and

$$\|u\| \left| \int_V (\|u\| - |u|(t)) dt \right| < \frac{\varepsilon^2}{18} m(V).$$

We have therefore for $u \in U_\tau(u_0; \eta, f_1, \dots, f_n)$ and every $x \in G$

$$|m(V)^{-1} \Phi_V * u(x) - u(x)| < \varepsilon/3.$$

Let $S(0, 1 + \|u_0\|)$ denote the sphere $\{u \in B(G) \mid \|u\| \leq 1 + \|u_0\|\}$; the map of $(S(0, 1 + \|u_0\|), \sigma(B(G), L^1(G)))$ into $(C^b(G), \text{compact-open topology})$ defined by $u \mapsto m(V)^{-1} \Phi_V * u$ is continuous in u_0 . It is therefore possible to find $\eta' > 0$ and $\{g_1, \dots, g_m\} \subset L^1(G)$ such that

$$u \in U(u_0; \eta', g_1, \dots, g_m) \cap S(0, 1 + \|u_0\|)$$

implies

$$|m(V)^{-1} \Phi_V * u(x) - m(V)^{-1} \Phi_V * u_0(x)| < \varepsilon/3$$

for every $x \in K$. If we set $\lambda = \min(\eta, \eta')$ and $\{h_1, \dots, h_k\} = \{f_1, \dots, f_n, g_1, \dots, g_m\}$, writing

$$\begin{aligned} |u(x) - u_0(x)| &\leq |u(x) - m(V)^{-1} \Phi_V * u(x)| + |m(V)^{-1} \Phi_V * u(x) \\ &\quad - m(V)^{-1} \Phi_V * u_0(x)| + |m(V)^{-1} \Phi_V * u_0(x) - u_0(x)|, \end{aligned}$$

we obtain finally

$$U_\tau(u_0; \lambda, h_1, \dots, h_k) \subset U(u_0; \varepsilon, K). \quad \text{q.e.d.}$$

COROLLARY. On $B_0 = \{u \in B(G) \mid \|u\| = 1\}$ the weak topology $\sigma(B(G), L^1(G))$ and the compact-open topology coincide.

Proof. On every bounded set in $B(G)$ the compact-open topology is stronger than the weak topology.

Remark. The corollary generalizes the following result due to D. A. Raikov: on $\{u \in P(G) \mid u(e) = 1\}$ the above topologies coincide ([2] p. 260–261).

4. Approximate Identity in the Fourier Algebra of a Locally Compact Group

We recall ([5]) some definitions and properties concerning $B(G)$. Let $L^1_\rho(G)$ denote the vector space $L^1(G)$ with the spectral norm. Then $(L^1_\rho(G))'$ is isomorphic ([5] p. 192) to a closed subspace $B_\rho(G)$ of $B(G)$ (for the norm topology). The set $\{k * l \mid k, l \in L^2(G)\}$ is contained in $B_\rho(G)$ and $\|k * l\| \leq \|k\|_2 \|l\|_2$ ([5] p. 207). The closure of $B(G) \cap C_{00}(G)$ in $B_\rho(G)$ (or in $B(G)$) is called the Fourier algebra $A(G)$ of G .

THEOREM 7. *The Fourier algebra $A(G)$ has an approximate identity of the following type: for every $u \in A(G)$ and $\varepsilon > 0$ there exists $v \in A(G)$ such that $\|u - uv\| < \varepsilon$ and $\|v\| = 1$ if and only if G has the property (R).*

Proof. The property (R) implies ([7] p. 172 and 168) that G has property P_1 . By ([4]) it follows (non trivially!) that for every $K \in \mathfrak{K}$ and $\varepsilon > 0$ there exists $U \in \mathfrak{K}$ with $m(U) > 0$ and $m(KU) < (1 + \varepsilon)m(U)$. Set $\varphi = m(U)^{-1} \Phi_{KU} * \tilde{\Phi}_U$. The function φ is contained in $B(G) \cap C_{00}(G)$ and $\|\varphi\| \leq m(U)^{-1} \|\Phi_{KU}\|_2 \|\tilde{\Phi}_U\|_2$; using $m(KU) < (1 + \varepsilon)m(U)$ we obtain $\|\varphi\| < 1 + \varepsilon$ and by definition $\text{Res}_K \varphi = 1$. We have therefore proved that for every $K \in \mathfrak{K}$ and every $\varepsilon > 0$ there is $\varphi \in B(G) \cap C_{00}(G)$ with $\|\varphi\| < 1 + \varepsilon$ and $\text{Res}_K \varphi = 1$. Let u be an arbitrary element of $A(G)$ and ε an arbitrary positive real number. We can assume $\|u\| > \varepsilon$. There exists by definition $w \in B(G) \cap C_{00}(G)$ with $\|u - w\| < \varepsilon/8$. We can find $\varphi \in B(G) \cap C_{00}(G)$ with $\text{Res}_{\text{supp } w} \varphi = 1$ ($\text{supp } w \neq \emptyset$) and $\|\varphi\| < 1 + \eta$ where

$$\eta = \min \left(1, \frac{\varepsilon}{2(1 + \|u\|)} \right).$$

From $\|u - u\varphi\| \leq \|u - w\| + \|\varphi\| \|u - w\|$ follows $\|u - u\varphi\| < \varepsilon/2$. If we set

$$v = \frac{\varphi}{\|\varphi\|}$$

we get

$$\|u - uv\| \leq \|\varphi\| \|u - u\varphi\| \leq \|u\varphi - u\| + \|u - \|\varphi\| u\| < \varepsilon.$$

To prove the converse, it is enough to find $k \in C_{00}(G)$ for every $\varepsilon > 0$ and $K \in \mathfrak{K}$ such that $|1 - k * \tilde{k}(x)| < \varepsilon$ on K .

By theorem 6 we can find $\varepsilon_1 > 0$ and a finite subset F_1 of $L^1(G)$ such that

$$U(1; \varepsilon_1, F_1) \cap B_0 \subset U(1; \varepsilon/4, K).$$

By the remark of prop. 1 there exists $\varepsilon_2 > 0$ and F_2 finite subset of $L^1(G)$ for which $u \in U(1; \varepsilon_2, F_2) \cap B_0$ implies $|u| \in U(1; \varepsilon_1, F_1)$. It is also possible to find $\varepsilon_3 > 0$ and $K' \in \mathfrak{K}$ with

$$U(1; \varepsilon_3, K') \cap B_0 \subset U(1; \varepsilon_2, F_2).$$

Let u be an element of $B(G) \cap C_{00}(G)$ with $\text{Res}_K u = 1$. By assumption there exists $v' \in A(G)$ with $\|v'\| = 1$ and $\|u - uv'\| < \varepsilon_3/6$; if we take $v'' \in B(G) \cap C_{00}(G)$ such that

$$\|v' - v''\| < \min\left(\frac{1}{2}, \frac{\varepsilon_3}{6(1 + \|u\|)}\right)$$

and put $v = v''/\|v''\|$ we obtain $\|u - uv\| < \varepsilon_3$ and therefore $|1 - v(x)| < \varepsilon_3$ for every $x \in K'$ i.e. $v \in U(1, \varepsilon_3, K') \cap B_0$. From above it follows $|v| \in U(1; \varepsilon/4, K)$. The function v is in $B_\varrho(G)$. This implies that $|v| \in B_\varrho(G)$ (by definition) and then by ([1] Theorem 4.3) $|v|$ must be contained in the closure of $P(G) \cap C_{00}(G)$ (for the compact-open topology) in $P(G)$. Therefore there is $w \in P(G) \cap C_{00}(G)$ such that $w \in U(|v|; \varepsilon/4, K)$. It is well known that we can choose $k \in C_{00}(G)$ with $\text{Sup}_{x \in G} |w(x) - k * \tilde{k}(x)| < \varepsilon/2$, then for every $x \in K$ we have

$$|1 - k * \tilde{k}(x)| \leq |1 - |v|(x)| + ||v|(x) - w(x)| + |w(x) - k * \tilde{k}(x)| < \varepsilon.$$

Consequence. For $G = \text{SL}(n, \mathbf{R})$ (with $n \geq 2$) and more generally for every connected semi-simple non compact Lie group with finite center, $A(G)$ does not have an approximate identity of the above type.

PROPOSITION 8. *A sufficient condition for G to satisfy the property (R) is that $A(G)$ has an approximate identity with $v \in A(G) \cap P(G)$.*

Proof. For every $K \in \mathfrak{K}$ there exists $u \in B(G) \cap C_{00}(G)$ with $\text{Res}_K u = 1$. By assumption for every $\varepsilon > 0$ one can find $v' \in A(G) \cap P(G)$ such that $\|u - uv'\| < \varepsilon/4$. For the same reason as above v' must be contained in the closure of $P(G) \cap C_{00}(G)$ (for the compact-open topology) in $P(G)$. We can therefore find $v \in P(G) \cap C_{00}(G)$ with $v \in U(v'; \varepsilon/4, K)$. Then choosing $k \in C_{00}(G)$ such that $\text{Sup}_{x \in G} |v(x) - k * \tilde{k}(x)| < \varepsilon/2$ we conclude as at the end of proof of theorem 6.

5. A Multiplier Theorem

Let Σ be the set of all unitary continuous representations of G . For $\pi \in \Sigma$ and an

arbitrary bounded Radon measure μ , we denote by $\|\pi(\mu)\|$ the norm of the operator $\int \pi(x) d\mu(x)$, and by $\|\mu\|_Z$ the $\sup \{\|\pi(\mu)\| \mid \pi \in \Sigma\}$.

From the proof of theorem 7 it follows that property (R) is equivalent to the following condition: for every $K \in \mathfrak{K}$ and $\varepsilon > 0$, there exists $f \in B(G) \cap C_{00}(G)$ with $\text{Res}_K f = 1$ and $\|f\| < 1 + \varepsilon$. This remark permits us to prove:

THEOREM 9. *If group G has property (R), and if $u \in C^G$ and $uf \in B(G)$ for every $f \in A(G)$, then $u \in B(G)$. Moreover, $\|u\| = \sup \{\|uf\| \mid f \in A(G), \|f\| \leq 1\}$.*

Proof. As in the abelian case ([8] p. 74) we can find a positive constant C such that $\|uf\| \leq C\|f\|$ for every $f \in A(G)$. Given $\{x_1, \dots, x_n\} \subset G$ and $\varepsilon > 0$ there exists $f \in B(G) \cap C_{00}(G)$ such that $\|f\| < 1 + \varepsilon$ and $f(x_j) = 1$ for $1 \leq j \leq n$. For arbitrary $\{c_1, \dots, c_n\} \subset \mathbb{C}$ we have

$$\left| \sum_j c_j u(x_j) \right| \leq \|fu\| \left\| \sum_j c_j \delta_{x_j} \right\|_Z$$

(δ_a denotes the evaluation at point a); thus

$$\left| \sum_j c_j u(x_j) \right| < C(1 + \varepsilon) \left\| \sum_j c_j \delta_{x_j} \right\|_Z$$

and therefore

$$\left| \sum_j c_j u(x_j) \right| \leq C \left\| \sum_j c_j \delta_{x_j} \right\|_Z.$$

From the continuity of u and from ([5] p. 202) it follows that $u \in B(G)$.

Let $\|u\|$ be the $\sup \{\|uf\| \mid f \in A(G), \|f\| \leq 1\}$. We have to show that $\|u\| \leq \|u\|$ (the converse inequality is clear). For arbitrary $h \in C_{00}(G)$ with $\|h\|_Z \leq 1$ and arbitrary $\varepsilon > 0$ there exists $w \in B(G) \cap C_{00}(G)$ with $\text{Res}_{\text{supp} h} w = 1$ and $\|w\| < 1 + \varepsilon$ (if $\text{supp} h$ is empty we choose $w = 0$). We obtain

$$\left| \int u(x) h(x) dx \right| = \left| \int u(x) w(x) h(x) dx \right| < \|u\| (1 + \varepsilon)$$

and therefore $\left| \int u(x) h(x) dx \right| \leq \|u\|$ for every $h \in C_{00}(G)$ with $\|h\|_Z \leq 1$, that is $\|u\| \leq \|u\|$.

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