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# Some Results on the Fourier-Stieltjes Algebra of a Locally Compact Group

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## 1. Introduction

This paper is concerned with the study of the set  $B(G)$  of all finite complex linear combinations of continuous positive definite functions over an arbitrary locally compact group  $G$ . We prove (corollary of theorem 6) that on the boundary of the unit sphere of  $B(G)$  ([5]), the compact-open topology and the weak topology  $\sigma(B(G), L^1(G))$  coincide (we consider  $L^1(G)$  with left Haar measure  $dx$ ). This extends the following theorem due to D. A. Raikov ([2] p. 260–261): the above topologies coincide on the set of all continuous positive definite functions with value 1 at the unit element  $e$  of  $G$ . We need, for this purpose, some general results on  $C^*$ -algebras which are developed in section 2.

As an application we give a new proof in section 4 of a result of H. Leptin ([6]) concerning the existence of an approximate identity in the Fourier algebra  $A(G)$  of  $G$  ([5]). More precisely, we obtain directly the existence of an approximate identity of the following type: for every  $u \in A(G)$  and  $\varepsilon > 0$  there exists  $v \in A(G)$  with  $\|v\| = 1$  such that  $\|u - uv\| < \varepsilon$  implies that the group  $G$  has property  $(R)$ . We recall that a locally compact group  $G$  is said to have property  $(R)$  if the constant 1 on  $G$  can be approximated uniformly on compact sets by functions of the form  $x \mapsto \int_G k(xy) \overline{k(y)} dy$ , where  $k$  is a continuous complex-valued function on  $G$  with compact support.

Finally in section 5 we extend the following multiplier theorem to every locally compact group  $G$ , satisfying property  $(R)$ : if  $u \in C^G$  and if  $uf \in B(G)$  for every  $f \in A(G)$ , then  $u \in B(G)$ . Moreover, Prof. P. Eymard told me that, under the same assumptions, we have

$$\|u\| = \text{Sup} \{ \|uf\| \mid f \in A(G), \|f\| \leq 1 \}.$$

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## 2. Some Remarks on the Polar Decomposition of Linear Continuous Functionals on a $C^*$ -Algebra

Let  $A$  be a  $C^*$ -algebra,  $A'$  the set of all linear continuous functionals on  $A$ . For

$f \in A'$  define

$$\|f\| = \text{Sup} \{ |f(x)| \mid \|x\| \leq 1, x \in A \}.$$

By ([2] p. 243) there is only one positive linear functional  $|f|$  on  $A$ , such that  $\|f\| = \||f|\|$ , and  $|f(x)|^2 \leq \|f\| \cdot |f|(xx^*)$ . Consider on  $A'$  the locally convex topology  $\tau$  defined by the following systems of sets:

$$U_\tau(f_0; \varepsilon, x_1, \dots, x_n) = \{f \in A' \mid |f(x_j) - f_0(x_j)| < \varepsilon \quad 1 \leq j \leq n, \|\|f\| - \|f_0\|\| < \varepsilon\}$$

where  $f_0 \in A'$ ,  $\varepsilon > 0$  and  $\{x_1, \dots, x_n\} \subset A$ .

**PROPOSITION 1.** *The map  $f \mapsto |f|$  is a continuous map of  $(A', \tau)$  in  $(A', \sigma(A', A))$ .*

This proposition was proved in ([3] Lemma 3.5).

*Remark.* The result remains valid if for the definition of the topology  $\tau$  one takes the  $\{x_1, \dots, x_n\}$  in an arbitrary dense subspace of  $A$ .

**PROPOSITION 2.** *To every  $f \in A'$  there exists a \*-representation of  $A$   $(\pi, H)$  with cyclic vector  $\xi$  such that  $|f|(x) = (\pi(x)\xi, \xi)$ . There also exists a unique  $\zeta \in H$  with  $f(x) = (\pi(x)\zeta, \xi)$ . We have*

$$\|f\|^{1/2} = \|\zeta\| = \|\xi\|.$$

*Proof.* On the dense subspace  $\pi(A)\xi$  consider the linear functional defined by

$$F(\pi(x^*)\xi) = \overline{f(x)}.$$

The inequality

$$|f(x)|^2 \leq \|f\| \|\pi(x^*)\xi\|^2$$

implies the continuity of  $F$  and therefore the existence and unicity of  $\zeta \in H$  with

$$\overline{f(x)} = (\pi(x^*)\xi, \zeta)$$

i.e.

$$f(x) = (\pi(x)\zeta, \xi).$$

From

$$\text{Sup}_{\|\pi(x^*)\xi\| \leq 1} |(\pi(x^*)\xi, \zeta)| = \|\zeta\|$$

there follows  $\|\zeta\| \leq \|f\|^{1/2}$ . On the other hand, the inequalities

$$|f(x)| \leq \|\pi(x^*)\xi\| \|\zeta\| \leq \|x\| \|\xi\| \|\zeta\|$$

imply

$$|f(x)| \leq \|x\| \|f\|^{1/2} \|\zeta\|,$$

that is  $\|f\|^{1/2} \leq \|\zeta\|$ , and finally  $\|f\|^{1/2} = \|\zeta\|$ .

*Remark.* Suppose that  $(\pi', H')$  is a \*-representation of  $A$  with cyclic vector  $\xi'$ , such that  $|f|(x) = (\pi'(x) \xi', \xi')$ . Let be  $\zeta'$  the corresponding vector of  $H'$  with  $f(x) = (\pi'(x) \zeta', \xi')$ . It is well known that there exists only one linear isometry  $\psi$  of  $H$  onto  $H'$  with  $\psi\pi(x) \xi = \pi'(x) \xi'$ . It is easy to verify that  $\psi(\zeta) = \zeta'$ .

**PROPOSITION 3.** *For every  $f \in A'$ , and  $x, y \in A$ , we have*

$$|f(x) - f(y)|^2 \leq \|f\| [\|f\|(\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f|(yx^*)].$$

*Proof.* Consider  $\pi, H, \xi$  and  $\zeta$  as in proposition 2. We have

$$f(x) = (\zeta, \pi(x^*) \xi);$$

that is

$$|f(x - y)|^2 = |(\zeta, \pi(x^* - y^*) \xi)|^2 \leq \|\zeta\|^2 \|\pi(x^* - y^*) \xi\|^2.$$

From

$$\|\pi(x^* - y^*) \xi\|^2 = \|\pi(x^*) \xi\|^2 + \|\pi(y^*) \xi\|^2 - 2 \operatorname{Re} (\pi(x^*) \xi, \pi(y^*) \xi)$$

and

$$(\pi(x^*) \xi, \pi(y^*) \xi) = |f|(yx^*)$$

we obtain

$$\|\pi(x^* - y^*) \xi\|^2 \leq \|\xi\|^2 (\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f|(yx^*)$$

and therefore

$$|f(x - y)|^2 \leq \|f\| [\|f\|(\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} |f|(yx^*)].$$

*Remark.* To every  $f \in A'$  there corresponds a unique positive linear functional  $p(f)$  on  $A$  such that

$$\|f\| = \|p(f)\| \quad \text{and} \quad |f(x)|^2 \leq \|f\| p(f)(x^* x).$$

We have  $p(f) = |f^*|$ ; for  $p(f)$  proposition 1 remains valid. Proposition 2 has to be modified in the following way: Let  $(\pi_p, H_p)$  a \*-representation of  $A$  with cyclic vector  $\xi_p$  such that  $p(f)(x) = (\pi_p(x) \xi_p, \xi_p)$ . Then there exists a unique  $\zeta_p \in H_p$  with  $f(x) = (\pi_p(x) \xi_p, \zeta_p)$  etc. ... The inequality of proposition 3 becomes

$$|f(x) - f(y)|^2 \leq \|f\| [\|f\|(\|x\|^2 + \|y\|^2) - 2 \operatorname{Re} p(f)(y^* x)].$$

### 3. Topologies on the Fourier-Stieltjes Algebra of a Locally Compact Group

In the sequel,  $G$  is an arbitrary locally compact group,  $C_{00}(G)$  the set of all complex-valued continuous functions on  $G$  with compact support,  $C^b(G)$  the set of all continuous complex-valued bounded functions on  $G$  and  $P(G)$  the set of all continuous positive definite functions. For  $f \in C^b(G)$  let  $\tilde{f}$  denote the function  $\tilde{f}(x) = \overline{f(x^{-1})}$ . Let  $C^*(G)$  be the  $C^*$ -algebra of the group  $G$ . The isomorphy of  $B(G)$  and  $C^*(G)'$  permits us (see [5] p. 192–193) to define on  $B(G)$  a norm  $u \mapsto \|u\|$  and an "absolute value"  $u \mapsto |u|$ .

**PROPOSITION 4.** *To every  $u \in B(G)$  there corresponds a unitary continuous representation  $(\pi, H)$  with cyclic vector  $\xi$ , such that  $|u|(x) = (\pi(x) \xi, \xi)$  and with a unique vector  $\zeta \in H$  such that  $u(x) = (\pi(x) \zeta, \xi)$ . We have, moreover,*

$$\|\zeta\| = \|u\|^{1/2} = \|\xi\| = |u|(e)^{1/2}.$$

This proposition is a direct consequence of prop. 2. A similar result is proved directly in ([5] p. 195).

**PROPOSITION 5.** *Every  $u \in B(G)$  satisfies the inequality*

$$|u(x) - u(y)|^2 \leq 2 \|u\| [|u|(e) - \operatorname{Re} |u|(yx^{-1})],$$

for all  $x, y \in G$ .

*Proof.* The proof is similar to that of prop. 3. It is enough to choose  $\pi, H, \xi$  and  $\zeta$  as in prop. 4. From

$$u(x) = (\zeta, \pi(x^{-1}) \xi)$$

follows

$$|u(x) - u(y)|^2 \leq \|\zeta\|^2 \|\pi(x^{-1}) \xi - \pi(y^{-1}) \xi\|^2.$$

On the other hand,

$$\|\pi(x^{-1}) \xi - \pi(y^{-1}) \xi\|^2 = 2 \|\xi\|^2 - 2 \operatorname{Re} (\pi(x^{-1}) \xi, \pi(y^{-1}) \xi)$$

and

$$\|\xi\|^2 = |u|(e) = \|\zeta\|^2 = \|u\|$$

imply the required inequality.

#### Remarks

- (1) The same inequality is well known for elements of  $P(G)$ .
- (2) It is possible to consider proposition 5 as a special case of proposition 3.

Consider the topology (denoted  $\tau$ ) on  $B(G)$  defined by the following systems of sets:

$$U_\tau(u_0; \varepsilon, f_1, \dots, f_n) = \{u \in B(G) \mid \left| \int f_j(x) (u(x) - u_0(x)) dx \right| < \varepsilon \quad 1 \leq j \leq n$$

and

$$\left\{ \|u\| - \|u_0\| < \varepsilon \right\}$$

where  $u_0 \in B(G)$ ,  $\varepsilon > 0$  and  $\{f_1, \dots, f_n\} \subset L^1(G)$ .

**THEOREM 6.** *The topology  $\tau$  on  $B(G)$  is stronger than the compact-open topology.*

*Proof.* Given  $u_0 \in B(G)$ ,  $\varepsilon > 0$  and  $K \in \mathfrak{K}$  ( $\mathfrak{K}$  denotes the set of all compact subsets of  $G$ ), we have to show the existence of  $\lambda \in \mathbf{R}^+$  and  $\{h_1, \dots, h_k\} \subset L^1(G)$  such that

$$\begin{aligned} U_\tau(u_0; \lambda, h_1, \dots, h_k) &\subset U(u_0; \varepsilon, K) \\ &= \{u \in B(G) \mid |u(x) - u_0(x)| < \varepsilon \quad \text{for every } x \in K\}. \end{aligned}$$

Consider a compact neighbourhood  $V$  of  $e$ , such that  $x \in V$  implies

$$|u_0(x) - u_0(e)| < \frac{\varepsilon^2}{54(1 + \|u_0\|)}.$$

Let  $\Phi_V$  denote the function on  $G$  defined as 1 on  $V$  and 0 outside. For every  $x \in G$  and  $u \in B(G)$  we have

$$m(V)^{-1} \Phi_V * u(x) - u(x) = m(V)^{-1} \int \Phi_V(t) (u(t^{-1}x) - u(x)) dt$$

(where  $m(V)$  is  $\int \Phi_V(t) dt$ ). From prop. 5 follows

$$|m(V)^{-1} \Phi_V * u(x) - u(x)| \leq 2^{1/2} m(V)^{-1} \int_V \|u\|^{1/2} (|u|(e) - \operatorname{Re}|u|(t))^{1/2} dt$$

and using the Hölder's inequality we obtain

$$|m(V)^{-1} \Phi_V * u(x) - u(x)| \leq 2^{1/2} m(V)^{-1/2} \|u\|^{1/2} \left| \int_V (|u|(e) - u(t)) dt \right|^{1/2}.$$

From the remark of prop. 1 follows the possibility of finding  $\{f_1, \dots, f_n\} \subset L^1(G)$  and  $\eta > 0$  such that  $u \in U_\tau(u_0; \eta, f_1, \dots, f_n)$  implies

$$|u| \in U \left( |u_0|, \frac{\varepsilon^2}{54(1 + \|u_0\|)}, \frac{\Phi_V}{m(V)} \right).$$

We can assume that

$$\eta \leq \min \left( \frac{\varepsilon^2}{54(1 + \|u_0\|)}, 1 \right).$$

Using the inequality

$$\begin{aligned} \left| \int_V (\|u\| - |u|(t)) dt \right| &\leq \left| \int_V (\|u\| - \|u_0\|) dt \right| + \left| \int_V (\|u_0\| - |u_0|(t)) dt \right| \\ &+ \left| \int \Phi_V(t) (|u_0|(t) - |u|(t)) dt \right|, \end{aligned}$$

we get for

$$u \in U_\tau(u_0; \eta, f_1, \dots, f_n) / \left| \int_V (\|u\| - |u|(t)) dt \right| < \frac{\varepsilon^2}{18(1 + \|u_0\|)} m(V)$$

and

$$\|u\| \left| \int_V (\|u\| - |u|(t)) dt \right| < \frac{\varepsilon^2}{18} m(V).$$

We have therefore for  $u \in U_\tau(u_0; \eta, f_1, \dots, f_n)$  and every  $x \in G$

$$|m(V)^{-1} \Phi_V * u(x) - u(x)| < \varepsilon/3.$$

Let  $S(0, 1 + \|u_0\|)$  denote the sphere  $\{u \in B(G) \mid \|u\| \leq 1 + \|u_0\|\}$ ; the map of  $(S(0, 1 + \|u_0\|), \sigma(B(G), L^1(G)))$  into  $(C^b(G), \text{compact-open topology})$  defined by  $u \mapsto m(V)^{-1} \Phi_V * u$  is continuous in  $u_0$ . It is therefore possible to find  $\eta' > 0$  and  $\{g_1, \dots, g_m\} \subset L^1(G)$  such that

$$u \in U(u_0; \eta', g_1, \dots, g_m) \cap S(0, 1 + \|u_0\|)$$

implies

$$|m(V)^{-1} \Phi_V * u(x) - m(V)^{-1} \Phi_V * u_0(x)| < \varepsilon/3$$

for every  $x \in K$ . If we set  $\lambda = \min(\eta, \eta')$  and  $\{h_1, \dots, h_k\} = \{f_1, \dots, f_n, g_1, \dots, g_m\}$ , writing

$$\begin{aligned} |u(x) - u_0(x)| &\leq |u(x) - m(V)^{-1} \Phi_V * u(x)| + |m(V)^{-1} \Phi_V * u(x) \\ &\quad - m(V)^{-1} \Phi_V * u_0(x)| + |m(V)^{-1} \Phi_V * u_0(x) - u_0(x)|, \end{aligned}$$

we obtain finally

$$U_\tau(u_0; \lambda, h_1, \dots, h_k) \subset U(u_0; \varepsilon, K). \quad \text{q.e.d.}$$

**COROLLARY.** *On  $B_0 = \{u \in B(G) \mid \|u\| = 1\}$  the weak topology  $\sigma(B(G), L^1(G))$  and the compact-open topology coincide.*

*Proof.* On every bounded set in  $B(G)$  the compact-open topology is stronger than the weak topology.

*Remark.* The corollary generalizes the following result due to D. A. Raikov: on  $\{u \in P(G) \mid u(e) = 1\}$  the above topologies coincide ([2] p. 260–261).

#### 4. Approximate Identity in the Fourier Algebra of a Locally Compact Group

We recall ([5]) some definitions and properties concerning  $B(G)$ . Let  $L_e^1(G)$  denote the vector space  $L^1(G)$  with the spectral norm. Then  $(L_e^1(G))'$  is isomorphic ([5] p. 192) to a closed subspace  $B_e(G)$  of  $B(G)$  (for the norm topology). The set  $\{k * \tilde{l} \mid k, l \in L^2(G)\}$  is contained in  $B_e(G)$  and  $\|k * \tilde{l}\| \leq \|k\|_2 \|l\|_2$  ([5] p. 207). The closure of  $B(G) \cap C_{00}(G)$  in  $B_e(G)$  (or in  $B(G)$ ) is called the Fourier algebra  $A(G)$  of  $G$ .

**THEOREM 7.** *The Fourier algebra  $A(G)$  has an approximate identity of the following type: for every  $u \in A(G)$  and  $\varepsilon > 0$  there exists  $v \in A(G)$  such that  $\|u - uv\| < \varepsilon$  and  $\|v\| = 1$  if and only if  $G$  has the property (R).*

*Proof.* The property (R) implies ([7] p. 172 and 168) that  $G$  has property  $P_1$ . By ([4]) it follows (non trivially!) that for every  $K \in \mathfrak{K}$  and  $\varepsilon > 0$  there exists  $U \in \mathfrak{K}$  with  $m(U) > 0$  and  $m(KU) < (1 + \varepsilon) m(U)$ . Set  $\varphi = m(U)^{-1} \Phi_{KU} * \tilde{\Phi}_U$ . The function  $\varphi$  is contained in  $B(G) \cap C_{00}(G)$  and  $\|\varphi\| \leq m(U)^{-1} \|\Phi_{KU}\|_2 \|\tilde{\Phi}_U\|_2$ ; using  $m(KU) < (1 + \varepsilon) m(U)$  we obtain  $\|\varphi\| < 1 + \varepsilon$  and by definition  $\text{Res}_K \varphi = 1$ . We have therefore proved that for every  $K \in \mathfrak{K}$  and every  $\varepsilon > 0$  there is  $\varphi \in B(G) \cap C_{00}(G)$  with  $\|\varphi\| < 1 + \varepsilon$  and  $\text{Res}_K \varphi = 1$ . Let  $u$  be an arbitrary element of  $A(G)$  and  $\varepsilon$  an arbitrary positive real number. We can assume  $\|u\| > \varepsilon$ . There exists by definition  $w \in B(G) \cap C_{00}(G)$  with  $\|u - w\| < \varepsilon/8$ . We can find  $\varphi \in B(G) \cap C_{00}(G)$  with  $\text{Res}_{\text{supp } w} \varphi = 1$  ( $\text{supp } w \neq \emptyset$ ) and  $\|\varphi\| < 1 + \eta$  where

$$\eta = \min \left( 1, \frac{\varepsilon}{2(1 + \|u\|)} \right).$$

From  $\|u - u\varphi\| \leq \|u - w\| + \|\varphi\| \|u - w\|$  follows  $\|u - u\varphi\| < \varepsilon/2$ . If we set

$$v = \frac{\varphi}{\|\varphi\|}$$

we get

$$\|u - uv\| \leq \|\varphi\| \|u - u\varphi\| \leq \|u\varphi - u\| + \|u - u\varphi\| < \varepsilon.$$

To prove the converse, it is enough to find  $k \in C_{00}(G)$  for every  $\varepsilon > 0$  and  $K \in \mathfrak{K}$  such that  $|1 - k * \tilde{k}(x)| < \varepsilon$  on  $K$ .

By theorem 6 we can find  $\varepsilon_1 > 0$  and a finite subset  $F_1$  of  $L^1(G)$  such that

$$U(1; \varepsilon_1, F_1) \cap B_0 \subset U(1; \varepsilon/4, K).$$

By the remark of prop. 1 there exists  $\varepsilon_2 > 0$  and  $F_2$  finite subset of  $L^1(G)$  for which  $u \in U(1; \varepsilon_2, F_2) \cap B_0$  implies  $|u| \in U(1; \varepsilon_1, F_1)$ . It is also possible to find  $\varepsilon_3 > 0$  and  $K' \in \mathfrak{K}$  with

$$U(1; \varepsilon_3, K') \cap B_0 \subset U(1; \varepsilon_2, F_2).$$

Let  $u$  be an element of  $B(G) \cap C_{00}(G)$  with  $\text{Res}_{K'} u = 1$ . By assumption there exists  $v' \in A(G)$  with  $\|v'\| = 1$  and  $\|u - uv'\| < \varepsilon_3/6$ ; if we take  $v'' \in B(G) \cap C_{00}(G)$  such that

$$\|v' - v''\| < \min\left(\frac{1}{2}, \frac{\varepsilon_3}{6(1 + \|u\|)}\right)$$

and put  $v = v''/\|v''\|$  we obtain  $\|u - uv\| < \varepsilon_3$  and therefore  $|1 - v(x)| < \varepsilon_3$  for every  $x \in K'$  i.e.  $v \in U(1, \varepsilon_3, K') \cap B_0$ . From above it follows  $|v| \in U(1; \varepsilon/4, K)$ . The function  $v$  is in  $B_\varepsilon(G)$ . This implies that  $|v| \in B_\varepsilon(G)$  (by definition) and then by ([1] Theorem 4.3)  $|v|$  must be contained in the closure of  $P(G) \cap C_{00}(G)$  (for the compact-open topology) in  $P(G)$ . Therefore there is  $w \in P(G) \cap C_{00}(G)$  such that  $w \in U(|v|; \varepsilon/4, K)$ . It is well known that we can choose  $k \in C_{00}(G)$  with  $\text{Sup}_{x \in G} |w(x) - k * \tilde{k}(x)| < \varepsilon/2$ , then for every  $x \in K$  we have

$$|1 - k * \tilde{k}(x)| \leq |1 - |v|(x)| + | |v|(x) - w(x)| + |w(x) - k * \tilde{k}(x)| < \varepsilon.$$

*Consequence.* For  $G = \text{SL}(n, \mathbf{R})$  (with  $n \geq 2$ ) and more generally for every connected semi-simple non compact Lie group with finite center,  $A(G)$  does not have an approximate identity of the above type.

**PROPOSITION 8.** *A sufficient condition for  $G$  to satisfy the property (R) is that  $A(G)$  has an approximate identity with  $v \in A(G) \cap P(G)$ .*

*Proof.* For every  $K \in \mathfrak{K}$  there exists  $u \in B(G) \cap C_{00}(G)$  with  $\text{Res}_K u = 1$ . By assumption for every  $\varepsilon > 0$  one can find  $v' \in A(G) \cap P(G)$  such that  $\|u - uv'\| < \varepsilon/4$ . For the same reason as above  $v'$  must be contained in the closure of  $P(G) \cap C_{00}(G)$  (for the compact-open topology) in  $P(G)$ . We can therefore find  $v \in P(G) \cap C_{00}(G)$  with  $v \in U(v'; \varepsilon/4, K)$ . Then choosing  $k \in C_{00}(G)$  such that  $\text{Sup}_{x \in G} |v(x) - k * \tilde{k}(x)| < \varepsilon/2$  we conclude as at the end of proof of theorem 6.

## 5. A Multiplier Theorem

Let  $\Sigma$  be the set of all unitary continuous representations of  $G$ . For  $\pi \in \Sigma$  and an

arbitrary bounded Radon measure  $\mu$ , we denote by  $\|\pi(\mu)\|$  the norm of the operator  $\int \pi(x) d\mu(x)$ , and by  $\|\mu\|_\Sigma$  the sup  $\{\|\pi(\mu)\| \mid \pi \in \Sigma\}$ .

From the proof of theorem 7 it follows that property (R) is equivalent to the following condition: for every  $K \in \mathfrak{K}$  and  $\varepsilon > 0$ , there exists  $f \in B(G) \cap C_{00}(G)$  with  $\text{Res}_K f = 1$  and  $\|f\| < 1 + \varepsilon$ . This remark permits us to prove:

**THEOREM 9.** *If group  $G$  has property (R), and if  $u \in \mathbf{C}^G$  and  $uf \in B(G)$  for every  $f \in A(G)$ , then  $u \in B(G)$ . Moreover,  $\|u\| = \text{Sup} \{\|uf\| \mid f \in A(G), \|f\| \leq 1\}$ .*

*Proof.* As in the abelian case ([8] p. 74) we can find a positive constant  $C$  such that  $\|uf\| \leq C \|f\|$  for every  $f \in A(G)$ . Given  $\{x_1, \dots, x_n\} \subset G$  and  $\varepsilon > 0$  there exists  $f \in B(G) \cap C_{00}(G)$  such that  $\|f\| < 1 + \varepsilon$  and  $f(x_j) = 1$  for  $1 \leq j \leq n$ . For arbitrary  $\{c_1, \dots, c_n\} \subset \mathbf{C}$  we have

$$\left| \sum_j c_j u(x_j) \right| \leq \|f u\| \left\| \sum_j c_j \delta_{x_j} \right\|_\Sigma$$

( $\delta_a$  denotes the evaluation at point  $a$ ); thus

$$\left| \sum_j c_j u(x_j) \right| < C(1 + \varepsilon) \left\| \sum_j c_j \delta_{x_j} \right\|_\Sigma$$

and therefore

$$\left| \sum_j c_j u(x_j) \right| \leq C \left\| \sum_j c_j \delta_{x_j} \right\|_\Sigma.$$

From the continuity of  $u$  and from ([5] p. 202) it follows that  $u \in B(G)$ .

Let  $\|u\|$  be the sup  $\{\|uf\| \mid f \in A(G), \|f\| \leq 1\}$ . We have to show that  $\|u\| \leq \|u\|$  (the converse inequality is clear). For arbitrary  $h \in C_{00}(G)$  with  $\|h\|_\Sigma \leq 1$  and arbitrary  $\varepsilon > 0$  there exists  $w \in B(G) \cap C_{00}(G)$  with  $\text{Res}_{\text{supp } h} w = 1$  and  $\|w\| < 1 + \varepsilon$  (if  $\text{supp } h$  is empty we choose  $w = 0$ ). We obtain

$$\left| \int u(x) h(x) dx \right| = \left| \int u(x) w(x) h(x) dx \right| < \|u\|(1 + \varepsilon)$$

and therefore  $\left| \int u(x) h(x) dx \right| \leq \|u\|$  for every  $h \in C_{00}(G)$  with  $\|h\|_\Sigma \leq 1$ , that is  $\|u\| \leq \|u\|$ .

## BIBLIOGRAPHY

- [1] DARSOW, W. F.: *Positive definite functions and states*, Annals of Math. 60, 447–453 (1954).
- [2] DIXMIER, J.: *Les  $C^*$ -algèbres et leurs représentations* (Gauthiers-Villars, Paris 1964).
- [3] EFFROS, E. G.: *Order ideals in a  $C^*$ -algebra and its dual*, Duke Math. J. 30, 391–411 (1963).
- [4] EMERSON, W. R. and GREENLEAF, F. P.: *Covering properties and Følner conditions for locally compact groups*, Math. Zeit. 102, 370–384 (1967).
- [5] EYMARD, P.: *L’algèbre de Fourier d’un groupe localement compact*, Bull. Soc. math. France 92, 181–236 (1964).

- [6] LEPTIN, H.: *Sur l'algèbre de Fourier d'un groupe localement compact*, C. R. Acad. Sc. Paris, 266, p. 1180–1182 (17 juin 1968).
- [7] REITER, H.: *Classical harmonic analysis and locally compact groups* (Clarendon Press, Oxford 1968).
- [8] RUDIN, W.: *Fourier Analysis on groups* (Interscience, New York 1962).

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