

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 45 (1970)

Artikel: An Algebraic Classification of some Knots of Codimension Two.
Autor: Levine, J.
DOI: <https://doi.org/10.5169/seals-34652>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 28.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

An Algebraic Classification of Some Knots of Codimension Two¹⁾

by J. LEVINE

An n -knot will denote a smooth oriented submanifold K of the $(n+2)$ -sphere S^{n+2} , where K is *homeomorphic* to S^n . If n is odd, one can associate to K a square integral matrix A , called a *Seifert matrix* of K , using a submanifold of S^{n+2} bounded by K (see [13] for $n=1$, and [4] or [8] in general). When $n=1$, it is known that two Seifert matrices of isotopic knots are related by certain algebraic "moves" (see [11], [16]). In this paper we will generalize this fact to all n . We then consider, for n odd, n -knots (referred to as simple) whose complements are of the same $((n-1)/2)$ type as a circle, i.e. $\Pi_q(S^{n+2}-K) \approx \Pi_q(S^1)$ for $q \leq (n-1)/2$. This is the most that can be asked without making K unknotted (see [7]). We will show that two simple n -knots ($n \geq 3$) are isotopic if and only if their Seifert matrices are related by such "moves". Thus it will follow that the semi-group of isotopy classes of simple n -knots depends only on the residue class, mod 4, of n for $n \geq 4$.

By contrast, Lashof and Shaneson [6] (and, independently, Browder) have shown that the isotopy class of an n -knot ($n \geq 3$) K , whose complement is of the same 1-type as a circle is determined by the homotopy type of its exterior pair $(X, \partial X)$, where X is the complement of an open tubular neighborhood of K in S^{n+2} — *except* for one other possible knot $\tau(K)$, obtained from K by removing a tubular neighborhood twisting, and reinserting in S^{n+2} . It is not known whether $\tau(K)$ is ever different from K . As a straightforward application, we will show that $\tau(K)$ is isotopic to K if K is simple.

We conclude with some remarks on the algebraic problems which arise.

1. Let K be a $(2q-1)$ -knot in S^{2q+1} . We recall the definition of a Seifert matrix of K . Let M be a smooth oriented submanifold of S^{2q+1} bounded by K . The l -pairing of M :

$$\theta: H_q(M) \otimes H_q(M) \rightarrow \mathbf{Z}$$

is defined by letting $\theta(\alpha \otimes \beta)$ be the linking number $L(z_1, z_2)$, where z_1 is a cycle in M representing α and z_2 is the translate in the positive normal direction off M of a cycle in M representing β . A Seifert matrix A of K is then a representative matrix of θ with respect to a basis of the torsion-free part of $H_q(M)$ — see e.g. [8].

We recall also the formula [8]:

$$\theta(\alpha \otimes \beta) + (-1)^q \theta(\beta \otimes \alpha) = \alpha \cdot \beta$$

¹⁾ This work was done while the author was partially supported by NSF GP 8885.

where $\alpha \cdot \beta$ is the intersection number in M . Thus $A + (-1)^q A^T$ is unimodular (A^T is the transpose of A) and, if $q=2$, $A + A^T$ has signature a multiple of 16 (see [9]).

2. Let A be a square integral matrix. Any matrix of the form:

$$\left(\begin{array}{c|cc} \underline{A} & 0 & \\ \hline \alpha & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) \quad \text{or} \quad \left(\begin{array}{c|cc} \underline{A} & \beta & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

where α is a row vector, β a column vector, will be called an *elementary enlargement* of A . A is an *elementary reduction* of any of its elementary enlargements. Two matrices (or their associated pairings) are equivalent if they can be connected by a chain of elementary enlargements, reductions and unimodular congruences. It is proved in [11] that Seifert matrices of isotopic 1-knots are equivalent. We shall prove:

THEOREM 1: *Seifert matrices of isotopic knots of any (odd) dimension are equivalent.*

THEOREM 2: *Let q be a positive integer and A a square integral matrix such that $A + (-1)^q A^T$ is unimodular and, if $q=2$, $A + A^T$ has signature a multiple of 16. If $q \neq 2$, there is a simple $(2q-1)$ -knot with Seifert matrix A ; if $q=2$, there is a simple 3-knot with Seifert matrix equivalent to A .*

THEOREM 3: *Let $q \geq 2$ and K_1, K_2 simple $(2q-1)$ -knots with equivalent Seifert matrices. Then K_1 is isotopic to K_2 .*

3. Proof of Theorem 1: Suppose K_1, K_2 are isotopic $(2q-1)$ -knots bounding manifolds M_1, M_2 , respectively, of S^{2q+1} . We first construct a submanifold V (with corners) of $I \times S^{2q+1}$ meeting $0 \times S^{2q+1}$ along $0 \times M_1$ and $1 \times S^{2q+1}$ along $1 \times M_2$ with boundary the union of $0 \times M_1; 1 \times M_2$ and the trace X of an isotopy from K_1 to K_2 . We use the Pontriagin-Thom construction as follows. First construct a normal vector field to $(0 \times M_1) \cup X \cup (1 \times M_2) = Y$ in $I \times S^{2q+1}$, which is tangent to $I \times S^{2q+1}$ along $0 \times M_1 \cup 1 \times M_2$. If $q \neq 1$, there is no obstruction. If $q=1$, the obstruction to extending such a vector field from $0 \times M_1 \cup 1 \times M_2$ over Y is the difference in its winding numbers about K_1 and K_2 . But since the field is defined over M_1 and M_2 , these winding numbers are zero.

Let T be a tubular neighborhood of X . We can "translate" $(X, v | X)$ to a framed submanifold of ∂T which agrees with the framed submanifold $(0 \times M_1 \cup 1 \times M_2, v)$ on $\partial T \cap (I \times S^{2q+1})$. Let $W = \overline{I \times S^{2q+1}} - T$; the Pontriagin-Thom construction on the above framed submanifolds of ∂W determines a map $\partial W \rightarrow S^1$. An extension of

this map over W will determine the desired V . The obstruction lies in

$$H^2(W, \partial W) \approx H^2(I \times S^{2q+1}, X \cup I \times S^{2q+1}) \approx H^1(X \cup I \times S^{2q+1}) = 0.$$

4. Now let $\Phi': V \rightarrow I$ be the ‘‘height’’ function defined by the restriction of the projection $I \times S^{2q+1} \rightarrow I$. We may assume Φ' has no critical points in a neighborhood of ∂V (omitting corners). Let Φ be a C^2 -approximation to Φ' which agrees with Φ' in a neighborhood of ∂V and has only non-degenerate critical points (except at corners) which are mapped one-one into I (see e.g. [10]). We can move V so that Φ becomes the new height function. In fact if $p: V \rightarrow S^{2q+1}$ is defined by the projection $I \times S^{2q+1} \rightarrow S^{2q+1}$ and Φ is a close enough approximation to Φ' , then $x \mapsto (\Phi(x), p(x))$ defines a new imbedding $V \rightarrow I \times S^{2q+1}$ which agrees with the original inclusion near ∂V and has Φ as its new height function.

5. Let $0 = t_0 < t_1 < \dots < t_k = 1$ be a partition of I satisfying

- (i) each t_i is a regular value of Φ ,
- (ii) at most one critical value of Φ lies in each interval (t_i, t_{i+1}) .

Let $\Phi^{-1}(t_i) = t_i \times M'_i$; then each M'_i is bounded by a knot isotopic to K_0 and K_1 , and $M'_0 = M_1, M'_k = M_2$. This shows that it suffices to consider the case where Φ has only one critical point.

LEMMA 1: *Let $\alpha, \alpha' \in H_q(M_1)$ and $\beta, \beta' \in H_q(M_2)$ and suppose that α is homologous to β and α' homologous to β' in V . Then $\theta_1(\alpha, \alpha') = \theta_2(\beta, \beta')$, where θ_i is the l -pairing of M_i .*

Proof: Let C, C' be $(q+1)$ -chains in V such that $\partial C = \alpha - \beta, \partial C' = \alpha' - \beta'$. Then it follows from the definition of θ_1, θ_2 that $\theta_1(\alpha, \alpha') - \theta_2(\beta, \beta')$ is the intersection number of C and the translate of C' off V in the positive normal direction – but this is obviously zero.

6. Now consider the following diagram:

$$\begin{array}{ccccccc} & & & & H_{q+1}(V, M_2) & & \\ & & & & \downarrow & & \\ & & & & H_q(M_2) & & \\ & & & & \downarrow & & \\ H_{q+1}(V, M_1) & \rightarrow & H_q(M_1) & \rightarrow & H_q(V) & \rightarrow & H_q(V, M_1) \\ & & & & \downarrow & & \\ & & & & H_q(V, M_2) & & \end{array}$$

consisting of the exact homology sequences of (V, M_1) and (V, M_2) . If the index of

the critical point of Φ is not q or $q+1$, then

$$H_q(V, M_1) = H_{q+1}(V, M_1) = H_q(V, M_2) = H_{q+1}(V, M_2) = 0$$

and we have

$$H_q(M_1) \approx H_q(V) \approx H_q(M_2).$$

It follows from Lemma 1 that θ_1 and θ_2 are congruent.

If the index of the critical point of Φ is q , then

$$H_q(V, M_1) \approx H_{q+1}(V, M_2) \approx \mathbf{Z}$$

and

$$H_q(V, M_2) = H_{q+1}(V, M_1) = 0.$$

If $\alpha \in H_q(M_2)$ is the image of a generator of $H_{q+1}(V, M_2)$, then the composite

$$H_q(M_2) \rightarrow H_q(V) \rightarrow H_q(V, M_1) \approx \mathbf{Z}$$

can be defined by $\beta \rightarrow \alpha \cdot \beta =$ intersection number in M_2 (see [5]). If α has finite order, then it follows that $H_q(M_1) \approx H_q(V) \approx H_q(M_2)$, modulo torsion, and, therefore, θ_1 and θ_2 are congruent modulo torsion.

7. Suppose α has infinite order; then α is a multiple of a primitive element α_0 and there exists $\beta_0 \in H_q(M_2)$ with $\alpha_0 \cdot \beta_0 = 1$. Suppose $\gamma'_1, \dots, \gamma'_s \in H_q(M_2)$ such that:

- (i) γ'_i is homologous to γ_i in V , and
- (ii) $\alpha_0, \beta_0, \gamma'_1, \dots, \gamma'_s$ is a basis of $H_q(M_2)$, modulo torsion.

We now examine θ_2 on the elements $\alpha_0, \beta_0, \gamma'_1, \dots, \gamma'_s$. By Lemma 1 we can conclude from (i) that $\theta_2(\gamma'_i, \gamma'_j) = \theta_1(\gamma_i, \gamma_j)$. Since α is null-homologous in V , $\theta_2(\alpha, \gamma'_i) = \theta_1(0, \gamma_i) = 0$ and $\theta_2(\alpha, \alpha) = \theta_2(0, 0) = 0$. Thus $\theta_2(\alpha_0, \gamma'_i) = \theta_2(\alpha_0, \alpha_0) = 0$; similarly $\theta_2(\gamma'_i, \alpha_0) = 0$. We also recall that (§ 1):

$$\theta_2(\alpha_0, \beta_0) + (-1)^q \theta_2(\beta_0, \alpha_0) = -\alpha_0 \cdot \beta_0 = -1$$

8. We may summarize this as follows. Let A be the matrix representative of θ_1 with respect to the basis $\gamma_1, \dots, \gamma_s$. The the matrix representative of θ_2 with respect to the basis $\gamma'_1, \dots, \gamma'_s, \alpha_0, \beta_0$ has the form:

$$B = \left(\begin{array}{ccc|cc} \mathbf{A} & & & 0 & \\ & & & \vdots & \eta \\ & & & 0 & \\ \hline 0 \dots 0 & & & 0 & x \\ \xi & & & x' & y \end{array} \right)$$

where x, y are integers, $x + (-1)^q x' = -1$, ξ is a row vector and η is a column vector.

Recall from e.g. [8] that the polynomial $\Delta_A(t) = \det(tA + (-1)^q A^T)$, where A is a Seifert matrix for a knot K , is an invariant of the isotopy class of K (up to multiplication by a unit in $\mathbf{Z}[t, t^{-1}]$). But it is easily verified that:

$$\Delta_B(t) = (tx + (-1)^q x')(tx' + (-1)^q x) \Delta_A(t)$$

Thus x (or x') is zero, since $x \pm x' = -1$, then x' (or x) is ± 1 . It now is easily checked that B is congruent to an elementary enlargement of A .

9. If the index of the critical point of Φ is $q + 1$; then its index as a critical point of $-\Phi$ is q . The preceding arguments apply to show that θ_2 is congruent to θ_1 , or has, as representative matrix, an elementary reduction of a representative matrix of θ_1 .

This completes the proof of Theorem 1.

10. *Proof of Theorem 2:* For $q \neq 2$, this is proved in [4] (see also [9]). For $q = 2$, we must show that A is equivalent to a matrix B , where $B + B^T$ is a matrix representative of the intersection pairing of some simply-connected closed 4-manifold. By an argument in [9], such a B can be obtained by adding enough blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to A ; but this is a sequence of elementary enlargements of A , so B is equivalent to A .

11. *Proof of Theorem 3:* We reduce Theorem 3 to two lemmas. Recall (see [7]) that a simple $(2q - 1)$ -knot bounds a $(q - 1)$ -connected submanifold of S^{2q+1} . A Seifert matrix obtained from the l -pairing of such a submanifold will be called *special*.

LEMMA 2: *Let K be a simple $(2q - 1)$ -knot with a special Seifert matrix A . If B is an elementary enlargement of A , then B is also a special Seifert matrix of K .*

LEMMA 3: *If $q \geq 2$, then simple $(2q - 1)$ -knots admitting identical special Seifert matrices are isotopic.*

12. We first show that Theorem 3 follows from Lemmas 2 and 3. Let K, K' be simple $(2q - 1)$ -knots with equivalent Seifert matrices, $q \geq 2$. Let A, A' be special Seifert matrices of K, K' , respectively. Thus there exists a sequence: $A = A_1, A_2, \dots, A_k = A'$, where each A_{i+1} is unimodularly congruent to an elementary enlargement or reduction of A_i . It follows from Theorem 2 that, for $q > 2$, each A_i is a special Seifert matrix of a simple $(2q - 1)$ -knot K_i (actually the proof of Theorem 2 (see [4]) realizes S as a special Seifert matrix of a simple knot). If $q = 2$, we can enlarge each A_i by adding a constant number of blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to obtain a new sequence A'_1, A'_2, \dots, A'_k . Each A'_{i+1} is again congruent to an elementary enlargement or reduction of A'_i – and it

now follows from the argument in § 10 that each A'_i is a special Seifert matrix of a simple 3-knot K_i .

We now prove that each K_i is isotopic to K_{i+1} . Suppose A_{i+1} is congruent to an elementary enlargement of A_i . It follows from Lemma 2 that A_{i+1} (or A'_{i+1}) is a special Seifert matrix of K_i . Then Lemma 3 implies K_i and K_{i+1} , both of which now admit A_{i+1} (or A'_{i+1}) as a special Seifert matrix, are isotopic. If A_{i+1} is congruent to an elementary reduction of A_i , the same argument works, switching the roles of K_i and K_{i+1} .

We may as well have chosen $K_1 = K$ and $K_k = K'$ if $q > 2$, but if $q = 2$ we need to show that K_1 is isotopic to K and K_k is isotopic to K' . It follows from Lemma 2 that A'_1 is a special Seifert matrix of K , since A'_1 is obtained from A_1 by a sequence of elementary enlargements. Then Lemma 3 implies K and K_1 are isotopic – similarly for K' and K_k .

13. Proof of Lemma 2: Let M be a $(q-1)$ -connected submanifold of S^{2q+1} bounded by K , and $\alpha_1, \dots, \alpha_n$ a basis of $H_q(M)$, modulo torsion, such that A is the corresponding matrix representative of the l -pairing of M . Let x_1, \dots, x_n be an arbitrary sequence of integers. It follows from Alexander duality that there exists a cycle $z \in H_q(S^{2q+1} - M)$ such that the linking numbers $L(z, \alpha_i) = x_i$, for $i = 1, \dots, n$. Now $S^{2q+1} - M$ is $(q-1)$ -connected and so z is spherical; by general position, z can be represented by an imbedded q -sphere $\sigma \subset S^{2q+1} - M$. The normal bundle to σ is trivial and so a tubular neighborhood T can be identified with $\sigma \times D^{q+1}$ – we may assume T disjoint from M . Orient ∂T so that the positive normal direction in S^{2q+1} points into T and let M' be the connected sum in S^{2q+1} of M and ∂T . Then $H_q(M')$ has rank two greater than the rank of $H_q(M)$, and $\alpha_1, \dots, \alpha_n$ may be extended to a basis of $H_q(M')$, modulo torsion, by adjoining the homology classes β_1, β_2 of $\sigma \times y_0$ and $x_0 \times S^q \subset \sigma \times S^q = \partial T$, respectively. The representative matrix of the l -pairing of M' with respect to the basis $\alpha_1, \dots, \alpha_n, \beta_1, \beta_2$ is

$$\left(\begin{array}{c|cc} \mathbf{A} & \pm x_1 & 0 \\ & \vdots & \vdots \\ & \pm x_n & 0 \\ \hline x_1 \dots x_n & x & 0 \\ 0 \dots 0 & \pm 1 & 0 \end{array} \right)$$

which is congruent to:

$$\left(\begin{array}{c|cc} \mathbf{A} & 0 & 0 \\ & \vdots & \vdots \\ & 0 & 0 \\ \hline x_1 \dots x_n & 0 & 0 \\ 0 \dots 0 & 1 & 0 \end{array} \right)$$

If z is chosen so that $L(\alpha_i, z) = x_i$ for $i = 1, \dots, n$, and ∂T is oriented so that the positive normal direction points out from T , then the representative matrix of the l -pairing of M' with respect to $\alpha_1, \dots, \alpha_n, \beta_1, \beta_2$ is:

$$\left(\begin{array}{c|cc} \mathbf{A} & x_1 & 0 \\ & \vdots & \vdots \\ & x_n & 0 \\ \hline \pm x_1 \dots \pm x_n & x & \pm 1 \\ 0 \dots 0 & 0 & 0 \end{array} \right)$$

which is congruent to:

$$\left(\begin{array}{c|cc} \mathbf{A} & x_1 & 0 \\ & \vdots & \vdots \\ & x_n & 0 \\ \hline 0 \dots 0 & 0 & 1 \\ 0 \dots 0 & 0 & 0 \end{array} \right)$$

Thus we can realize any elementary enlargement of A as a special Seifert matrix of K .

14. Proof of Lemma 3: Suppose K and K' are $(2q-1)$ -knots bounding $(q-1)$ -connected submanifolds M and M' of S^{2q+1} with l -pairings θ and θ' . Suppose also that there exists an isomorphism $\Phi: H_q(M) \rightarrow H_q(M')$ preserving the l -pairings, i.e. $\theta = \theta' \circ (\Phi \otimes \Phi)$.

Let us assume, for now, $q > 2$; we will show that M and M' are isotopic submanifolds of S^{2q+1} . According to [15], M and M' have handle decompositions:

$$\begin{aligned} M &= D^{2q} \cup h_1 \cup \dots \cup h_r \\ M' &= D^{2q} \cup h'_1 \cup \dots \cup h'_r \end{aligned}$$

where each h_i, h'_i is a handle of index q – diffeomorphic to $D^q \times D^q$. The $h_i (h'_i)$ are attached to D^{2q} by disjoint imbeddings $S^{q-1} \times D^q \rightarrow \partial D^{2q}$. Let $C_i (C'_i)$ be the “core” of $h_i (h'_i)$, i.e. the submanifold corresponding to $D^q \times 0$ – then $\partial C_i = C_i \cap \partial D^{2q}$.

The imbedded disks $(C_i, \partial C_i) \subset (M, D^{2q})$ represent a basis $\{\alpha_i\}$ of $H_q(M, D^{2q}) \approx H_q(M)$. According to handle body theory (see [17]), we can choose a handle-decomposition realizing any prescribed basis $\{\alpha_i\}$. Thus if $\{\alpha'_i\}$ is the basis of $H_q(M')$ defined by $(C'_i, \partial C'_i) \subset (M', D^{2q})$, we may, by setting $\alpha'_i = \Phi(\alpha_i)$, assume $\theta(\alpha_i, \alpha_j) = \theta'(\alpha'_i, \alpha'_j)$.

15. Now consider the links $\{\partial C_i\}$ and $\{\partial C'_i\}$ in ∂D^{2q} ; by [17] and § 1 we have:

$$L(\partial C_i, \partial C_j) = \alpha_i \cdot \alpha_j = -\theta(\alpha_i, \alpha_j) - (-1)^q \theta(\alpha_j, \alpha_i);$$

similarly for $L(\partial C'_i, \partial C'_j)$. Therefore $L(\partial C_i, \partial C_j) = L(\partial C'_i, \partial C'_j)$, for $i \neq j$, and, since $q > 2$, the links $\{\partial C_i\}$ and $\{\partial C'_i\}$ are isotopic in ∂D^{2q} .

Clearly we may assume that the base disks D^{2q} in the handle decompositions of M and M' coincide as imbedded in S^{2q+1} . Thus the cores C_i and C'_i , as imbedded

in S^{2q+1} , may be assumed to coincide on their boundaries: $\partial C_i = \partial C'_i$. We next show how to isotopically deform $\{C_i\}$ onto $\{C'_i\}$, keeping $\{\partial C_i\}$ fixed and avoiding any intersections with D^{2q} (except, of course, along ∂C_i).

Assume inductively that $C_i = C'_i$ for $i < k$. We will isotopically deform C_k to C'_k , avoiding intersections with $D^{2q} \cup C_1 \cup \dots \cup C_{k-1}$. Given such an isotopy, we can extend it to an isotopy of $h_k \cup h_{k+1} \cup \dots \cup h_i$ in

$$S^{2q+1} - (D^{2q} \cup h_1 \cup \dots \cup h_{k-1});$$

the result is an isotopy of M to a new imbedding satisfying $C_i = C'_i$ for $i \leq k$. We begin with an isotopy of C_k to C'_k , rel ∂C_k , avoiding D^{2q} ; this exists according to Wu [20] because the imbeddings of C_k and C'_k and $S^{2q+1} - \text{int } D^{2q}$ are homotopic rel $\partial C_k = \partial C'_k$ and $S^{2q+1} - \text{int } D^{2q}$ is simply-connected. We would then like to use Whitney's procedure, as in [20], to remove the intersections of this isotopy with $I \times C_i$ ($i = 1, \dots, k-1$) in $S^{2q+1} - D^{2q}$, since $q \geq 2$ and $S^{2q+1} - D^{2q}$ is simply-connected. The only obstruction to this is the intersection number, which is easily seen to be (up to sign) $\theta(\alpha_i, \alpha_k) - \theta'(\alpha'_i, \alpha'_k) = 0$.

16. We now have achieved $C_i = C'_i$ for $i = 1, \dots, r$. By the tubular neighborhood theorem we may assume $h_i \cap D^{2q} = h'_i \cap D^{2q}$. Let $v_i(v'_i)$ be the positive unit normal field to $h_i(h'_i)$ on $C_i = C'_i$.

By the tubular neighborhood theorem, we may assume that $h_i(h'_i)$ is the orthogonal complement of $v_i(v'_i)$ in a normal disk bundle neighborhood N of $C_i = C'_i$ in S^{2q+1} . Therefore if we can homotopically deform v_i to v'_i , rel ∂C_i , we obtain an isotopy, rel $h_i \cap D^{2q}$, of h_i to h'_i within N . Doing this for all i achieves, finally, an isotopy of M to M' .

Since $v_i = v'_i$ along ∂C_i , v_i differs from v'_i by an element of $\Pi_q(S^q) \approx Z$ (the normal space to C_i in S^{2q+1} has dimension $q+1$). But this element can be identified with $\theta(\alpha_i, \alpha_i) - \theta'(\alpha'_i, \alpha'_i) = 0$, and so Lemma 3 is proved - for $q > 2$.

17. For $q = 2$, more work is required to repair those parts of the preceding argument which are no longer valid. First of all M and M' are not necessarily diffeomorphic. On the other hand, they are simply-connected 4-manifolds with boundaries diffeomorphic to S^3 and isomorphic intersection pairings (since their l -pairings are isomorphic). It then follows from [19] that, after adding on a number of copies of $S^2 \times S^2$, M and M' will be diffeomorphic. Since these enlargements of M and M' can be realized by adding to the Seifert matrices blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, as is demonstrated in § 13, we may as well assume that M and M' are diffeomorphic to start with. In fact, by [18], there is a diffeomorphism $f: M \rightarrow M'$ preserving the l -pairings i.e. $\theta = \theta' \circ (f_* \otimes f_*)$, where $f_*: H_2(M) \rightarrow H_2(M')$ is the induced homomorphism.

We reformulate the situation so far as follows. M is a simply-connected 4-manifold, ∂M diffeomorphic to S^3 , and we have imbeddings $g, g': M \rightarrow S^5$ such that $g(\partial M) = K$ and $g'(\partial M) = K'$. The l -pairings of $g(M)$ and $g'(M)$ are identical, as pairings on $H_2(M)$.

18. Now let

$$M = D^4 \cup h_1^1 \cup \dots \cup h_k^1 \cup h_1^2 \cup \dots \cup h_l^2 \cup h_1^3 \cup \dots \cup h_m^3 *$$

be a handle decomposition of M , where h_j^i is a handle of index i . Since $H_1(M) = 0$ we can choose the handles of index 2 in such a way that the first k of them – h_1^2, \dots, h_k^2 – homologically cancel out the handles of index 1 (see e.g. [10], Theorem 7.6) i.e. if $V = D^4 \cup h_1^1 \cup \dots \cup h_k^1$, then the boundary operator $H_2(M, V) \rightarrow H_1(V)$ maps the subgroup of $H_2(M, V)$ generated by the “cores” of h_1^2, \dots, h_k^2 isomorphically onto $H_1(V)$. Then

$$\Delta = D^4 \cup h_1^1 \cup \dots \cup h_k^1 \cup h_1^2 \cup \dots \cup h_k^2$$

is acyclic. Set

$$M_0 = \Delta \cup h_{k+1}^2 \cup \dots \cup h_l^2.$$

We will show that $g|_{M_0}$ and $g'|_{M_0}$ are isotopic.

First we show that $g|_{\Delta}$ and $g'|_{\Delta}$ are isotopic by extending $g' \circ g^{-1}: g(\Delta) \rightarrow g'(\Delta)$ to an orientation preserving diffeomorphism of S^5 . Begin by extending it to a tubular neighborhood T of $g(\Delta)$ diffeomorphic to $g(\Delta) \times I$ (with corners rounded). Now ∂T is a homology 4-sphere bounding the contractible 5-manifold $\overline{S^5 - T}$ – similarly for T' a tubular neighborhood of $g'(\Delta)$. That $\overline{S^5 - T}$ is acyclic follows from Alexander duality; that $\overline{S^5 - T}$ is simply-connected follows from the fact that Δ collapses onto a 2-dimensional polyhedron (it has only handles of index one and two) which has codimension > 2 in S^5 . We now invoke the case $n = 5$ of the following lemma (stated by Kato for the PL case in [3]) since $\Gamma^5 = 0$, to obtain the extension over S^5 .

19. LEMMA 4: *If C_1, C_2 are contractible smooth manifolds of dimension $n \geq 5$, then any diffeomorphism $d: \partial C_1 \rightarrow \partial C_2$ extends to a diffeomorphism of C_1 onto C_2 , after perhaps changing d on an $(n - 1)$ -disks in ∂C_1 .*

Proof of Lemma 4: Consider $W = C_1 \cup_d C_2$, a homotopy n -sphere. Since $n \geq 5$, W is homeomorphic to S^n ([15]) and, by changing d , we can insure that W is diffeomorphic to S^n . Then W bounds a copy of D^{n+1} which determines an h -cobordism

*) In fact an argument of A. Wallace, communicated to me by C. T. C. Wall, shows that only handles of index 2 are needed after connected sum with enough copies of $S^2 \times S^2$. This obviates the need for the arguments of § 18, 19 and 21, since $\Delta = D^4$ and $M_0 = M$.

from C_1 to C_2 , trivial from ∂C_1 to ∂C_2 (identifying them by d). By the relative version of the h -cobordism theorem ([15, Cor. 3.2]), d extends to a diffeomorphism from C_1 onto C_2 .

20. Now M_0 is obtained from Δ by attaching handles of index two. Since we may assume that $g|_{\Delta} = g'|_{\Delta}$, the problem of showing $g|M_0$ isotopic to $g'|M_0$ is similar to (but not exactly the same as) the argument above in § 15, 16. We need first to know that the l -pairings on $H_2(M_0)$, defined from g and g' , are identical, but this is because they are induced from the l -pairings on $H_2(M)$ by the inclusion $M_0 \subset M$. The argument in § 15 will then serve to show that g and g' are isotopic on the cores of the handles. To extend the isotopy to the entire handles, it will suffice to show that the normal 2-fields to the imbedded cores in S^5 , defined by applying the differentials of g and g' to the standard normal 2-fields to the cores in the handles of M_0 , are homotopic. But the obstructions are elements of $\Pi_2(V_{3,2})=0$, where $V_{3,2}$ is the Stiefel manifold of 2-fields in 3-space.

21. We now have shown that $g|M_0$ is isotopic to $g'|M_0$. The proof of Lemma 3 for $q=2$ will be completed by showing that ∂M can be isotopically deformed, in M , inside M_0 . This will use the engulfing theorem in its most naive form ([12, Lemma 2.7]). Let N be the closed simply-connected 4-manifold obtained from M by attaching a 4-disk D to ∂M . Given any handle-decomposition of N it follows from the engulfing theorem that the handles of index one are contained in a 4-disk imbedded in N . Applying this to the *dual* handle decomposition of that postulated in § 18, we find that $N - M_0$ is contained in a 4-disk D' in N . Since any two similarly oriented n -disks in an unbounded n -manifold are isotopic, D and D' are isotopic in N . It also may be arranged that any given point in $(\text{int } D) \cap (\text{int } D')$ is fixed during the isotopy. It is then easy to see that $\partial D (= \partial M)$ and $\partial D'$ are isotopic in $M = \overline{N - D}$. This completes the proof of Lemma 3, and so Theorem 3.

22. Let $f: S^n \rightarrow R^{n+2}$ be a smooth imbedding. Since its normal bundle is trivial, f extends to an imbedding $F: S^n \times D^2 \rightarrow R^{n+2}$ whose isotopy class is uniquely determined by f , if $n > 1$. Let $h: S^n \times S^1 \rightarrow S^n \times S^1$ be the diffeomorphism defined by $(x, y) \rightarrow (\Phi(y) \cdot x, y)$ where $\Phi: S^1 \rightarrow SO(n+1)$ represents the non-zero element of $\Pi_1(SO(n+1))$. Define

$$R_0 = S^n \times D^2 \cup_{F,h} \overline{R^{n+2} - F(S^n \times D^2)}$$

Representing R^{n+2} as the interior of D^{n+2} , the construction of R_0 represents R_0 as the interior of a compact manifold D_0 with boundary diffeomorphic to S^{n+1} . By [15], D_0 is diffeomorphic to D^{n+2} if $n \geq 3$; therefore R_0 is diffeomorphic to R^{n+2} . Consider

the knot $S^n \times 0 \subset S^n \times D^2 \subset R_0 \approx R^{n+2}$, which we denote by $\tau(K)$, if K is the knot $f(S^n)$. It follows easily that the isotopy class of $\tau(K)$ depends only on that of K .

The interest of $\tau(K)$ is that its complement is diffeomorphic to the complement of K and, besides K itself, is, up to isotopy, the only knot with this property (see [2] and [6]). It is not known whether $\tau(K)$ is ever *not* isotopic to K . We can prove:

COROLLARY 1: *If K is a simple $(2q-1)$ -knot, $q \geq 2$, then $\tau(K)$ is isotopic to K .*

Proof: Let M be a submanifold of $R^{n+2} \subset S^{n+2}$ bounded by $K=f(S^n)$ – we may assume that $M \cap F(S^n \times D^2) = F(S^n \times \varrho)$ where ϱ is any ray from the origin in D^2 . Since $h(S^n \times x_0) = S^n \times x_0$, for any $x_0 \in S^1$, we may define

$$M' = S^n \times \varrho \cup (M \cap \overline{R^{n+2} - F(S^n \times D^2)}),$$

a submanifold of R_0 bounded by $\tau(K)$. It is obvious that M' is diffeomorphic to M and the l -pairings coincide. Thus, by Theorem 3 (or even Lemma 3), K and $\tau(K)$ are isotopic.

23. Another consequence of Theorem 3 – not surprisingly – is the unknotting theorem of [7] and [14].* For if K is a $(2q-1)$ -knot with complement X and universal abelian covering \tilde{X} , and A is a Seifert matrix for K , then $tA + (-1)^q A^T$ is a relation matrix for $H_q(\tilde{X}; Q)$, as a module over the rational group ring $Q[Z] = Q[t, t^{-1}]$ (see [8]). Now A is equivalent to a non-singular matrix A' (allowing $A' = 0$) by Proposition 1 in § 24. But if A' has rank r , it follows easily that $H_q(\tilde{X}; Q)$ has dimension r as a Q -module. Thus, if $H_q(\tilde{X}; Q) = 0$, A must be equivalent to 0, which implies, by Theorem 3, that K is unknotted for $q \geq 2$.

24. We now turn to the algebraic problem presented by the notion of equivalence of matrices. Results are very incomplete, and most of them are contained implicitly in [16].

PROPOSITION 1: *Any matrix A such that $A + A^T$ is unimodular is equivalent to a non-singular matrix (i.e. with non-zero determinant) or zero.*

Proof: By the argument in [16, p. 484], if A is singular it admits an elementary reduction. Thus by a sequence of elementary reductions (and congruences) we may make A non-singular (or zero).

25. PROPOSITION 2: *Suppose that A and B are equivalent non-singular matrices. Then $\det A = \det B = d$, and A is congruent to B over any ring R in which d is a unit.*

*) The argument here applies, of course, only for odd dimensional knots. The case $q = 2$ was also announced by C. T. C. Wall: Proc. Camb. Phil. Soc. 63 (1967), p. 6.

Proof: This is proved implicitly in [16, Theorem 2] more or less as follows. Consider the matrix $tA - A^T$, with entries considered as elements of $Z[t, t^{-1}] = \Lambda$, and the Λ -module H_A with $tA - A^T$ as relation matrix. Consider also the bilinear form $[,]: H \otimes_{\Lambda} H \rightarrow Q(\Lambda)/\Lambda = S(\Lambda)$ ($Q(\Lambda)$ is the quotient field of Λ) defined, with respect to the same generators of H_A as $tA - A^T$ is a relation matrix, by the matrix $(tA - A^T)^{-1}$. Note that $tA - A^T$ is non-singular over $Q(\Lambda)$ because $\Delta_A(t) = \det(tA - A^T)$ is non-zero. In fact the leading coefficient of $\Delta_A(t)$ is $\det A \neq 0$.

It is not hard to see that the isomorphism class of $(H_A; [,])$ is an invariant of the equivalence class of A . Furthermore the element $\Delta_A(t) = \det(tA - A^T)$ is an invariant of the equivalence class, up to multiplication by powers of t . From this it follows that $\det A = \det B$.

If A is unimodular over R , then $t - A^{-1}A^T$ is a presentation matrix of $H_A \otimes R (\otimes = \otimes_Z)$, so $H_A \otimes R$ is a free R -module of the same rank as A and $(tA - A^T)^{-1}$ is the matrix for $[,]$ with respect to an R -basis for $H_A \otimes R$. From this it follows that there exists a matrix P with entries in, and unimodular over, R such that

$$P(tA - A^T)^{-1} P^T = (tB - B^T)^{-1} \quad \text{in } S(\Lambda) \otimes R.$$

Let $\overline{(tA - A^T)}$ be the "adjoint" of $tA - A^T$ ([1, p. 305]). Then

$$\left. \begin{aligned} P \overline{(tA - A^T)} P^T &= \overline{tB - B^T} \quad \text{in } (\Lambda/(\Delta(t)) \otimes R, \\ &\text{where } \Delta(t) = \Delta_A(t) = \Delta_B(t). \end{aligned} \right\} \quad (*)$$

Since A and B are unimodular over R , $\Delta(t)$ has as leading coefficient a unit of R . From this it follows that there exists a unique well-defined R -linear map $\gamma: \Lambda/(\Delta(t)) \otimes R \rightarrow R$ defined by the properties $\gamma(1) = 1$ and $\gamma(t^i) = 0$ for $0 < i < \text{degree } \Delta(t)$. Now every entry of $\overline{tA - A^T}$ and $\overline{tB - B^T}$ has degree $< \text{rank } A = \text{degree } \Delta(t)$. Therefore, applying γ to equation (*) we find that $PAP^T = B$ in R .

26. We cannot strengthen the conclusion of Proposition 2 to conclude that A and B are congruent over K , as the following example shows. Set:

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

We first show that A and B are not congruent over Z . Consider the solutions X of $X^T A X = X^T B X = 2$; they are $X = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$. If $P^T A P = B$, it follows that $PX = \pm X$ (say $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$). Now $BX = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and so $Y^T B X$ is even, for any Y . Choose Y so that $PY = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; then $Y^T B X = Y^T P^T A P X = \pm (PY)^T A X$ which one can calculate to be ± 1 .

To see that A and B are equivalent, we consider the following elementary enlargements, respectively, of A and B :

$$A' = \left(\begin{array}{c|cc} \mathbf{A} & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \quad B' = \left(\begin{array}{c|cc} \mathbf{B} & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right)$$

A' and B' are congruent; in fact, $PA'P^T = B'$, where

$$P = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

27. On the other hand, the converse of Proposition 2 is false. Consider the following matrices for $\varepsilon = \pm 1$:

$$A = \begin{pmatrix} 0 & \varepsilon x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & p^2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p(1 + \varepsilon) + 1 \\ 0 & 0 & 0 & 0 & 0 & p + 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \varepsilon x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 \\ p^4 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p(1 + \varepsilon) + 1 \\ 0 & 0 & 0 & 0 & 0 & p + 1 \end{pmatrix}$$

where p is any odd prime, and $x = \frac{1}{4}(p^4 - 1)$. It may be checked directly that $A + \varepsilon A^T$ and $B + \varepsilon B^T$ are unimodular and $\det A = \det B$ is divisible by p . But A and B are congruent over any ring in which p is a unit. In fact $PAP^T = B$ where:

$$P = \begin{pmatrix} p & & & & & \\ & 1/p & & & & \\ & & \mathbf{0} & & & \\ & & & p & & \\ & & & & 1/p & \\ \mathbf{0} & & & & & 1 \\ & & & & & & 1 \end{pmatrix}$$

Finally, A and B are not equivalent. To see this consider $A - \varepsilon A^T$ and $B - \varepsilon B^T$ over Z_p . It follows from Witts Theorem (see e.g. [21]) for $\varepsilon = -1$, or the well-known classification of skew-symmetric forms (see e.g. [1]) for $\varepsilon = +1$, that the congruence class of $A - \varepsilon A^T$ over Z_p is an invariant of the equivalence class of A . But $A - \varepsilon A^T$ and $B - \varepsilon B^T$ have ranks 2 and 4, respectively, over Z_p .

28. Finally we remark that the genus of the non-degenerate quadratic form $A + A^T$ is an invariant of the equivalence class of A (see [16, Prop. 5.1]).

REFERENCES

- [1] BIRKHOFF, G. and MACLANE, S.: *A survey of modern algebra* (MacMillan Co., New York 1953).
- [2] BROWDER, W.: *Diffeomorphisms of 1-connected manifolds*, Trans. Amer. Math. Soc. 128 (1967), 155–63.
- [3] KATO, M.: *Embedding spheres and balls in codimension ≤ 2* (mimeographed notes).
- [4] KERVAIRE, M.: *Les nœuds de dimensions supérieures*, Bull. Soc. Math. France, 93 (1965), 225–71.
- [5] KERVAIRE, M. and MILNOR, J.: *Groups of homotopy spheres*, I, Annals of Math. 77 (1963), 504–37.
- [6] LASHOF, R. and SHANESON, J.: *Classification of knots in codimension two*, Bull. Amer. Math. Soc. 75 (1969), 171–5.
- [7] LEVINE, J.: *Unknotting spheres in codimension two*, Topology 4 (1965), 9–16.
- [8] LEVINE, J.: *Polynomial invariants of knots of codimension two*, Annals of Math. 84 (1966), 537–54.
- [9] LEVINE, J.: *Knot cobordism groups in codimension two*, Comment. Math. Helv. 44 (1969), 229–44.
- [10] MILNOR, J.: *Lectures on the h-cobordism theorem*, Princeton Mathematical Notes (Princeton U. Press, Princeton 1965).
- [11] MURASUGI, K.: *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965), 387–422.
- [12] PENROSE, R., WHITEHEAD, H. H. C., and ZEEMAN, E. C.: *Imbeddings of manifolds in Euclidean space*, Annals of Math. 73 (1961), 613–23.
- [13] SEIFERT, H.: *Über das Geschlecht von Knoten*, Math. Ann. 110 (1934), 571–92.
- [14] SHANESON, J.: *Embeddings of spheres in spheres of codimension two and h-cobordism of $S^1 \times S^3$* , Bull. Amer. Math. Soc. 74 (1968), 972–4.
- [15] SMALE, S.: *On the structure of manifolds*, Amer. J. Math. 84 (1962), 387–99.
- [16] TROTTER, H.: *Homology of groups system with applications to knot theory*, Annals of Math. 76 (1962), 464–98.
- [17] WALL, C. T. C.: *Classification of $(n - 1)$ -connected $2n$ -manifolds*, Annals of Math. 75 (1962), 163–89.
- [18] WALL, C. T. C.: *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. 39 (1964), 131–40.
- [19] WALL, C. T. C.: *On simply-connected 4-manifolds*, J. London Math. Soc. 39 (1964), 141–9.
- [20] WU, W. T.: *On the isotopy of C^r -manifolds of dimension n in Euclidean $(2n + 1)$ -space*, Science Record, New Series II (1958), 271–5.

Received December 1, 1969