**Zeitschrift:** Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

**Band:** 45 (1970)

**Artikel:** An Algebraic Classification of some Knots of Codimension Two.

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**DOI:** https://doi.org/10.5169/seals-34652

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# An Algebraic Classification of Some Knots of Codimension Two1)

by J. LEVINE

An *n*-knot will denote a smooth oriented submanifold K of the (n+2)-sphere  $S^{n+2}$ , where K is homeomorphic to  $S^n$ . If n is odd, one can associate to K a square integral matrix A, called a Seifert matrix of K, using a submanifold of  $S^{n+2}$  bounded by K (see [13] for n=1, and [4] or [8] in general). When n=1, it is known that two Seifert matrices of isotopic knots are related by certain algebraic "moves" (see [11], [16]). In this paper we will generalize this fact to all n. We then consider, for n odd, n-knots (referred to as simple) whose complements are of the same ((n-1)/2) type as a circle, i.e.  $\Pi_q(S^{n+2}-K)\approx \Pi_q(S^1)$  for  $q\leqslant (n-1)/2$ . This is the most that can be asked without making K unknotted (see [7]). We will show that two simple n-knots  $(n\geqslant 3)$  are isotopic if and only if their Seifert matrices are related by such "moves". Thus it will follow that the semi-group of isotopy classes of simple n-knots depends only on the residue class, mod 4, of n for  $n\geqslant 4$ .

By contrast, Lashof and Shaneson [6] (and, independently, Browder) have shown that the isotopy class of an n-knot ( $n \ge 3$ ) K, whose complement is of the same 1-type as a circle is determined by the homotopy type of its exterior pair  $(X, \partial X)$ , where X is the complement of an open tubular neighborhood of K in  $S^{n+2}$ -except for one other possible knot  $\tau(K)$ , obtained from K by removing a tubular neighborhood twisting, and reinserting in  $S^{n+2}$ . It is not known whether  $\tau(K)$  is ever different from K. As a straightforward application, we will show that  $\tau(K)$  is isotopic to K if K is simple.

We conclude with some remarks on the algebraic problems which arise.

1. Let K be a (2q-1)-knot in  $S^{2q+1}$ . We recall the definition of a Seifert matrix of K. Let M be a smooth oriented submanifold of  $S^{2q+1}$  bounded by K. The *l-pairing* of M:

$$\theta: H_q(M) \otimes H_q(M) \to \mathbb{Z}$$

is defined by letting  $\theta(\alpha \otimes \beta)$  be the linking number  $L(z_1, z_2)$ , where  $z_1$  is a cycle in M representing  $\alpha$  and  $z_2$  is the translate in the positive normal direction off M of a cycle in M representing  $\beta$ . A Seifert matrix A of K is then a representative matrix of  $\theta$  with respect to a basis of the torsion-free part of  $H_a(M)$  – see e.g. [8].

We recall also the formula [8]:

$$\theta(\alpha \otimes \beta) + (-1)^q \theta(\beta \otimes \alpha) = \alpha \cdot \beta$$

<sup>1)</sup> This work was done while the author was partially supported by NSF GP 8885.

where  $\alpha \cdot \beta$  is the intersection number in M. Thus  $A + (-1)^q A^T$  is unimodular  $(A^T)$  is the transpose of A) and, if q = 2,  $A + A^T$  has signature a multiple of 16 (see [9]).

2. Let A be a square integral matrix. Any matrix of the form:

$$\begin{pmatrix} \underline{A} & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \underline{A} & \beta & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where  $\alpha$  is a row vector,  $\beta$  a column vector, will be called an *elementary enlargement* of A. A is an *elementary reduction* of any of its elementary enlargements. Two matrices (or their associated pairings) are equivalent if they can be connected by a chain of elementary enlargements, reductions and unimodular congruences. It is proved in [11] that Seifert matrices of isotopic 1-knots are equivalent. We shall prove:

THEOREM 1: Seifert matrices of isotopic knots of any (odd) dimension are equivalent.

THEOREM 2: Let q be a positive integer and A a square integral matrix such that  $A + (-1)^q A^T$  is unimodular and, if q = 2,  $A + A^T$  has signature a multiple of 16. If  $q \neq 2$ , there is a simple (2q-1)-knot with Seifert matrix A; if q = 2, there is a simple 3-knot with Seifert matrix equivalent to A.

THEOREM 3: Let  $q \ge 2$  and  $K_1$ ,  $K_2$  simple (2q-1)-knots with equivalent Seifert matrices. Then  $K_1$  is isotopic to  $K_2$ .

3. Proof of Theorem 1: Suppose  $K_1$ ,  $K_2$  are isotopic (2q-1)-knots bounding manifolds  $M_1$ ,  $M_2$ , respectively, of  $S^{2q+1}$ . We first construct a submanifold V (with corners) of  $I \times S^{2q+1}$  meeting  $0 \times S^{2q+1}$  along  $0 \times M_1$  and  $1 \times S^{2q+1}$  along  $1 \times M_2$  with boundary the union of  $0 \times M_1$ ;  $1 \times M_2$  and the trace X of an isotopy from  $K_1$  to  $K_2$ . We use the Pontriagin-Thom construction as follows. First construct a normal vector field to  $(0 \times M_1) \cup X \cup (1 \times M_2) = Y$  in  $I \times S^{2q+1}$ , which is tangent to  $I \times S^{2q+1}$  along  $0 \times M_1 \cup 1 \times M_2$ . If  $q \ne 1$ , there is no obstruction. If q = 1, the obstruction to extending such a vector field from  $0 \times M_1 \cup 1 \times M_2$  over Y is the difference in its winding numbers about  $K_1$  and  $K_2$ . But since the field is defined over  $M_1$  and  $M_2$ , these winding numbers are zero.

Let T be a tubular neighborhood of X. We can "translate"  $(X, v \mid X)$  to a framed submanifold of  $\partial T$  which agress with the framed submanifold  $(0 \times M_1 \cup 1 \times M_2, v)$  on  $\partial T \cap (I \times S^{2q+1})$ . Let  $W = I \times S^{2q+1} - T$ ; the Pontriagin-Thom construction on the above framed submanifolds of  $\partial W$  determines a map  $\partial W \to S^1$ . An extension of

this map over W will determine the desired V. The obstruction lies in

$$H^2(W,\partial W)\approx H^2(I\times S^{2q+1},X\cup\dot{I}\times S^{2q+1})\approx H^1(X\cdot\cup\dot{I}\times S^{2q+1})=0.$$

- 4. Now let  $\Phi': V \to I$  be the "height" function defined by the restriction of the projection  $I \times S^{2q+1} \to I$ . We may assume  $\Phi'$  has no critical points in a neighborhood of  $\partial V$  (omitting corners). Let  $\Phi$  be a  $C^2$ -approximation to  $\Phi'$  which agrees with  $\Phi'$  in a neighborhood of  $\partial V$  and has only non-degenerate critical points (except at corners) which are mapped one-one into I (see e.g. [10]). We can move V so that  $\Phi$  becomes the new height function. In fact if  $p: V \to S^{2q+1}$  is defined by the projection  $I \times S^{2q+1} \to S^{2q+1}$  and  $\Phi$  is a close enough approximation to  $\Phi'$ , then  $x \mapsto (\Phi(x), p(x))$  defines a new imbedding  $V \to I \times S^{2q+1}$  which agrees with the original inclusion near  $\partial V$  and has  $\Phi$  as its new height function.
- 5. Let  $0 = t_0 < t_1 < \dots < t_k = 1$  be a partition of I satisfying
  - (i) each  $t_i$  is a regular value of  $\Phi$ ,
  - (ii) at most one critical value of  $\Phi$  lies in each interval  $(t_i, t_{i+1})$ .

Let  $\Phi^{-1}(t_i) = t_i \times M_i'$ ; then each  $M_i'$  is bounded by a knot isotopic to  $K_0$  and  $K_1$ , and  $M_0' = M_1$ ,  $M_k' = M_2$ . This shows that it suffices to consider the case where  $\Phi$  has only one critical point.

LEMMA 1: Let  $\alpha$ ,  $\alpha' \in H_q(M_1)$  and  $\beta$ ,  $\beta' \in H_q(M_2)$  and suppose that  $\alpha$  is homologous to  $\beta$  and  $\alpha'$  homologous to  $\beta'$  in V. Then  $\theta_1(\alpha, \alpha') = \theta_2(\beta, \beta')$ , where  $\theta_i$  is the l-pairing of  $M_i$ .

*Proof:* Let C, C' be (q+1)-chains in V such that  $\partial C = \alpha - \beta$ ,  $\partial C' = \alpha' - \beta'$ . Then it follows from the definition of  $\theta_1$ ,  $\theta_2$  that  $\theta_1(\alpha, \alpha') - \theta_2(\beta, \beta')$  is the intersection number of C and the translate of C' off V in the positive normal direction – but this is obviously zero.

6. Now consider the following diagram:

$$H_{q+1}(V, M_2)$$

$$\downarrow$$

$$H_q(M_2)$$

$$\downarrow$$

$$H_{q+1}(V, M_1) \rightarrow H_q(M_1) \rightarrow H_q(V) \rightarrow H_q(V, M_1)$$

$$\downarrow$$

$$H_q(V, M_2)$$

consisting of the exact homology sequences of  $(V, M_1)$  and  $(V, M_2)$ . If the index of

the critical point of  $\Phi$  is not q or q+1, then

$$H_a(V, M_1) = H_{a+1}(V, M_1) = H_a(V, M_2) = H_{a+1}(V, M_2) = 0$$

and we have

$$H_a(M_1) \approx H_a(V) \approx H_a(M_2)$$
.

It follows from Lemma 1 that  $\theta_1$  and  $\theta_2$  are congruent.

If the index of the critical point of  $\Phi$  is q, then

$$H_a(V, M_1) \approx H_{a+1}(V, M_2) \approx Z$$

and

$$H_q(V, M_2) = H_{q+1}(V, M_1) = 0.$$

If  $\alpha \in H_q(M_2)$  is the image of a generator of  $H_{q+1}(V, M_2)$ , then the composite

$$H_q(M_2) \to H_q(V) \to H_q(V, M_1) \approx \mathbb{Z}$$

can be defined by  $\beta \to \alpha \cdot \beta =$  intersection number in  $M_2$  (see [5]). If  $\alpha$  has finite order, then it follows that  $H_q(M_1) \approx H_q(V) \approx H_q(M_2)$ , modulo torsion, and, therefore,  $\theta_1$  and  $\theta_2$  are congruent modulo torsion.

- 7. Suppose  $\alpha$  has infinite order; then  $\alpha$  is a multiple of a primitive element  $\alpha_0$  and there exists  $\beta_0 \in H_q(M_2)$  with  $\alpha_0 \cdot \beta_0 = 1$ . Suppose  $\gamma_1', ..., \gamma_s' \in H_q(M_2)$  such that:
  - (i)  $\gamma'_i$  is homologous to  $\gamma_i$  in V, and
  - (ii)  $\alpha_0, \beta_0, \gamma'_1, ..., \gamma'_s$  is a basis of  $H_a(M_2)$ , modulo torsion.

We now examine  $\theta_2$  on the elements  $\alpha_0$ ,  $\beta_0$ ,  $\gamma_1'$ , ...,  $\gamma_s'$ . By Lemma 1 we can conclude from (i) that  $\theta_2(\gamma_i', \gamma_j') = \theta_1(\gamma_i, \gamma_j)$ . Since  $\alpha$  is null-homologous in V,  $\theta_2(\alpha, \gamma_i') = \theta_1(0, \gamma_i) = 0$  and  $\theta_2(\alpha, \alpha) = \theta_2(0, 0) = 0$ . Thus  $\theta_2(\alpha_0, \gamma_i') = \theta_2(\alpha_0, \alpha_0) = 0$ ; similarly  $\theta_2(\gamma_i', \alpha_0) = 0$ . We also recall that (§ 1):

$$\theta_2(\alpha_0, \beta_0) + (-1)^q \theta_2(\beta_0, \alpha_0) = -\alpha_0 \cdot \beta_0 = -1$$

8. We may summarize this as follows. Let A be the matrix representative of  $\theta_1$  with respect to the basis  $\gamma_1, ..., \gamma_s$ . The the matrix representative of  $\theta_2$  with respect to the basis  $\gamma_1', ..., \gamma_s', \alpha_0, \beta_0$  has the form:

$$B = \begin{pmatrix} \mathbf{A} & 0 & \\ \vdots & \eta \\ 0 & 0 & \\ \hline 0 \dots 0 & 0 & x \\ \xi & x' & y \end{pmatrix}$$

where x, y are integers,  $x + (-1)^q x' = -1$ ,  $\xi$  is a row vector and  $\eta$  is a column vector.

Recall from e.g. [8] that the polynomial  $\Delta_A(t) = \det(tA + (-1)^q A^T)$ , where A is a Seifert matrix for a knot K, is an invariant of the isotopy class of K (up to multiplication by a unit in  $\mathbb{Z}[t, t^{-1}]$ ). But it is easily verified that:

$$\Delta_B(t) = (tx + (-1)^q x')(tx' + (-1)^q x) \Delta_A(t)$$

Thus x (or x') is zero, since  $x \pm x' = -1$ , then x' (or x) is  $\pm 1$ . It now is easily checked that B is congruent to an elementary enlargement of A.

- 9. If the index of the critical point of  $\Phi$  is q+1; then its index as a critical point of  $-\Phi$  is q. The preceding arguments apply to show that  $\theta_2$  is congruent to  $\theta_1$ , or has, as representative matrix, an elementary reduction of a representative matrix of  $\theta_1$ . This completes the proof of Theorem 1.
- 10. Proof of Theorem 2: For  $q \neq 2$ , this is proved in [4] (see also [9]). For q = 2, we must show that A is equivalent to a matrix B, where  $B + B^T$  is a matrix representative of the intersection pairing of some simply-connected closed 4-manifold. By an argument in [9], such a B can be obtained by adding enough blocks  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  to A; but this is a sequence of elementary enlargements of A, so B is equivalent to A.
- 11. Proof of Theorem 3: We reduce Theorem 3 to two lemmas. Recall (see [7]) that a simple (2q-1)-knot bounds a (q-1)-connected submanifold of  $S^{2q+1}$ . A Seifert matrix obtained from the *l*-pairing of such a submanifold will be called *special*.
- LEMMA 2: Let K be a simple (2q-1)-knot with a special Seifert matrix A. If B is an elementary enlargement of A, then B is also a special Seifert matrix of K.
- LEMMA 3: If  $q \ge 2$ , then simple (2q-1)-knots admitting identical special Seifert matrices are isotopic.
- 12. We first show that Theorem 3 follows from Lemmas 2 and 3. Let K, K' be simple (2q-1)-knots with equivalent Seifert matrices,  $q \ge 2$ . Let A, A' be special Seifert matrices of K, K', respectively. Thus there exists a sequence:  $A = A_1, A_2, ..., A_k = A'$ , where each  $A_{i+1}$  is unimodularly congruent to an elementary enlargement or reduction. of  $A_i$  It follows from Theorem 2 that, for q > 2, each  $A_i$  is a special Seifert matrix of a simple (2q-1)-knot  $K_i$  (actually the proof of Theorem 2 (see [4]) realizes S as a special Seifert matrix of a simple knot). If q = 2, we can enlarge each  $A_i$  by adding a constant number of blocks  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  to obtain a new sequence  $A'_1, A'_2, ..., A'_k$ . Each  $A'_{i+1}$  is again congruent to an elementary enlargement or reduction of  $A'_i$  and it

now follows from the argument in § 10 that each  $A'_i$  is a special Seifert matrix of a simple 3-knot  $K_i$ .

We now prove that each  $K_i$  is isotopic to  $K_{i+1}$ . Suppose  $A_{i+1}$  is congruent to an elementary enlargement of  $A_i$ . It follows from Lemma 2 that  $A_{i+1}$  (or  $A'_{i+1}$ ) is a special Seifert matrix of  $K_i$ . Then Lemma 3 implies  $K_i$  and  $K_{i+1}$ , both of which now admit  $A_{i+1}$  (or  $A'_{i+1}$ ) as a special Seifert matrix, are isotopic. If  $A_{i+1}$  is congruent to an elementary reduction of  $A_i$ , the same argument works, switching the roles of  $K_i$  and  $K_{i+1}$ .

We may as well have chosen  $K_1 = K$  and  $K_k = K'$  if q > 2, but if q = 2 we need to show that  $K_1$  is isotopic to K and  $K_k$  is isotopic to K'. It follows from Lemma 2 that  $A'_1$  is a special Seifert matrix of K, since  $A'_1$  is obtained from  $A_1$  by a sequence of elementary enlargements. Then Lemma 3 implies K and  $K_1$  are isotopic – similarly for K' and  $K_k$ .

13. Proof of Lemma 2: Let M be a (q-1)-connected submanifold of  $S^{2q+1}$  bounded by K, and  $\alpha_1, ..., \alpha_n$  a basis of  $H_q(M)$ , modulo torsion, such that A is the corresponding matrix representative of the l-pairing of M. Let  $x_1, ..., x_n$  be an arbitrary sequence of integers. It follows from Alexander duality that there exists a cycle  $z \in H_q(S^{q+1}-M)$  such that the linking numbers  $L(z, \alpha_i) = x_i$ , for i = 1, ..., n. Now  $S^{2q+1} - M$  is (q-1)-connected and so z is spherical; by general position, z can be represented by an imbedded q-sphere  $\sigma \subset S^{2q+1} - M$ . The normal bundle to  $\sigma$  is trivial and so a tubular neighborhood T can be identified with  $\sigma \times D^{q+1}$  – we may assume T disjoint from M. Orient  $\partial T$  so that the positive normal direction in  $S^{2q+1}$  points into T and let M' be the connected sum in  $S^{2q+1}$  of M and  $\partial T$ . Then  $H_q(M')$  has rank two greater than the rank of  $H_q(M)$ , and  $\alpha_1, ..., \alpha_n$  may be extended to a basis of  $H_q(M')$ , modulo torsion, by adjoining the homology classes  $\beta_1, \beta_2$  of  $\sigma \times y_0$  and  $x_0 \times S^q \subset \sigma \times S^q = \partial T$ , respectively. The representative matrix of the l-pairing of M' with respect to the basis  $\alpha_1, ..., \alpha_n, \beta_1, \beta_2$  is

$$\begin{bmatrix}
A & \pm x_1 & 0 \\
\vdots & \vdots \\
\pm x_n & 0 \\
x_1 \dots x_n & x & 0 \\
0 \dots 0 & \pm 1 & 0
\end{bmatrix}$$

which is congruent to:

$$\begin{bmatrix}
A & 0 & 0 \\
\vdots & \vdots \\
x_1 \dots x_n & 0 & 0 \\
0 \dots 0 & 1 & 0
\end{bmatrix}$$

If z is chosen so that  $L(\alpha_i, z) = x_i$  for i = 1, ..., n, and  $\partial T$  is oriented so that the positive normal direction points out from T, then the representative matrix of the *l*-pairing of M' with respect to  $\alpha_1, ..., \alpha_n, \beta_1, \beta_2$  is:

$$\begin{bmatrix}
A & x_1 & 0 \\
\vdots & \vdots & \vdots \\
x_n & 0 \\
\hline
\pm x_1 \dots \pm x_n & x & \pm 1 \\
0 \dots 0 & 0 & 0
\end{bmatrix}$$

which is congruent to:

$$\begin{bmatrix}
A & x_1 & 0 \\
\vdots & \vdots \\
x_n & 0 \\
0 \dots 0 & 0 & 1 \\
0 \dots 0 & 0 & 0
\end{bmatrix}$$

Thus we can realize any elementary enlargement of A as a special Seifert matrix of K.

14. Proof of Lemma 3: Suppose K and K' are (2q-1)-knots bounding (q-1)-connected submanifolds M and M' of  $S^{2q+1}$  with l-pairings  $\theta$  and  $\theta'$ . Suppose also that there exists an isomorphism  $\Phi: H_q(M) \to H_q(M')$  preserving the l-pairings, i.e.  $\theta = \theta' \circ (\Phi \otimes \Phi)$ .

Let us assume, for now, q>2; we will show that M and M' are isotopic submanifolds of  $S^{2q+1}$ . According to [15], M and M' have handle decompositions:

$$M = D^{2q} \cup h_1 \cup \dots \cup h_r$$
  
$$M' = D^{2q} \cup h'_1 \cup \dots \cup h'_r$$

where each  $h_i$ ,  $h'_i$  is a handle of index q – diffeomorphic to  $D^q \times D^q$ . The  $h_i(h'_i)$  are attached to  $D^{2q}$  by disjoint imbeddings  $S^{q-1} \times D^q \to \partial D^{2q}$ . Let  $C_i(C'_i)$  be the "core" of  $h_i(h'_i)$ , i.e. the submanifold corresponding to  $D^q \times 0$  – then  $\partial C_i = C_i \cap D^{2q}$ .

The imbedded disks  $(C_i, \partial C_i) \subset (M, D^{2q})$  represent a basis  $\{\alpha_i\}$  of  $H_q(M, D^{2q}) \approx H_q(M)$ . According to handle body theory (see [17]), we can choose a handle-decomposition realizing any prescribed basis  $\{\alpha_i\}$ . Thus if  $\{\alpha_i'\}$  is the basis of  $H_q(M')$  defined by  $(C_i', \partial C_i') \subset (M', D^{2q})$ , we may, by setting  $\alpha_i' = \Phi(\alpha_i)$ , assume  $\theta(\alpha_i, \alpha_j) = \theta'(\alpha_i', \alpha_j')$ .

15. Now consider the links  $\{\partial C_i\}$  and  $\{\partial C_i'\}$  in  $\partial D^{2q}$ ; by [17] and § 1 we have:

$$L(\partial C_i, \partial C_j) = \alpha_i \cdot \alpha_j = -\theta(\alpha_i, \alpha_j) - (-1)^q \theta(\alpha_j, \alpha_i);$$

similarly for  $L(\partial C_i', \partial C_j')$ . Therefore  $L(\partial C_i, \partial C_j) = L(\partial C_i', \partial C_j')$ , for  $i \neq j$ , and, since q > 2, the links  $\{\partial C_i\}$  and  $\{\partial C_i'\}$  are isotopic in  $\partial D^q$ .

Clearly we may assume that the base disks  $D^{2q}$  in the handle decompositions of M and M' coincide as imbedded in  $S^{2q+1}$ . Thus the cores  $C_i$  and  $C'_i$ , as imbedded

in  $S^{2q+1}$ , may be assumed to coincide on their boundaries:  $\partial C_i = \partial C_i'$ . We next show how to isotopically defrom  $\{C_i\}$  onto  $\{C_i'\}$ , keeping  $\{\partial C_i\}$  fixed and avoiding any intersections with  $D^{2q}$  (except, of course, along  $\partial C_i$ ).

Assume inductively that  $C_i = C_i'$  for i < k. We will isotopically deform  $C_k$  to  $C_k'$ , avoiding intersections with  $D^{2q} \cup C_1 \cup \cdots \cup C_{k-1}$ . Given such an isotopy, we can extend it to an isotopy of  $h_k \cup h_{k+1} \cup \cdots \cup h_i$  in

$$S^{2q+1} - (D^{2q} \cup h_1 \cup \cdots \cup h_{k-1});$$

the result is an isotopy of M to a new imbedding satisfying  $C_i = C_i'$  for  $i \le k$ . We begin with an isotopy of  $C_k$  to  $C_k'$ , rel $\partial C_k$ , avoiding  $D^{2q}$ ; this exists according to Wu [20] because the imbeddings of  $C_k$  and  $C_k'$  and  $S^{2q+1} - \operatorname{int} D^{2q}$  are homotopic rel $\partial C_k = \partial C_k'$  and  $S^{2q+1} - \operatorname{int} D^{2q}$  is simply-connected. We would then like to use Whitney's procedure, as in [20], to remove the intersections of this isotopy with  $I \times C_i$   $(i=1,\ldots,k-1)$  in  $S^{2q+1} - D^{2q}$ , since  $q \ge 2$  and  $S^{2q+1} - D^{2q}$  is simply-connected. The only obstruction to this is the intersection number, which is easily seen to be (up to sign)  $\theta(\alpha_i, \alpha_k) - \theta'(\alpha_i', \alpha_k') = 0$ .

16. We now have achieved  $C_i = C'_i$  for i = 1, ..., r. By the tubular neighborhood theorem we may assume  $h_i \cap D^{2q} = h'_i \cap D^{2q}$ . Let  $v_i(v'_i)$  be the positive unit normal field to  $h_i(h'_i)$  on  $C_i = C'_i$ .

By the tubular neighborhood theorem, we may assume that  $h_i(h'_i)$  is the orthogonal complement of  $v_i(v'_i)$  in a normal disk bundle neighborhood N of  $C_i = C'_i$  in  $S^{2q+1}$ . Therefore if we can homotopically deform  $v_i$  to  $v'_i$ , rel $\partial C_i$ , we obtain an isotopy, rel $h_i \cap D^{2q}$ , of  $h_i$  to  $h'_i$  within N. Doing this for all i achieves, finally, an isotopy of M to M'.

Since  $v_i = v_i'$  along  $\partial C_i$ ,  $v_i$  differs from  $v_i'$  by an element of  $\Pi_q(S^q) \approx Z$  (the normal space to  $C_i$  in  $S^{2q+1}$  has dimension q+1). But this element can be identified with  $\theta(\alpha_i, \alpha_i) - \theta'(\alpha_i', \alpha_i') = 0$ , and so Lemma 3 is proved  $-for \ q > 2$ .

17. For q=2, more work is required to repair those parts of the preceding argument which are no longer valid. First of all M and M' are not necessarily diffeomorphic. On the other hand, they are simply-connected 4-manifolds with boundaries diffeomorphic to  $S^3$  and isomorphic intersection pairings (since their l-pairings are isomorphic). It then follows from [19] that, after adding on a number of copies of  $S^2 \times S^2$ , M and M' will be diffeomorphic. Since these enlargements of M and M' can be realized by adding to the Seifert matrices blocks  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ , as is demonstrated in § 13, we may as well assume that M and M' are diffeomorphic to start with. In fact, by [18], there is a diffeomorphism  $f: M \rightarrow M'$  preserving the l-pairings i.e.  $\theta = \theta' \circ (f_* \otimes f_*)$ , where  $f_*: H_2(M) \rightarrow H_2(M')$  is the induced homomorphism.

We reformulate the situation so far as follows. M is a simply-connected 4-manifold,  $\partial M$  diffeomorphic to  $S^3$ , and we have imbeddings  $g, g': M \rightarrow S^5$  such that  $g(\partial M) = K$  and  $g'(\partial M) = K'$ . The l-pairings of g(M) and g'(M) are identical, as pairings on  $H_2(M)$ .

## 18. Now let

$$M = D^4 \cup h_1^1 \cup \dots \cup h_k^1 \cup h_1^2 \cup \dots \cup h_l^2 \cup h_1^3 \cup \dots \cup h_m^3 *$$

be a handle decomposition of M, where  $h_j^i$  is a handle of index i. Since  $H_1(M)=0$  we can choose the handles of index 2 in such a way that the first k of them  $-h_1^2, ..., h_k^2$  homologically cancel out the handles of index 1 (see e.g. [10], Theorem 7.6]) i.e. if  $V = D^4 \cup h_1^1 \cup ... \cup h_k^1$ , then the boundary operator  $H_2(M, V) \to H_1(V)$  maps the subgroup of  $H_2(M, V)$  generated by the "cores" of  $h_1^2, ..., h_k^2$  isomorphically onto  $H_1(V)$ . Then

$$\Delta = D^4 \cup h_1^1 \cup \cdots \cup h_k^1 \cup h_1^2 \cup \cdots \cup k_k^2$$

is acyclic. Set

$$M_0 = \Delta \cup h_{k+1}^2 \cup \cdots \cup h_l^2$$
.

We will show that  $g \mid M_0$  and  $g' \mid M_0$  are isotopic.

First we show that  $g \mid \Delta$  and  $g' \mid \Delta$  are isotopic by extending  $g' \circ g^{-1} : g(\Delta) \to g'(\Delta)$  to an orientation preserving diffeomorphism of  $S^5$ . Begin by extending it to a tubular neighborhood T of  $g(\Delta)$  diffeomorphic to  $g(\Delta) \times I$  (with corners rounded). Now  $\partial T$  is a homology 4-sphere bounding the contractible 5-manifold  $S^5 - T$  – similarly for T' a tubular neighborhood of  $g'(\Delta)$ . That  $S^5 - T$  is acyclic follows from Alexander duality; that  $\overline{S^5 - T}$  is simply-connected follows from the fact that  $\Delta$  collapses onto a 2-dimensional polyhedron (it has only handles of index one and two) which has codimension > 2 in  $S^5$ . We now invoke the case n = 5 of the following lemma (stated by Kato for the PL case in [3]) since  $\Gamma^5 = 0$ , to obtain the extension over  $S^5$ .

19. LEMMA 4: If  $C_1$ ,  $C_2$  are contractible smooth manifolds of dimension  $n \ge 5$ , then any diffeomorphism  $d: \partial C_1 \to \partial C_2$  extends to a diffeomorphism of  $C_1$  onto  $C_2$ , after perhaps changing d on an (n-1)-disks in  $\partial C_1$ .

**Proof of Lemma 4:** Consider  $W = C_1 \cup_d D_2$ , a homotopy *n*-sphere. Since  $n \ge 5$ , W is homeomorphic to  $S^n$  ([15]) and, by changing d, we can insure that W is diffeomorphic to  $S^n$ . Then W bounds a copy of  $D^{n+1}$  which determines an h-cobordism

<sup>\*)</sup> In fact an argument of A. Wallace, communicated to me by C. T. C. Wall, shows that only handles of index 2 are needed after connected sum with enough copies of  $S^2 \times S^2$ . This obviates the need for the arguments of § 18, 19 and 21, since  $\Delta = D^4$  and  $M_0 = M$ .

from  $C_1$  to  $C_2$ , trivial from  $\partial C_1$  to  $\partial C_2$  (identifying them by d). By the relative version of the h-cobordism theorem ([15, Cor. 3.2]), d extends to a diffeomorphism from  $C_1$  onto  $C_2$ .

- 20. Now  $M_0$  is obtained from  $\Delta$  by attaching handles of index two. Since we may assume that  $g \mid \Delta = g' \mid \Delta$ , the problem of showing  $g \mid M_0$  isotopic to  $g' \mid M_0$  is similar to (but not exactly the same as) the argument above in § 15, 16. We need first to know that the l-pairings on  $H_2(M_0)$ , defined from g and g', are identical, but this is because they are induced from the l-pairings on  $H_2(M)$  by the inclusion  $M_0 \subset M$ . The argument in § 15 will then serve to show that g and g' are isotopic on the cores of the handles. To extend the isotopy to the entire handles, it will suffice to show that the normal 2-fields to the imbedded cores in  $S^5$ , defined by applying the differentials of g and g' to the standard normal 2-fields to the cores in the handles of  $M_0$ , are homotopic. But the obstructions are elements of  $\Pi_2(V_{3,2})=0$ , where  $V_{3,2}$  is the Stiefel manifold of 2-fields in 3-space.
- 21. We now have shown that  $g | M_0$  is isotopic to  $g' | M_0$ . The proof of Lemma 3 for q=2 will be completed by showing that  $\partial M$  can be isotopically deformed, in M, inside  $M_0$ . This will use the engulfing theorem in its most naive form ([12, Lemma 2.7]). Let N be the closed simply-connected 4-manifold obtained from M by attaching a 4-disk D to  $\partial M$ . Given any handle-decomposition of N it follows from the engulfing theorem that the handles of index one are contained in a 4-disk imbedded in N. Applying this to the *dual* handle decomposition of that postulated in § 18, we find that  $N-M_0$  is contained in a 4-disk D' in N. Since any two similarly oriented n-disks in an unbounded n-manifold are isotopic, D and D' are isotopic in N. It also may be arranged that any given point in  $(\text{int } D) \cap (\text{int } D')$  is fixed during the isotopy. It is then easy to see that  $\partial D (=\partial M)$  and  $\partial D'$  are isotopic in  $M=\overline{N-D}$ . This completes the proof of Lemma 3, and so Theorem 3.
- **22.** Let  $f: S^n \to R^{n+2}$  be a smooth imbedding. Since its normal bundle is trivial, f extends to an imbedding  $F: S^n \times D^2 \to R^{n+2}$  whose isotopy class is uniquely determined by f, if n > 1. Let  $h: S^n \times S^1 \to S^n \times S^1$  be the diffeomorphism defined by  $(x, y) \to \Phi(y) \cdot x$ , y where  $\Phi: S^1 \to S0(n+1)$  represents the non-zero element of  $\Pi_1(S0(n+1))$ . Define

$$R_0 = S^n \times D^2 \cup_{F \circ h} \overline{R^{n+2} - F(S^n \times D^2)}$$

Representing  $R^{n+2}$  as the interior of  $D^{n+2}$ , the construction of  $R_0$  represents  $R_0$  as the interior of a compact manifold  $D_0$  with boundary diffeomorphic to  $S^{n+1}$ . By [15],  $D_0$  is diffeomorphic to  $D^{n+2}$  if  $n \ge 3$ ; therefore  $R_0$  is diffeomorphic to  $R^{n+2}$ . Consider

the knot  $S^n \times 0 \subset S^n \times D^2 \subset R_0 \approx R^{n+2}$ , which we denote by  $\tau(K)$ , if K is the knot  $f(S^n)$ . It follows easily that the isotopy class of  $\tau(K)$  depends only on that of K. The interest of  $\tau(K)$  is that its complement is diffeomorphic to the complement of K and, besides K itself, is, up to isotopy, the only knot with this property (see [2] and [6]). It is not known whether  $\tau(K)$  is ever *not* isotopic to K. We can prove:

COROLLARY 1: If K is a simple (2q-1)-knot,  $q \ge 2$ , then  $\tau(K)$  is isotopic to K

*Proof:* Let M be a submanifold of  $R^{n+2} \subset S^{n+2}$  bounded by  $K = f(S^n)$  – we may assume that  $M \cap F(S^n \times D^2) = F(S^n \times \varrho)$  where  $\varrho$  is any ray from the origin in  $D^2$ . Since  $h(S^n \times x_0) = S^n \times x_0$ , for any  $x_0 \in S^1$ , we may define

$$M' = S^{n} \times \varrho \cup (M \cap \overline{R^{n+2} - F(S^{n} \times D^{2})}),$$

a submanifold of  $R_0$  bounded by  $\tau(K)$ . It is obvious that M' is diffeomorphic to M and the l-pairings coincide. Thus, by Theorem 3 (or even Lemma 3), K and  $\tau(K)$  are isotopic.

- 23. Another consequence of Theorem 3 not surprisingly is the unknotting theorem of [7] and [14].\* For if K is a (2q-1)-knot with complement X and universal abelian covering  $\tilde{X}$ , and A is a Seifert matrix for K, then  $tA+(-1)^qA^T$  is a relation matrix for  $H_q(\tilde{X};Q)$ , as a module over the rational group ring  $Q[Z]=Q[t,t^{-1}]$  (see [8]). Now A is equivalent to a non-singular matrix A' (allowing A'=0) by Proposition 1 in § 24. But if A' has rank r, it follows easily that  $H_q(\tilde{X};Q)$  has dimension r as a Q-module. Thus, if  $H_q(\tilde{X};Q)=0$ , A must be equivalent to 0, which implies, by Theorem 3, that K is unknotted for  $q\geqslant 2$ .
- 24. We now turn to the algebraic problem presented by the notion of equivalence of matrices. Results are very incomplete, and most of them are contained implicitly in [16].

PROPOSITION 1: Any matrix A such that  $A + A^{T}$  is unimodular is equivalent to a non-singular matrix (i.e. with non-zero determinant) or zero.

*Proof:* By the argument in [16, p. 484], if A is singular it admits an elementary reduction. Thus by a sequence of elementary reductions (and congruences) we may make A non-singular (or zero).

25. PROPOSITION 2: Suppose that A and B are equivalent non-singular matrices. Then  $\det A = \det B = d$ , and A is congruent to B over any ring R in which d is a unit.

<sup>\*)</sup> The argument here applies, of course, only for odd dimensional knots. The case q=2 was also announced by C. T. C. Wall: Proc. Camb. Phil. Soc. 63 (1967), p. 6.

**Proof:** This is proved implicitly in [16, Theorem 2] more or less as follows. Consider the matrix  $tA - A^T$ , with entries considered as elements of  $Z[t, t^{-1}] = \Lambda$ , and the  $\Lambda$ -module  $H_A$  with  $tA - A^T$  as relation matrix. Consider also the bilinear form  $[,]: H \otimes_{\Lambda} H \to Q(\Lambda)/\Lambda = S(\Lambda)$   $(Q(\Lambda))$  is the quotient field of  $\Lambda$ ) defined, with respect to the same generators of  $H_A$  as  $tA - A^T$  is a relation matrix, by the matrix  $(tA - A^T)^{-1}$ . Note that  $tA - A^T$  is non-singular over  $Q(\Lambda)$  because  $A_A(t) = \det(tA - A^T)$  is non-zero. In fact the leading coefficient of  $A_A(t)$  is  $\det A \neq 0$ .

It is not hard to see that the isomorphism class of  $(H_A; [,])$  is an invariant of the equivalence class of A. Furthermore the element  $\Delta_A(t) = \det(tA - A^T)$  is an invariant of the equivalence class, up to multiplication by powers of t. From this it follows that  $\det A = \det B$ .

If A is unimodular over R, then  $t-A^{-1}A^T$  is a presentation matrix of  $H_A \otimes R (\otimes = \otimes_Z)$ , so  $H_A \otimes R$  is a free R-module of the same rank as A and  $(tA - A^T)^{-1}$  is the matrix for [,] with respect to an R-basis for  $H_A \otimes R$ . From this it follows that there exists a matrix P with entries in, and unimodular over, R such that

$$P(tA - A^T)^{-1} P^T = (tB - B^T)^{-1}$$
 in  $S(\Lambda) \otimes R$ .

Let  $\overline{(tA-A^T)}$  be the "adjoint" of  $tA-A^T$  ([1, p. 305)]. Then

$$P(\overline{tA - A^{T}}) P^{T} = t\overline{B - B^{T}} \text{ in } (\Lambda/(\Delta(t)) \otimes R,$$

$$\text{where } \Delta(t) = \Delta_{A}(t) = \Delta_{B}(t).$$
(\*)

Since A and B are unimodular over R,  $\Delta(t)$  has as leading coefficient a unit of R. From this it follows that there exists a unique well-defined R-linear map  $\gamma: \Lambda/(\Delta(t)) \otimes R \to R$  defined by the properties  $\gamma(1) = 1$  and  $\gamma(t^i) = 0$  for 0 < i < degree  $\Delta(t)$ . Now every entry of  $\overline{tA - A^T}$  and  $\overline{tB - B^T}$  has degree  $< \text{rank } A = \text{degree } \Delta(t)$ . Therefore, applying  $\gamma$  to equation (\*) we find that  $PAP^T = B$  in R.

26. We cannot strengthen the conclusion of Proposition 2 to conclude that A and B are congruent over K, as the following example shows. Set:

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

We first show that A and B are not congruent over Z. Consider the solutions X of  $X^TAX = X^TBX = 2$ ; they are  $X = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$ . If  $P^TAP = B$ , it follows that  $PX = \pm X \left( \text{say} \right)$ 

 $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Now  $BX = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$  and so  $Y^T BX$  is even, for any Y. Choose Y so that  $PY = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ; then  $Y^T BX = Y^T P^T APX = \pm (PY)^T AX$  which one can calculate to be  $\pm 1$ .

To see that A and B are equivalent, we consider the following elementary enlargements, respectively, of A and B:

$$A' = \begin{pmatrix} \mathbf{A} & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} \mathbf{B} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A' and B' are congruent; in fact,  $PA'P^T = B'$ , where

$$P = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

27. On the other hand, the converse of Proposition 2 is false. Consider the following matrices for  $\varepsilon = \pm 1$ :

$$A = \begin{pmatrix} 0 & \varepsilon x & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & p^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p(1+\varepsilon)+1 \\ 0 & 0 & 0 & 0 & p+1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \varepsilon x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 \\ p^4 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p(1+\varepsilon)+1 \\ 0 & 0 & 0 & 0 & 0 & p+1 \end{pmatrix}$$

where p is any odd prime, and  $x = \frac{1}{4}(p^4 - 1)$ . It may be checked directly that  $A + \varepsilon A^T$  and  $B + \varepsilon B^T$  are unimodular and  $\det A = \det B$  is divisible by p. But A and B are congruent over any ring in which p is a unit. In fact  $PAP^T = B$  where:

$$P = \begin{pmatrix} p & & & \\ 1/p & & & \\ p & & & \\ 0 & & 1 \end{pmatrix}$$

Finally, A and B are not equivalent. To see this consider  $A - \varepsilon A^T$  and  $B - \varepsilon B^T$  over  $Z_p$ . It follows from Witts Theorem (see e.g. [21]) for  $\varepsilon = -1$ , or the well-known classification of skew-symmetric forms (see e.g. [1]) for  $\varepsilon = +1$ , that the congruence class of  $A - \varepsilon A^T$  over  $Z_p$  is an invariant of the equivalence class of A. But  $A - \varepsilon A^T$  and  $B - \varepsilon B^T$  have ranks 2 and 4, respectively, over  $Z_p$ .

**28.** Finally we remark that the genus of the non-degenerate quadratic form  $A + A^T$  is an invariant of the equivalence class of A (see [16, Prop. 5.1]).

#### REFERENCES

- [1] BIRKHOFF, G. and MACLANE, S.: A survey of modern algebra (MacMillan Co., New York 1953).
- [2] Browder, W.: Diffeomorphisms of 1-connected manifolds, Trans. Amer. Math. Soc. 128 (1967), 155-63.
- [3] Kato, M.: Embedding spheres and balls in codimension  $\leq 2$  (mimeographed notes).
- [4] Kervaire, M.: Les nœuds de dimensions superieures, Bull. Soc. Math. France, 93 (1965), 225-71.
- [5] KERVAIRE, M. and MILNOR, J.: Groups of homotopy spheres, I, Annals of Math. 77 (1963), 504-37.
- [6] Lashof, R. and Shaneson, J.: Classification of knots in codimension two, Bull. Amer. Math. Soc. 75 (1969), 171-5.
- [7] LEVINE, J.: Unknotting spheres in codimension two, Topology 4 (1965), 9-16.
- [8] LEVINE, J.: Polynomial invariants of knots of codimension two, Annals of Math. 84 (1966), 537-54.
- [9] LEVINE, J.: Knot cobordism groups in codimension two, Comment. Math. Helv. 44 (1969), 229-44.
- [10] MILNOR, J.: Lectures on the h-cobordism theorem, Princeton Mathematical Notes (Princeton U. Press, Princeton 1965).
- [11] Murasugi, K.: On a certain numerical invariant of link types, Trans. Amer. Math. Soc. 117 (1965), 387-422.
- [12] Penrose, R., Whitehead, H. H. C., and Zeeman, E. C.: Imbeddings of manifolds in Euclidean space, Annals of Math. 73 (1961), 613-23.
- [13] SEIFERT, H.: Über das Geschlecht von Knoten, Math. Ann. 110 (1934), 571-92.
- [14] Shaneson, J.: Embeddings of spheres in spheres of codimension two and h-cobordism of  $S^1 \times S^3$ , Bull. Amer. Math. Soc. 74 (1968), 972–4.
- [15] SMALE, S.: On the structure of manifolds, Amer. J. Math. 84 (1962), 387-99.
- [16] TROTTER, H.: Homology of groups system with applications to knot theory, Annals of Math. 76 (1962), 464-98.
- [17] Wall, C. T. C.: Classification of (n-1)-connected 2n-manifolds, Annals of Math. 75 (1962), 163-89.
- [18] Wall, C. T. C.: Diffeomorphisms of 4-manifolds, J. London Math. Soc. 39 (1964), 131-40.
- [19] Wall, C. T. C.: On simply-connected 4-manifolds, J. London Math. Soc. 39 (1964), 141-9.
- [20] Wu, W. T.: On the isotopy of  $C^r$ -manifolds of dimension n in Euclidean (2n+1)-space, Science Record, New Series II (1958), 271-5.

Received December 1, 1969