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An Algebraic Classification of Some Knots of Codimension Two¹)

by J. LEVINE

An *n*-knot will denote a smooth oriented submanifold K of the (n+2)-sphere S^{n+2} , where K is homeomorphic to S^n . If n is odd, one can associate to K a square integral matrix A, called a Seifert matrix of K, using a submanifold of S^{n+2} bounded by K (see [13] for n=1, and [4] or [8] in general). When n=1, it is known that two Seifert matrices of isotopic knots are related by certain algebraic "moves" (see [11], [16]). In this paper we will generalize this fact to all n. We then consider, for n odd, n-knots (referred to as simple) whose complements are of the same ((n-1)/2) type as a circle, i.e. $\Pi_q(S^{n+2}-K) \approx \Pi_q(S^1)$ for $q \leq (n-1)/2$. This is the most that can be asked without making K unknotted (see [7]). We will show that two simple n-knots $(n \geq 3)$ are isotopic if and only if their Seifert matrices are related by such "moves". Thus it will follow that the semi-group of isotopy classes of simple n-knots depends only on the residue class, mod4, of n for $n \geq 4$.

By contrast, Lashof and Shaneson [6] (and, independently, Browder) have shown that the isotopy class of an *n*-knot ($n \ge 3$) K, whose complement is of the same 1-type as a circle is determined by the homotopy type of its exterior pair ($X, \partial X$), where X is the complement of an open tubular neighborhood of K in S^{n+2} -except for one other possible knot $\tau(K)$, obtained from K by removing a tubular neighborhood twisting, and reinserting in S^{n+2} . It is not known whether $\tau(K)$ is ever different from K. As a straightforward application, we will show that $\tau(K)$ is isotopic to K if K is simple.

We conclude with some remarks on the algebraic problems which arise.

1. Let K be a (2q-1)-knot in S^{2q+1} . We recall the definition of a Seifert matrix of K. Let M be a smooth oriented submanifold of S^{2q+1} bounded by K. The *l*-pairing of M:

 $\theta: H_a(M) \otimes H_a(M) \to \mathbb{Z}$

is defined by letting $\theta(\alpha \otimes \beta)$ be the linking number $L(z_1, z_2)$, where z_1 is a cycle in M representing α and z_2 is the translate in the positive normal direction off M of a cycle in M representing β . A Seifert matrix A of K is then a representative matrix of θ with respect to a basis of the torsion-free part of $H_a(M)$ – see e.g. [8].

We recall also the formula [8]:

 $\theta(\alpha \otimes \beta) + (-1)^q \, \theta(\beta \otimes \alpha) = \alpha \cdot \beta$

¹) This work was done while the author was partially supported by NSF GP 8885.

where $\alpha \cdot \beta$ is the intersection number in *M*. Thus $A + (-1)^q A^T$ is unimodular $(A^T$ is the transpose of *A*) and, if $q=2, A+A^T$ has signature a multiple of 16 (see [9]).

2. Let A be a square integral matrix. Any matrix of the form:

$$\begin{pmatrix} \underline{A} & 0 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \underline{A} & \beta & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where α is a row vector, β a column vector, will be called an *elementary enlargement* of A. A is an *elementary reduction* of any of its elementary enlargements. Two matrices (or their associated pairings) are equivalent if they can be connected by a chain of elementary enlargements, reductions and unimodular congruences. It is proved in [11] that Seifert matrices of isotopic 1-knots are equivalent. We shall prove:

THEOREM 1: Seifert matrices of isotopic knots of any (odd) dimension are equivalent.

THEOREM 2: Let q be a positive integer and A a square integral matrix such that $A + (-1)^q A^T$ is unimodular and, if q=2, $A + A^T$ has signature a multiple of 16. If $q \neq 2$, there is a simple (2q-1)-knot with Seifert matrix A; if q=2, there is a simple 3-knot with Seifert matrix equivalent to A.

THEOREM 3: Let $q \ge 2$ and K_1 , K_2 simple (2q-1)-knots with equivalent Seifert matrices. Then K_1 is isotopic to K_2 .

3. Proof of Theorem 1: Suppose K_1, K_2 are isotopic (2q-1)-knots bounding manifolds M_1, M_2 , respectively, of S^{2q+1} . We first construct a submanifold V (with corners) of $I \times S^{2q+1}$ meeting $0 \times S^{2q+1}$ along $0 \times M_1$ and $1 \times S^{2q+1}$ along $1 \times M_2$ with boundary the union of $0 \times M_1$; $1 \times M_2$ and the trace X of an isotopy from K_1 to K_2 . We use the Pontriagin-Thom construction as follows. First construct a normal vector field to $(0 \times M_1) \cup X \cup (1 \times M_2) = Y$ in $I \times S^{2q+1}$, which is tangent to $I \times S^{2q+1}$ along $0 \times M_1 \cup 1 \times M_2$. If $q \neq 1$, there is no obstruction. If q=1, the obstruction to extending such a vector field from $0 \times M_1 \cup 1 \times M_2$ over Y is the difference in its winding numbers about K_1 and K_2 . But since the field is defined over M_1 and M_2 , these winding numbers are zero.

Let T be a tubular neighborhood of X. We can "translate" $(X, v \mid X)$ to a framed submanifold of ∂T which agress with the framed submanifold $(0 \times M_1 \cup 1 \times M_2, v)$ on $\partial T \cap (I \times S^{2q+1})$. Let $W = \overline{I \times S^{2q+1} - T}$; the Pontriagin-Thom construction on the above framed submanifolds of ∂W determines a map $\partial W \rightarrow S^1$. An extension of this map over W will determine the desired V. The obstruction lies in

$$H^2(W, \partial W) \approx H^2(I \times S^{2q+1}, X \cup \dot{I} \times S^{2q+1}) \approx H^1(X \cup \dot{I} \times S^{2q+1}) = 0.$$

4. Now let $\Phi': V \to I$ be the "height" function defined by the restriction of the projection $I \times S^{2q+1} \to I$. We may assume Φ' has no critical points in a neighborhood of ∂V (omitting corners). Let Φ be a C^2 -approximation to Φ' which agrees with Φ' in a neighborhood of ∂V and has only non-degenerate critical points (except at corners) which are mapped one-one into I (see e.g. [10]). We can move V so that Φ becomes the new height function. In fact if $p: V \to S^{2q+1}$ is defined by the projection $I \times S^{2q+1} \to S^{2q+1}$ and Φ is a close enough approximation to Φ' , then $x \mapsto (\Phi(x), p(x))$ defines a new imbedding $V \to I \times S^{2q+1}$ which agrees with the original inclusion near ∂V and has Φ as its new height function.

5. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a partition of I satisfying

(i) each t_i is a regular value of Φ ,

(ii) at most one critical value of Φ lies in each interval (t_i, t_{i+1}) .

Let $\Phi^{-1}(t_i) = t_i \times M'_i$; then each M'_i is bounded by a knot isotopic to K_0 and K_1 , and $M'_0 = M_1, M'_k = M_2$. This shows that it suffices to consider the case where Φ has only one critical point.

LEMMA 1: Let α , $\alpha' \in H_q(M_1)$ and β , $\beta' \in H_q(M_2)$ and suppose that α is homologous to β and α' homologous to β' in V. Then $\theta_1(\alpha, \alpha') = \theta_2(\beta, \beta')$, where θ_i is the *l*-pairing of M_i .

Proof: Let C, C' be (q+1)-chains in V such that $\partial C = \alpha - \beta$, $\partial C' = \alpha' - \beta'$. Then it follows from the definition of θ_1 , θ_2 that $\theta_1(\alpha, \alpha') - \theta_2(\beta, \beta')$ is the intersection number of C and the translate of C' off V in the positive normal direction – but this is obviously zero.

6. Now consider the following diagram:

$$H_{q+1}(V, M_2)$$

$$\downarrow$$

$$H_q(M_2)$$

$$\downarrow$$

$$H_{q+1}(V, M_1) \rightarrow H_q(M_1) \rightarrow H_q(V) \rightarrow H_q(V, M_1)$$

$$\downarrow$$

$$H_q(V, M_2)$$

consisting of the exact homology sequences of (V, M_1) and (V, M_2) . If the index of

the critical point of Φ is not q or q+1, then

$$H_q(V, M_1) = H_{q+1}(V, M_1) = H_q(V, M_2) = H_{q+1}(V, M_2) = 0$$

and we have

 $H_q(M_1) \approx H_q(V) \approx H_q(M_2).$

It follows from Lemma 1 that θ_1 and θ_2 are congruent.

If the index of the critical point of Φ is q, then

$$H_q(V, M_1) \approx H_{q+1}(V, M_2) \approx Z$$

and

$$H_q(V, M_2) = H_{q+1}(V, M_1) = 0.$$

If $\alpha \in H_q(M_2)$ is the image of a generator of $H_{q+1}(V, M_2)$, then the composite

$$H_q(M_2) \rightarrow H_q(V) \rightarrow H_q(V, M_1) \approx \mathbb{Z}$$

can be defined by $\beta \to \alpha \cdot \beta =$ intersection number in M_2 (see [5]). If α has finite order, then it follows that $H_q(M_1) \approx H_q(V) \approx H_q(M_2)$, modulo torsion, and, therefore, θ_1 and θ_2 are congruent modulo torsion.

7. Suppose α has infinite order; then α is a multiple of a primitive element α_0 and there exists $\beta_0 \in H_q(M_2)$ with $\alpha_0 \cdot \beta_0 = 1$. Suppose $\gamma'_1, \dots, \gamma'_s \in H_q(M_2)$ such that:

(i) γ'_i is homologous to γ_i in V, and

(ii) $\alpha_0, \beta_0, \gamma'_1, ..., \gamma'_s$ is a basis of $H_q(M_2)$, modulo torsion.

We now examine θ_2 on the elements α_0 , β_0 , $\gamma'_1, \dots, \gamma'_s$. By Lemma 1 we can conclude from (i) that $\theta_2(\gamma'_i, \gamma'_j) = \theta_1(\gamma_i, \gamma_j)$. Since α is null-homologous in V, $\theta_2(\alpha, \gamma'_i) = \theta_1(0, \gamma_i) = 0$ and $\theta_2(\alpha, \alpha) = \theta_2(0, 0) = 0$. Thus $\theta_2(\alpha_0, \gamma'_i) = \theta_2(\alpha_0, \alpha_0) = 0$; similarly $\theta_2(\gamma'_i, \alpha_0) = 0$. We also recall that (§ 1):

$$\theta_2(\alpha_0,\beta_0) + (-1)^q \theta_2(\beta_0,\alpha_0) = -\alpha_0 \cdot \beta_0 = -1$$

8. We may summarize this as follows. Let A be the matrix representative of θ_1 with respect to the basis $\gamma_1, \ldots, \gamma_s$. The the matrix representative of θ_2 with respect to the basis $\gamma'_1, \ldots, \gamma'_s, \alpha_0, \beta_0$ has the form:



where x, y are integers, $x + (-1)^q x' = -1$, ξ is a row vector and η is a column vector.

Recall from e.g. [8] that the polynomial $\Delta_A(t) = \det(tA + (-1)^q A^T)$, where A is a Seifert matrix for a knot K, is an invariant of the isotopy class of K (up to multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$). But it is easily verified that:

$$\Delta_B(t) = \left(tx + (-1)^q x'\right) \left(tx' + (-1)^q x\right) \Delta_A(t)$$

Thus x (or x') is zero, since $x \pm x' = -1$, then x' (or x) is ± 1 . It now is easily checked that B is congruent to an elementary enlargement of A.

9. If the index of the critical point of Φ is q+1; then its index as a critical point of $-\Phi$ is q. The preceding arguments apply to show that θ_2 is congruent to θ_1 , or has, as representative matrix, an elementary reduction of a representative matrix of θ_1 .

This completes the proof of Theorem 1.

10. Proof of Theorem 2: For $q \neq 2$, this is proved in [4] (see also [9]). For q=2, we must show that A is equivalent to a matrix B, where $B+B^T$ is a matrix representative of the intersection pairing of some simply-connected closed 4-manifold. By an argument in [9], such a B can be obtained by adding enough blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to A; but this is a sequence of elementary enlargements of A, so B is equivalent to A.

11. Proof of Theorem 3: We reduce Theorem 3 to two lemmas. Recall (see [7]) that a simple (2q-1)-knot bounds a (q-1)-connected submanifold of S^{2q+1} . A Seifert matrix obtained from the *l*-pairing of such a submanifold will be called *special*.

LEMMA 2: Let K be a simple (2q-1)-knot with a special Seifert matrix A. If B is an elementary enlargement of A, then B is also a special Seifert matrix of K.

LEMMA 3: If $q \ge 2$, then simple (2q-1)-knots admitting identical special Seifert matrices are isotopic.

12. We first show that Theorem 3 follows from Lemmas 2 and 3. Let K, K' be simple (2q-1)-knots with equivalent Seifert matrices, $q \ge 2$. Let A, A' be special Seifert matrices of K, K', respectively. Thus there exists a sequence: $A = A_1, A_2, ..., A_k = A'$, where each A_{i+1} is unimodularly congruent to an elementary enlargement or reduction. of A_i It follows from Theorem 2 that, for q > 2, each A_i is a special Seifert matrix of a simple (2q-1)-knot K_i (actually the proof of Theorem 2 (see [4]) realizes S as a special Seifert matrix of a simple knot). If q=2, we can enlarge each A_i by adding a constant number of blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to obtain a new sequence $A'_1, A'_2, ..., A'_k$. Each A'_{i+1} is again congruent to an elementary enlargement or reduction of A'_i – and it

now follows from the argument in § 10 that each A'_i is a special Seifert matrix of a simple 3-knot K_i .

We now prove that each K_i is isotopic to K_{i+1} . Suppose A_{i+1} is congruent to an elementary enlargement of A_i . It follows from Lemma 2 that A_{i+1} (or A'_{i+1}) is a special Seifert matrix of K_i . Then Lemma 3 implies K_i and K_{i+1} , both of which now admit A_{i+1} (or A'_{i+1}) as a special Seifert matrix, are isotopic. If A_{i+1} is congruent to an elementary reduction of A_i , the same argument works, switching the roles of K_i and K_{i+1} .

We may as well have chosen $K_1 = K$ and $K_k = K'$ if q > 2, but if q = 2 we need to show that K_1 is isotopic to K and K_k is isotopic to K'. It follows from Lemma 2 that A'_1 is a special Seifert matrix of K, since A'_1 is obtained from A_1 by a sequence of elementary enlargements. Then Lemma 3 implies K and K_1 are isotopic – similarly for K' and K_k .

13. Proof of Lemma 2: Let M be a (q-1)-connected submanifold of S^{2q+1} bounded by K, and $\alpha_1, ..., \alpha_n$ a basis of $H_q(M)$, modulo torsion, such that A is the corresponding matrix representative of the *l*-pairing of M. Let $x_1, ..., x_n$ be an arbitrary sequence of integers. It follows from Alexander duality that there exists a cycle $z \in H_q(S^{q+1}-M)$ such that the linking numbers $L(z, \alpha_i) = x_i$, for i=1, ..., n. Now $S^{2q+1} - M$ is (q-1)connected and so z is spherical; by general position, z can be represented by an imbedded q-sphere $\sigma \subset S^{2q+1} - M$. The normal bundle to σ is trivial and so a tubular neighborhood T can be identified with $\sigma \times D^{q+1}$ – we may assume T disjoint from M. Orient ∂T so that the positive normal direction in S^{2q+1} points into T and let M' be the connected sum in S^{2q+1} of M and ∂T . Then $H_q(M')$ has rank two greater than the rank of $H_q(M)$, and $\alpha_1, ..., \alpha_n$ may be extended to a basis of $H_q(M')$, modulo torsion, by adjoining the homology classes β_1 , β_2 of $\sigma \times y_0$ and $x_0 \times S^q \subset \sigma \times S^q = \partial T$, respectively. The representative matrix of the *l*-pairing of M' with respect to the basis $\alpha_1, ..., \alpha_n, \beta_1, \beta_2$ is



which is congruent to:



If z is chosen so that $L(\alpha_i, z) = x_i$ for i = 1, ..., n, and ∂T is oriented so that the positive normal direction points out from T, then the representative matrix of the *l*-pairing of M' with respect to $\alpha_1, ..., \alpha_n, \beta_1, \beta_2$ is:



which is congruent to:



Thus we can realize any elementary enlargement of A as a special Seifert matrix of K.

14. Proof of Lemma 3: Suppose K and K' are (2q-1)-knots bounding (q-1)-connected submanifolds M and M' of S^{2q+1} with *l*-pairings θ and θ' . Suppose also that there exists an isomorphism $\Phi: H_q(M) \to H_q(M')$ preserving the *l*-pairings, i.e. $\theta = \theta' \circ (\Phi \otimes \Phi)$.

Let us assume, for now, q>2; we will show that M and M' are isotopic submanifolds of S^{2q+1} . According to [15], M and M' have handle decompositions:

$$M = D^{2q} \cup h_1 \cup \dots \cup h_r$$
$$M' = D^{2q} \cup h'_1 \cup \dots \cup h'_r$$

where each h_i , h'_i is a handle of index q – diffeomorphic to $D^q \times D^q$. The $h_i(h'_i)$ are attached to D^{2q} by disjoint imbeddings $S^{q-1} \times D^q \rightarrow \partial D^{2q}$. Let $C_i(C'_i)$ be the "core" of $h_i(h'_i)$, i.e. the submanifold corresponding to $D^q \times 0$ – then $\partial C_i = C_i \cap D^{2q}$.

The imbedded disks $(C_i, \partial C_i) \subset (M, D^{2q})$ represent a basis $\{\alpha_i\}$ of $H_q(M, D^{2q}) \approx \approx H_q(M)$. According to handle body theory (see [17]), we can choose a handle-decomposition realizing any prescribed basis $\{\alpha_i\}$. Thus if $\{\alpha'_i\}$ is the basis of $H_q(M')$ defined by $(C'_i, \partial C'_i) \subset (M', D^{2q})$, we may, by setting $\alpha'_i = \Phi(\alpha_i)$, assume $\theta(\alpha_i, \alpha_j) = \theta'(\alpha'_i, \alpha'_j)$.

15. Now consider the links $\{\partial C_i\}$ and $\{\partial C'_i\}$ in ∂D^{2q} ; by [17] and §1 we have:

 $L(\partial C_i, \partial C_j) = \alpha_i \cdot \alpha_j = -\theta(\alpha_i, \alpha_j) - (-1)^q \theta(\alpha_j, \alpha_i);$

similarly for $L(\partial C'_i, \partial C'_j)$. Therefore $L(\partial C_i, \partial C_j) = L(\partial C'_i, \partial C'_j)$, for $i \neq j$, and, since q > 2, the links $\{\partial C_i\}$ and $\{\partial C'_i\}$ are isotopic in ∂D^q .

Clearly we may assume that the base disks D^{2q} in the handle decompositions of M and M' coincide as imbedded in S^{2q+1} . Thus the cores C_i and C'_i , as imbedded

•

in S^{2q+1} , may be assumed to coincide on their boundaries: $\partial C_i = \partial C'_i$. We next show how to isotopically defrom $\{C_i\}$ onto $\{C'_i\}$, keeping $\{\partial C_i\}$ fixed and avoiding any intersections with D^{2q} (except, of course, along ∂C_i).

Assume inductively that $C_i = C'_i$ for i < k. We will isotopically deform C_k to C'_k , avoiding intersections with $D^{2q} \cup C_1 \cup \cdots \cup C_{k-1}$. Given such an isotopy, we can extend it to an isotopy of $h_k \cup h_{k+1} \cup \cdots \cup h_i$ in

 $S^{2q+1} - (D^{2q} \cup h_1 \cup \cdots \cup h_{k-1});$

the result is an isotopy of M to a new imbedding satisfying $C_i = C'_i$ for $i \le k$. We begin with an isotopy of C_k to C'_k , rel ∂C_k , avoiding D^{2q} ; this exists according to Wu [20] because the imbeddings of C_k and C'_k and $S^{2q+1} - \operatorname{int} D^{2q}$ are homotopic rel $\partial C_k = \partial C'_k$ and $S^{2q+1} - \operatorname{int} D^{2q}$ is simply-connected. We would then like to use Whitney's procedure, as in [20], to remove the intersections of this isotopy with $I \times C_i$ (i=1, ..., k-1) in $S^{2q+1} - D^{2q}$, since $q \ge 2$ and $S^{2q+1} - D^{2q}$ is simply-connected. The only obstruction to this is the intersection number, which is easily seen to be (up to sign) $\theta(\alpha_i, \alpha_k) - \theta'(\alpha'_i, \alpha'_k) = 0$.

16. We now have achieved $C_i = C'_i$ for i = 1, ..., r. By the tubular neighborhood theorem we may assume $h_i \cap D^{2q} = h'_i \cap D^{2q}$. Let $v_i(v'_i)$ be the positive unit normal field to $h_i(h'_i)$ on $C_i = C'_i$.

By the tubular neighborhood theorem, we may assume that $h_i(h'_i)$ is the orthogonal complement of $v_i(v'_i)$ in a normal disk bundle neighborhood N of $C_i = C'_i$ in S^{2q+1} . Therefore if we can homotopically deform v_i to v'_i , rel ∂C_i , we obtain an isotopy, rel $h_i \cap D^{2q}$, of h_i to h'_i within N. Doing this for all *i* achieves, finally, an isotopy of M to M'.

Since $v_i = v'_i$ along ∂C_i , v_i differs from v'_i by an element of $\Pi_q(S^q) \approx Z$ (the normal space to C_i in S^{2q+1} has dimension q+1). But this element can be identified with $\theta(\alpha_i, \alpha_i) - \theta'(\alpha'_i, \alpha'_i) = 0$, and so Lemma 3 is proved -for q > 2.

17. For q=2, more work is required to repair those parts of the preceding argument which are no longer valid. First of all M and M' are not necessarily diffeomorphic. On the other hand, they are simply-connected 4-manifolds with boundaries diffeomorphic to S^3 and isomorphic intersection pairings (since their *l*-pairings are isomorphic). It then follows from [19] that, after adding on a number of copies of $S^2 \times S^2$, M and M' will be diffeomorphic. Since these enlargements of M and M' can be realized by adding to the Seifert matrices blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, as is demonstrated in § 13, we may as well assume that M and M' are diffeomorphic to start with. In fact, by [18], there is a diffeomorphism $f: M \to M'$ preserving the *l*-pairings i.e. $\theta = \theta' \circ (f_* \otimes f_*)$, where $f_*: H_2(M) \to H_2(M')$ is the induced homomorphism.

We reformulate the situation so far as follows. M is a simply-connected 4-manifold, ∂M diffeomorphic to S^3 , and we have imbeddings $g, g': M \rightarrow S^5$ such that $g(\partial M) = K$ and $g'(\partial M) = K'$. The *l*-pairings of g(M) and g'(M) are identical, as pairings on $H_2(M)$.

18. Now let

 $M = D^4 \cup h_1^1 \cup \cdots \cup h_k^1 \cup h_1^2 \cup \cdots \cup h_l^2 \cup h_1^3 \cup \cdots \cup h_m^3 *$

be a handle decomposition of M, where h_j^i is a handle of index *i*. Since $H_1(M)=0$ we can choose the handles of index 2 in such a way that the first k of them $-h_1^2, ..., h_k^2$ - homologically cancel out the handles of index 1 (see e.g. [10], Theorem 7.6]) i.e. if $V=D^4 \cup h_1^1 \cup ... \cup h_k^1$, then the boundary operator $H_2(M, V) \rightarrow H_1(V)$ maps the subgroup of $H_2(M, V)$ generated by the "cores" of $h_1^2, ..., h_k^2$ isomorphically onto $H_1(V)$. Then

$$\Delta = D^4 \cup h_1^1 \cup \cdots \cup h_k^1 \cup h_1^2 \cup \cdots \cup k_k^2$$

is acyclic. Set

 $M_0 = \Delta \cup h_{k+1}^2 \cup \cdots \cup h_l^2.$

We will show that $g \mid M_0$ and $g' \mid M_0$ are isotopic.

First we show that $g|\Delta$ and $g'|\Delta$ are isotopic by extending $g' \circ g^{-1}: g(\Delta) \rightarrow g'(\Delta)$ to an orientation preserving diffeomorphism of S^5 . Begin by extending it to a tubular neighborhood T of $g(\Delta)$ diffeomorphic to $g(\Delta) \times I$ (with corners rounded). Now ∂T is a homology 4-sphere bounding the contractible 5-manifold $\overline{S^5 - T}$ - similarly for T' a tubular neighborhood of $g'(\Delta)$. That $\overline{S^5 - T}$ is acyclic follows from Alexander duality; that $\overline{S^5 - T}$ is simply-connected follows from the fact that Δ collapses onto a 2-dimensional polyhedron (it has only handles of index one and two) which has codimension >2 in S^5 . We now invoke the case n=5 of the following lemma (stated by Kato for the PL case in [3]) since $\Gamma^5 = 0$, to obtain the extension over S^5 .

19. LEMMA 4: If C_1 , C_2 are contractible smooth manifolds of dimension $n \ge 5$, then any diffeomorphism $d: \partial C_1 \rightarrow \partial C_2$ extends to a diffeomorphism of C_1 onto C_2 , after perhaps changing d on an (n-1)-disks in ∂C_1 .

Proof of Lemma 4: Consider $W = C_1 \cup_d D_2$, a homotopy *n*-sphere. Since $n \ge 5$, W is homeomorphic to S^n ([15]) and, by changing d, we can insure that W is diffeomorphic to S^n . Then W bounds a copy of D^{n+1} which determines an h-cobordism

^{*)} In fact an argument of A. Wallace, communicated to me by C. T. C. Wall, shows that only handles of index 2 are needed after connected sum with enough copies of $S^2 \times S^2$. This obviates the need for the arguments of § 18, 19 and 21, since $\Delta = D^4$ and $M_0 = M$.

from C_1 to C_2 , trivial from ∂C_1 to ∂C_2 (identifying them by d). By the relative version of the *h*-cobordism theorem ([15, Cor. 3.2]), d extends to a diffeomorphism from C_1 onto C_2 .

20. Now M_0 is obtained from Δ by attaching handles of index two. Since we may assume that $g | \Delta = g' | \Delta$, the problem of showing $g | M_0$ isotopic to $g' | M_0$ is similar to (but not exactly the same as) the argument above in § 15, 16. We need first to know that the *l*-pairings on $H_2(M_0)$, defined from g and g', are identical, but this is because they are induced from the *l*-pairings on $H_2(M)$ by the inclusion $M_0 \subset M$. The argument in § 15 will then serve to show that g and g' are isotopic on the cores of the handles. To extend the isotopy to the entire handles, it will suffice to show that the normal 2-fields to the imbedded cores in S^5 , defined by applying the differentials of g and g' to the standard normal 2-fields to the cores in the handles of M_0 , are homotopic. But the obstructions are elements of $\Pi_2(V_{3,2})=0$, where $V_{3,2}$ is the Stiefel manifold of 2-fields in 3-space.

21. We now have shown that $g|M_0$ is isotopic to $g'|M_0$. The proof of Lemma 3 for q=2 will be completed by showing that ∂M can be isotopically deformed, in M, inside M_0 . This will use the engulfing theorem in its most naive form ([12, Lemma 2.7]). Let N be the closed simply-connected 4-manifold obtained from M by attaching a 4-disk D to ∂M . Given any handle-decomposition of N it follows from the engulfing theorem that the handles of index one are contained in a 4-disk imbedded in N. Applying this to the *dual* handle decomposition of that postulated in § 18, we find that $N-M_0$ is contained in a 4-disk D' in N. Since any two similarly oriented *n*-disks in an unbounded *n*-manifold are isotopic, D and D' are isotopic in N. It also may be arranged that any given point in $(int D) \cap (int D')$ is fixed during the isotopy. It is then easy to see that $\partial D(=\partial M)$ and $\partial D'$ are isotopic in $M=\overline{N-D}$. This completes the proof of Lemma 3, and so Theorem 3.

22. Let $f: S^n \to R^{n+2}$ be a smooth imbedding. Since its normal bundle is trivial, f extends to an imbedding $F: S^n \times D^2 \to R^{n+2}$ whose isotopy class is uniquely determined by f, if n > 1. Let $h: S^n \times S^1 \to S^n \times S^1$ be the diffeomorphism defined by $(x, y) \to \to (\Phi(y) \cdot x, y)$ where $\Phi: S^1 \to S0(n+1)$ represents the non-zero element of $\Pi_1(S0(n+1))$. Define

 $R_0 = S^n \times D^2 \cup_{F \circ h} \overline{R^{n+2} - F(S^n \times D^2)}$

Representing \mathbb{R}^{n+2} as the interior of D^{n+2} , the construction of \mathbb{R}_0 represents \mathbb{R}_0 as the interior of a compact manifold D_0 with boundary diffeomorphic to S^{n+1} . By [15], D_0 is diffeomorphic to D^{n+2} if $n \ge 3$; therefore \mathbb{R}_0 is diffeomorphic to \mathbb{R}^{n+2} . Consider

the knot $S^n \times 0 \subset S^n \times D^2 \subset R_0 \approx R^{n+2}$, which we denote by $\tau(K)$, if K is the knot $f(S^n)$. It follows easily that the isotopy class of $\tau(K)$ depends only on that of K.

The interest of $\tau(K)$ is that its complement is diffeomorphic to the complement of K and, besides K itself, is, up to isotopy, the only knot with this property (see [2] and [6]). It is not known whether $\tau(K)$ is ever *not* isotopic to K. We can prove:

COROLLARY 1: If K is a simple (2q-1)-knot, $q \ge 2$, then $\tau(K)$ is isotopic to K

Proof: Let M be a submanifold of $\mathbb{R}^{n+2} \subset S^{n+2}$ bounded by $K=f(S^n)$ – we may assume that $M \cap F(S^n \times D^2) = F(S^n \times \varrho)$ where ϱ is any ray from the origin in D^2 . Since $h(S^n \times x_0) = S^n \times x_0$, for any $x_0 \in S^1$, we may define

 $M' = S^n \times \varrho \cup (M \cap \overline{R^{n+2} - F(S^n \times D^2)}),$

a submanifold of R_0 bounded by $\tau(K)$. It is obvious that M' is diffeomorphic to M and the *l*-pairings coincide. Thus, by Theorem 3 (or even Lemma 3), K and $\tau(K)$ are isotopic.

23. Another consequence of Theorem 3 – not surprisingly – is the unknotting theorem of [7] and [14].* For if K is a (2q-1)-knot with complement X and universal abelian covering \tilde{X} , and A is a Seifert matrix for K, then $tA + (-1)^q A^T$ is a relation matrix for $H_q(\tilde{X}; Q)$, as a module over the rational group ring $Q[Z] = Q[t, t^{-1}]$ (see [8]). Now A is equivalent to a non-singular matrix A' (allowing A'=0) by Proposition 1 in § 24. But if A' has rank r, it follows easily that $H_q(\tilde{X}; Q)$ has dimension r as a Q-module. Thus, if $H_q(\tilde{X}; Q) = 0$, A must be equivalent to 0, which implies, by Theorem 3, that K is unknotted for $q \ge 2$.

24. We now turn to the algebraic problem presented by the notion of equivalence of matrices. Results are very incomplete, and most of them are contained implicitly in [16].

PROPOSITION 1: Any matrix A such that $A + A^T$ is unimodular is equivalent to a non-singular matrix (i.e. with non-zero determinant) or zero.

Proof: By the argument in [16, p. 484], if A is singular it admits an elementary reduction. Thus by a sequence of elementary reductions (and congruences) we may make A non-singular (or zero).

25. PROPOSITION 2: Suppose that A and B are equivalent non-singular matrices. Then det $A = \det B = d$, and A is congruent to B over any ring R in which d is a unit.

^{*)} The argument here applies, of course, only for odd dimensional knots. The case q = 2 was also announced by C. T. C. Wall: Proc. Camb. Phil. Soc. 63 (1967), p. 6.

Proof: This is proved implicitly in [16, Theorem 2] more or less as follows. Consider the matrix $tA - A^T$, with entries considered as elements of $Z[t, t^{-1}] = A$, and the Λ -module H_A with $tA - A^T$ as relation matrix. Consider also the bilinear form $[,]: H \otimes_A H \rightarrow Q(\Lambda)/\Lambda = S(\Lambda) (Q(\Lambda) \text{ is the quotient field of } \Lambda)$ defined, with respect to the same generators of H_A as $tA - A^T$ is a relation matrix, by the matrix $(tA - A^T)^{-1}$. Note that $tA - A^T$ is non-singular over $Q(\Lambda)$ because $\Delta_A(t) = \det(tA - A^T)$ is non-zero. In fact the leading coefficient of $\Delta_A(t)$ is $\det A \neq 0$.

It is not hard to see that the isomorphism class of $(H_A; [,])$ is an invariant of the equivalence class of A. Furthermore the element $\Delta_A(t) = \det(tA - A^T)$ is an invariant of the equivalence class, up to multiplication by powers of t. From this it follows that $\det A = \det B$.

If A is unimodular over R, then $t-A^{-1}A^T$ is a presentation matrix of $H_A \otimes R(\otimes = \bigotimes_Z)$, so $H_A \otimes R$ is a free R-module of the same rank as A and $(tA - A^T)^{-1}$ is the matrix for [,] with respect to an R-basis for $H_A \otimes R$. From this it follows that there exists a matrix P with entries in, and unimodular over, R such that

$$P(tA - A^T)^{-1} P^T = (tB - B^T)^{-1}$$
 in $S(\Lambda) \otimes R$.

Let $\overline{(tA-A^T)}$ be the "adjoint" of $tA-A^T$ ([1, p. 305)]. Then

$$P(\overline{tA - A^{T}}) P^{T} = t\overline{B - B^{T}} \text{ in } (\Lambda/(\Delta(t)) \otimes R,$$

where $\Delta(t) = \Delta_{A}(t) = \Delta_{B}(t).$ (*)

Since A and B are unimodular over R, $\Delta(t)$ has as leading coefficient a unit of R. From this it follows that there exists a unique well-defined R-linear map $\gamma: \Lambda/(\Delta(t)) \otimes R \to R$ defined by the properties $\gamma(1)=1$ and $\gamma(t^i)=0$ for 0 < i < degree $\Delta(t)$. Now every entry of $\overline{tA-A^T}$ and $\overline{tB-B^T}$ has degree $< \text{rank } A = \text{degree } \Delta(t)$. Therefore, applying γ to equation (*) we find that $PAP^T = B$ in R.

26. We cannot strengthen the conclusion of Proposition 2 to conclude that A and B are congruent over K, as the following example shows. Set:

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

We first show that A and B are not congruent over Z. Consider the solutions X of

 $X^{T}AX = X^{T}BX = 2$; they are $X = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$. If $P^{T}AP = B$, it follows that $PX = \pm X \begin{pmatrix} \text{say} \\ X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Now $BX = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and so $Y^{T}BX$ is even, for any Y. Choose Y so that $PY = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; then $Y^{T}BX = Y^{T}P^{T}APX = \pm (PY)^{T}AX$ which one can calculate to be ± 1 .

To see that A and B are equivalent, we consider the following elementary enlargements, respectively, of A and B:

$$A' = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad B' = \begin{pmatrix} B & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A' and B' are congruent; in fact, $PA'P^T = B'$, where

$$P = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

27. On the other hand, the converse of Proposition 2 is false. Consider the following matrices for $\varepsilon = \pm 1$:

$$A = \begin{pmatrix} 0 & \varepsilon x & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & p^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p(1+\varepsilon)+1 \\ 0 & 0 & 0 & 0 & 0 & p+1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \varepsilon x & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 \\ p^4 & 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p(1+\varepsilon)+1 \\ 0 & 0 & 0 & 0 & 0 & p+1 \end{pmatrix}$$

where p is any odd prime, and $x = \frac{1}{4}(p^4 - 1)$. It may be checked directly that $A + \varepsilon A^T$ and $B + \varepsilon B^T$ are unimodular and det $A = \det B$ is divisible by p. But A and B are congruent over any ring in which p is a unit. In fact $PAP^T = B$ where:

$$P = \begin{pmatrix} p \\ 1/p \\ p \\ p \\ 0 \\ 1/p \\ 0 \\ 1 \end{pmatrix}$$

Finally, A and B are not equivalent. To see this consider $A - \varepsilon A^T$ and $B - \varepsilon B^T$ over Z_p . It follows from Witts Theorem (see e.g. [21]) for $\varepsilon = -1$, or the well-known classification of skew-symmetric forms (see e.g. [1]) for $\varepsilon = +1$, that the congruence class of $A - \varepsilon A^T$ over Z_p is an invariant of the equivalence class of A. But $A - \varepsilon A^T$ and $B - \varepsilon B^T$ have ranks 2 and 4, respectively, over Z_p .

28. Finally we remark that the genus of the non-degenerate quadratic form $A + A^T$ is an invariant of the equivalence class of A (see [16, Prop. 5.1]).

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