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On the Symplectic Cobordism Ring

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I. Introduction

This paper continues the study of Ω_*^{Sp} , the ring of cobordism classes of weakly symplectic manifolds along lines initiated by Liulevicius, [6] and Stong [10]. Ω_n^{Sp} is computed for $n < 25$ and, modulo extension problems, for $n < 30$ (along with information on the ring structure) and some information is obtained on the image of the map $\Omega_*^{Sp} \rightarrow \Omega_n^0$, to wit:

THEOREM 1.1. $\Omega_n^{Sp} \rightarrow \Omega_n^0$ is trivial for $n < 32$; for $n = 32$ the image is given by $[RP(2)]^{16}$.

The first stage in our work is the (partial) calculation of the E_2 term of the mod 2 Adams spectral sequence for $\Pi_*(MSp)$ using an algebraic spectral sequence essentially contained in Adams [1] and Liulevicius [6], [7].

In the second stage we employ "Riemann-Roch" relations on the cohomology characteristic number of symplectic manifolds to establish the impossibility of certain classes in the Adams spectral sequence persisting to E_∞ ; this leads immediately to a determination of certain differentials. Also applicable to the calculation of differentials (although dispensable for our range of calculation) are the Cartan formulae for higher order cohomology operations and complete knowledge of Ω_*^{SU} and its Adams spectral sequence (due to Anderson, Brown and Peterson [4]).

Finally, we make some remarks about the extension problem in the E_∞ term.

The author is happy to acknowledge his indebtedness to F. P. Peterson for his guidance in this work, to R. Stong for communication of his results and to D. W. Anderson for much helpful criticism and advice.

II. The E_2 Term

Let $BSp(n)$ denote the universal classifying space for $Sp(n)$ vector bundles and let $MSp(n)$ be the associated Thom space; MSp will designate the associated Thom spectrum. Note that MSp is a ringed spectrum and, according to Novikov [5] we have a ring isomorphism $\Omega_*^{Sp} \cong \Pi_*(MSp)$. Milnor [9] has shown that $\Pi_*(MSp)$ has no odd torsion; we therefore consider only the mod 2 Adams spectral sequence which we denote $E_*^{*,*}(MSp)$. To compute $E_2^{*,*}(MSp) = Ext_A^{*,*}(H^*(MSp), Z_2)$ [A will be

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the mod2 Steenrod algebra, and cohomology will be mod2 unless otherwise indicated] we first summarize results contained essentially in Liulevicius [6].

If H is a graded vector space let H' denote the vector space isomorphic to H as an ungraded vector space but such that the isomorphism $H \rightarrow H'$ doubles grading (and preserves any multiplication or comultiplication that H may possess.) Let A'' be given structure as a graded A -module by having A act via the map $A \xrightarrow{\alpha} A \xrightarrow{\alpha} A$ where α is the dual of the squaring map on the dual of A . Let S be the graded coalgebra over Z_2 such that $S^* \cong Z_2[x_2, x_4, x_5, \dots, x_i, \dots]$ x_i of grading $i, i \neq 2^a - 1$.

PROPOSITION 2.1. $H^*(MSp) \cong A'' \otimes S''$ as a graded coalgebra and A -module (A operating on S'' trivially).

PROPOSITION 2.2. $E_2^{**}(MSp) = Ext_A(A'', Z_2) \otimes S''^*$
 $= Ext_B(Z_2, Z_2) \otimes Z_2[V_2, V_4, V_5, \dots, V_i, \dots]$
 $(i \neq 2^a - 1, V_j \in Ext_A^{0, 4j}(H^*(MSp), Z_2))$

where B is the normal and Hopf subalgebra of A such that $A//B \cong A''$.

If we describe A in terms of the Milnor basis [8] B may be described as the Hopf subalgebra of A generated (additively) by $\{St^I\}, I = (C_0, C_1, \dots, C_j, \dots) 0 < C_j \leq 4$ for all j .

It is convenient to employ a special notation for elements of B .

Let $I = (C_0, C_1, \dots, C_j, \dots) (0 \leq C_j < 4)$. Substitute for each non-zero C_j the symbol

- 1_j if $C_j = 1$
- 2_j if $C_j = 2$
- $1_j 2_j$ if $C_j = 3$

delete zeros and commas and bracket the resulting concatenation; this will be taken as the new symbol for St^I , e.g. $St^{(1, 3, 2, 0, 1, 2)}$ becomes $[1_0 1_1 2_1 2_2 1_4 2_5]$.

Let $C(n)$ be the Hopf subalgebra of B generated (multiplicatively) by the elements $[1_0], [2_j]$ for $j < n, [1_j]$ for $j > n$. As a Hopf algebra $C(n) \cong 'C(n) \otimes ''C(n)$ where $'C(n)$ is generated by the elements $[1_j], [2_j]$ for $j < n$ and $''C(n)$ by the elements $[1_j], j > n$. Note that $Ext_B^{s,t}(Z_2, Z_2) \rightarrow Ext_{C(n)}^{s,t}(Z_2, Z_2)$ induced by the injection $C(n) \rightarrow B$ is an isomorphism onto for $t - s < 2^{n+2} - 2$.

$C(n)$ is normal in $C(n+1), 'C(n)$ is normal in $'C(n+1), C(n+1)//C(n) \cong 'C(n+1) // ''C(n) \cong K(n)$ where $K(n)$ is an exterior algebra on one generator of grading $2^{n+2} - 2$.

We now give a slight improvement of a spectral sequence due to Adams [3]:

Let Γ be a connected Hopf algebra over Z_2, A a normal (but not necessarily central) and Hopf subalgebra of Γ . Let $F(\Gamma^*)$ be the cobar construction [1] on Γ^* , the dual of Γ . By definition

$$F(\Gamma^*) = \sum_{s \geq 0} I(\Gamma^*)^{\otimes s}$$

where \otimes_s denotes s -fold tensor product and I denotes augmentation ideal. $F(\Gamma^*)$ is a bigraded cochain complex with cup products (see [3] for gradings, differentials and cup products) and its cohomology is just $Ext_{\Gamma^*}^{*,*}(Z_2, Z_2)$. We define a filtration, the Λ -filtration on $F(\Gamma^*)$ by setting

$$a_1 \otimes a_2 \cdots \otimes a_r \in F(\Gamma^*)^{(p)} \quad (a_i \in I(\Gamma^*))$$

if a_i annihilates $I(\Lambda)$ for p -values of i . Let $\Omega = \Gamma // \Lambda$.

PROPOSITION 2.3. The Λ -filtration of $F(\Gamma^*)$ gives rise to a trigraded spectral sequence with products, $E_*^{*,*,*}$ such that

$$E_1^{*,*,*} = Ext_{\Gamma^*}^{*,*}(Z_2, Z_2) \otimes F(\Omega^*)$$

$$E_1^{s,t,u} = \sum_{u'+u''=u} Ext_{\Omega^*}^{t,u'}(Z_2, Z_2) \otimes F(\Omega^*)^{s,u''}$$

$$d_r: E_r^{s,t,u} \rightarrow E_r^{s+r,t-r+1,u}$$

$$d_r(x \cdot y) = d_r(x) y + x d_r(y)$$

$$E_{r+1} = H(E_r; d_r);$$

$E_{\infty}^{s,n-s,t}$ gives a composition series for $Ext_{\Gamma^*}^{s,t}(Z_2, Z_2)$; the product structure of E_{∞} as the limit of the E_r agrees with the product structure of E_{∞} as the graded group associated to a filtration of $Ext_{\Gamma^*}^{*,*}(Z_2, Z_2)$.

Proof. We refer to the proof of Theorem 2.3.1 of [3]. We must verify that the condition Λ central in Γ may properly be weakened to Λ normal in Γ . Now actually the effect upon the proof in [3] of this weakening of the hypothesis is that the maps

$$\mu_*: \bar{B}(\Lambda) \otimes (\bar{\Gamma})^p \rightarrow \bar{B}(\Gamma)^{(p)}$$

$$\mu': B(\Omega) \otimes (\bar{\Omega})^p \rightarrow C^{(p)}$$

([3], p. 40) fail to be chain maps (that is they do not commute with the differentials). The reduction

$$\mu'_*: \bar{B}(\Lambda) \otimes (\bar{\Omega})^p \rightarrow \bar{B}(\Gamma)^{(p)} / \bar{B}(\Gamma)^{(p-1)}$$

will however remain a chain map since in fact μ' deviates from being a chain map precisely by an error term in $\bar{B}(\Gamma)^{(p-1)}$.

We could also give the E_2 term as $Ext_{\Omega}(Z_2, Ext_{\Gamma}(Z_2, Z_2))$ but this would require a prior knowledge of the action of Ω on $Ext_{\Gamma}(Z_2, Z_2)$; on the other hand the E_1 term requires $F(\Omega^*)$ which is awkward in the general case but very simple for our applications in which $\Omega = K(n)$.

In fact let $\Gamma = C(n+1)$, $\Lambda = C(n)$, $\Omega = \Gamma // \Lambda = K(n)$, and let the associated spectral sequence be denominated ${}_n E$. Then $F(K(n)^*)$ is simply a polynomial algebra $Z_2[k_n]$, k_n

of bigrading $(1, 2^{n+2} - 2)$, so we write

$$\begin{aligned} {}_nE_1 &= \text{Ext}_{C(n)}(Z_2, Z_2) \otimes Z_2 [k_n] \\ {}_nE_1^{s, t, u} &= \text{Ext}_{C(n)}^{t, u - s(2^{n+2} - 2)}(Z_2, Z_2) \otimes k_n^s. \end{aligned}$$

PROPOSITION 2.4. (Liulevicius [6]) $\text{Ext}_{C(1)}^{*,*}(Z_2, Z_2)$ has multiplicative generators $q_0, k_0, \tau_0, \omega_0, q_j (j > 1)$ of bigradings $(1, 1), (1, 2), (3, 7), (4, 12), (1, 2^{j+1} - 1)$ respectively subject to the relations

$$q_0 k_0 = 0 \quad k_0^3 = 0 \quad k_0 \tau_0 = 0 \quad \tau_0^2 = q_0^2 \omega_0;$$

$\text{Ext}_{C(1)}^{*,*}(Z_2, Z_2)$ thus is a free $Z_2[\omega_0, q_2, q_3, \dots, q_j, \dots]$ module.

This proposition may easily be reproven using ${}_0E$; $\text{Ext}_{C(0)}^{*,*}(Z_2, Z_2)$ is of course just $Z_2[q_0, q_1, \dots, q_j, \dots]$ since $C(0)$ is an exterior algebra.

We now consider ${}_nE, n > 0$. Since $C(n) = {}'C(n) \otimes {}''C(n)$ and ${}''C(n)$ is an exterior algebra we may write

$$\begin{aligned} \text{Ext}_{C(n)}(Z_2, Z_2) &= \text{Ext}_{C(n)} \otimes Z_2 [q_{n+1}] \otimes Z_2 [q_{n+2}, \dots, q_j, \dots] \\ \text{Ext}_{C(n+1)}(Z_2, Z_2) &= \text{Ext}_{C(n+1)} \otimes Z_2 [q_{n+2}, \dots, q_j, \dots] \\ {}_nE &= {}'_nE \otimes Z_2 [q_{n+2}, \dots, q_j, \dots] \end{aligned}$$

where $'_nE$ is the spectral sequence associated to $'C(n+1) // {}''C(n) \cong K(n)$.

THEOREM 2.5. All d_r in ${}_nE$ are trivial for $r > 3$. i.e. ${}_nE_4 \cong {}_nE_\infty$.

Proof. We define a second grading, the "n-grading" on $C(n+1) = {}'C(n+1) \otimes {}''C(n+1)$ by assigning n-grading 0 to $[1_0]$, to $[2_j], j < n$ and to $[1_j], j > n+1$ and n-grading 1 to $[2_n]$. It is readily verified that this gives a well defined grading on $C(n+1)$ and induces a fourth grading on ${}_nE$ so that

$$\begin{aligned} \text{Ext}_{C(n)}(Z_2, Z_2) &\in {}_nE_1^{*,*,*,0} \\ q_{n+1} &\in {}_nE_1^{1,0,2^{n+2}-1,1} \\ k_n &\in {}_nE_1^{0,1,2^{n+2}-2,1} \\ q_j &\in {}_nE_1^{1,0,2^{j+1}-1,0} \quad (j > n+1) \end{aligned}$$

and so that d_r preserves the fourth grading. Direct computation shows that q_{n+1}^4 persists to ${}_nE_\infty$ and that ${}_nE_r$ can be regarded as a free $Z_2[q_{n+1}^4]$ module for $r=1$; we show indirectly that this is true for all r . Assume then that ${}_nE_r$ is a free $Z_2[q_{n+1}^4]$ module. To show that ${}_nE_{r+1}$ is also a free $Z_2[q_{n+1}^4]$ module we must show that if $u \in {}_nE_r, d_r u = [q_{n+1}^4] \cdot w$ then there is a $u' \in {}_nE_r$ such that $d_r u' = w$. Since $d_r u = [q_{n+1}^4] \cdot w$ u has fourth grading greater by 4 than that of w and a second grading 2 less than that of w . In ${}_nE_1$ the fourth grading minus the second grading gives the power of q_{n+1} , hence u is the image of an element of ${}_nE_1$ which is there divisible by q_{n+1}^4 . By induction

this divisibility persists to ${}_n E_r$ so

$$u = [q_{n+1}^4] u', \quad d_r u = [q_{n+1}^4] d_r u' = [q_{n+1}^4 w]$$

and $u' = w$.

Now suppose $d_r u \neq 0, r > 3$. Then as before $u = [q_{n+1}^4] u', d_r u = [q_{n+1}^4] d_r u', d_r u \neq 0$ giving an absurd infinite regress and so proving the theorem.

Now let ω_n denote the class in $Ext_{C(n+1)}(Z_2, Z_2)$ corresponding to the image of q_{n+1}^4 in ${}_n E_\infty$. We will (in line with our general notational policy) also refer to the corresponding (unique) element of $Ext_B(Z_2, Z_2)$ as ω_n .

COROLLARY 2.6. *$Ext_B(Z_2, Z_2)$ is a free $Z_2[\omega_0, \omega_1, \dots, \omega_j, \dots]$ module. Note that we do have relations in $Ext_B(Z_2, Z_2)$ of the form $u \cdot v = w \omega_n$ where u, v are not divisible by ω_n .*

Let us consider the element k_j which arises in ${}_j E_1$ and persists by reason of grading considerations to ${}_j E_\infty$ to give an element of $Ext_{C(j+1)}^{1, 2^{j+2}-2}(Z_2, Z_2)$ which cannot (again for grading considerations) be killed in ${}_j E(j' > j)$ and so determines an element (also called) $k_j \in Ext_B(Z_2, Z_2)$. Direct computation shows that these k_j together with $q_0 \in Ext_B^{1,1}(Z_2, Z_2)$ complete the description of $Ext_B^{1,*}(Z_2, Z_2)$. (The elements $[2_j], [1_0]$ constitute a minimal set of polynomial generators of B ; k_j, q_0 are their duals.)

The element q_{j+1} must therefore perish; it can do so only in ${}_j E$ and only via $d_1 q_{j+1} = q_0 k_j$. However the class $q_{j+1} k_i (j > i)$ which arises in ${}_i E$ is not killed in ${}_j E$ as $d_1(q_{j+1} k_i) = q_0 k_i k_j$ but $q_0 k_i k_j = 0$ in ${}_j E$ as already in ${}_i E$ $q_0 k_i$ has been killed by q_{i+1} .

There is a well-defined element

$$\gamma_{i,j} \in Ext_B^{2, 2^{i+2}+2^{j+2}-3}(Z_2, Z_2)$$

corresponding to $q_{j+1} k_i$.

Similarly, the element $q_0 q_{j+1}^2$ is not killed in ${}_j E$ as while $d_2(q_{j+1}^2) = k_0 k_j^2, d_2(q_0 q_{j+1}^2) = q_0 k_0 k_j^2 = d_1(q_{j+1} k_0 k_j) = 0 \in {}_j E_2$, so $q_0 q_{j+1}^2$ determines a class

$$\tau_j \in Ext_B^{3, 2^{j+3}-1}(Z_2, Z_2).$$

(If $j=0$ replace d_2 by d_3 in the preceding). For $j > 0$ the definition of τ_j is subject to an indeterminacy of $k_j \gamma_{0,j}$. More generally let $j(i), 1 \leq i \leq n$ be a strictly increasing sequence of nonnegative integers. Corresponding to

$$q_0 q_{j(1)+1}^2 \cdots q_{j(i)+1}^2 \cdots q_{j(n)+1}^2$$

we have a class

$$\tau_{j(1), \dots, j(i), \dots, j(n)}$$

Now suppose $0 \leq i < j < k$. In ${}_k E$

$$\gamma_{j,k} \text{ is represented by } q_{k+1} k_j$$

$\gamma_{i,k}$ is represented by $q_{k+i}k_i$

$\gamma_{i,j}$ is represented by $\gamma_{i,j}$ (coming from $q_{j+1}k_i$ in ${}_jE$).

There are also elements $\gamma_{i,j}k_k$ and $q_k k_i k_j$ in ${}_kE$ (persisting to ${}_kE_\infty$). We see at once that

$$k_k \gamma_{i,j} \neq 0, \quad k_i \gamma_{j,k} \neq 0, \quad k_j \gamma_{i,k} \neq 0$$

and

$$k_i \gamma_{j,k} = k_j \gamma_{i,k} + c k_k \gamma_{i,j}$$

where $c \in Z_2$ represents an extension problem. But there is an automorphism Π on B as an ungraded Hopf algebra which interchanges $[1_{j+1}]$ and $[1_{k+1}]$ and $[2_j]$ and $[2_k]$. The corresponding (non-grading preserving) automorphism Π^* on $Ext_B(Z_2, Z_2)$ has

$$\begin{aligned} \Pi^* k_i &= k_i & \Pi^* \gamma_{i,j} &= \gamma_{i,k} \\ \Pi^* k_j &= k_k & \Pi^* \gamma_{i,k} &= \gamma_{i,j} \\ \Pi^* k_k &= k_j & \Pi^* \gamma_{j,k} &= \gamma_{j,k} \end{aligned}$$

so that

$$k_i \gamma_{j,k} = k_k \gamma_{i,j} + c k_j \gamma_{i,k}$$

Now $c=0$ would imply

$$k_i \gamma_{j,k} = k_j \gamma_{i,k} = k_k \gamma_{i,j}$$

but we have seen that $k_i \gamma_{j,k}, k_j \gamma_{i,k}, k_k \gamma_{i,j}$ must generate a two-dimensional Z_2 -vector space in $Ext_B(Z_2, Z_2)$ so $c=1$ and we have the relation

$$k_i \gamma_{j,k} + k_j \gamma_{i,k} + k_k \gamma_{i,j} = 0$$

The symmetries of B are thus very useful for the study of $Ext_B(Z_2, Z_2)$. They may be used for example to show that for $0 < i < j$,

$$(\gamma_{i,j})^4 = k_i^4 \omega_j + k_j^4 \omega_i$$

(It is also true, but more difficult to establish that $(\gamma_{0,j})^4 = k_j^4 \omega_0$.)

It is clear that $Ext_B(Z_2, Z_2)$ contains a free polynomial ring generated by the ω_i and the $k_j (j > 0)$. As an application we observe that the spectral sequences show that $k_2 \tau_{0,1}$ is the unique non-zero element in $Ext_B^{6,35}(Z_2, Z_2)$. On the other hand $[(\gamma_{0,1})^2 (\gamma_{0,2})]^4 = k_1^8 k_2^4 \omega_0^3 \neq 0$ so we have, using symmetry, $i, j > 0, i \neq j$

$$k_i \tau_{0,j} = (\gamma_{0,j})^2 (\gamma_{0,1}).$$

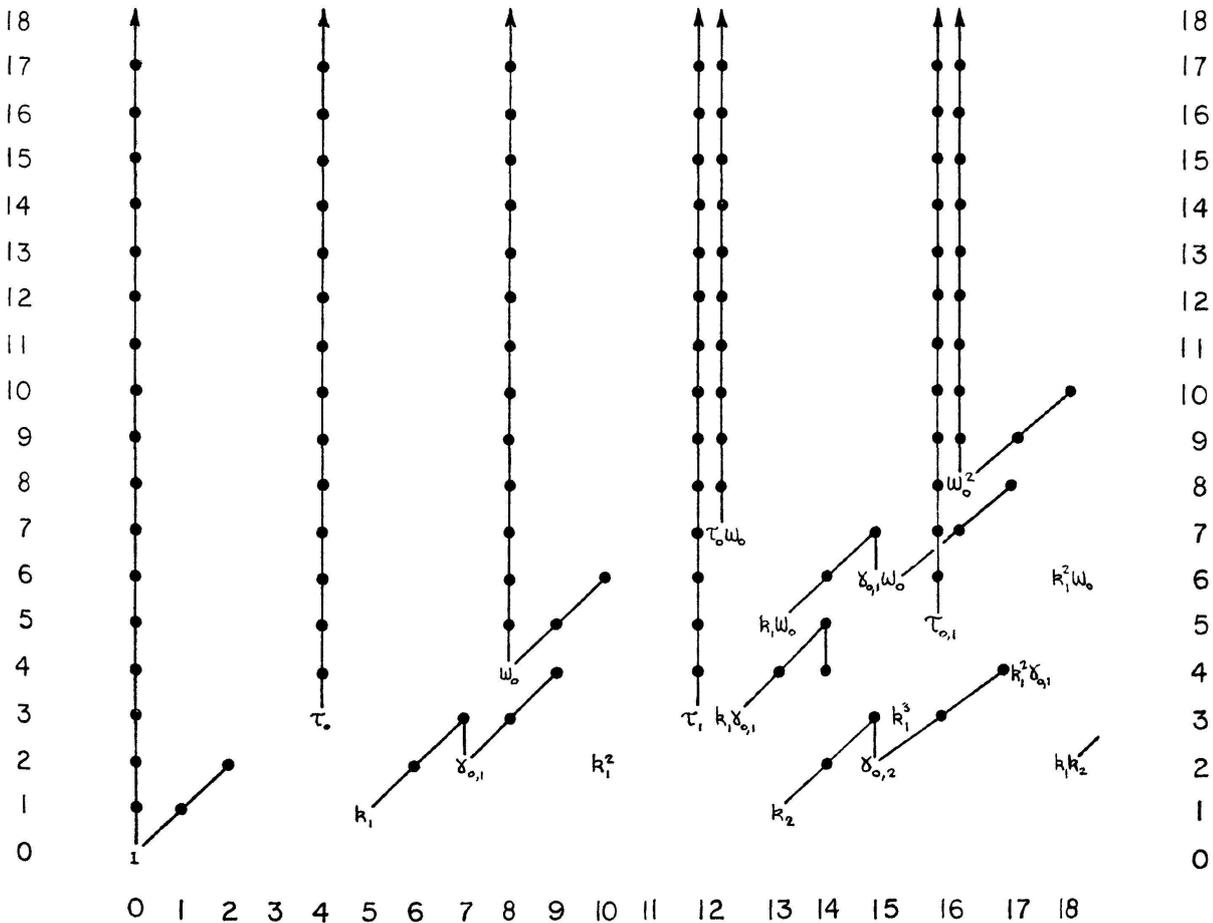
Let us fix $\tau_j, j > 0$, by requiring that $k_0 \tau_j = 0$. Then we have (by computations such as those above):

THEOREM 2.7. *The following are some of the relations holding in $Ext_B(Z_2, Z_2)$. (In each case $0 \leq i < j < k < l$.)*

$$\begin{aligned}
 q_0 k_i &= 0 & k_0 k_i^2 &= 0 & q_0 \gamma_{i,j} &= k_0 k_i k_j & k_0 k_i k_j k_k &= 0 & (\gamma_{i,j})^4 &= k_i^4 \omega_j + k_j^4 \omega_i \\
 k_i \gamma_{j,k} + k_j \gamma_{i,k} + k_k \gamma_{i,j} &= 0 & \gamma_{i,j} \gamma_{k,l} + \gamma_{i,k} \gamma_{j,l} + \gamma_{i,l} \gamma_{j,k} &= 0 & k_0 \tau_i &= 0 \\
 k_j \tau_j &\equiv 0 \pmod{k_j^2 \gamma_{0,j}} & k_j \tau_k &\equiv 0 \pmod{k_k^2 \gamma_{0,j}} & k_j \tau_0 &\equiv 0 \pmod{k_0^2 \gamma_{0,j}} & \tau_0^2 &= q_0^2 \omega_0 \\
 \tau_j^2 &\equiv q_0^2 \omega_j \pmod{k_j^2 (\gamma_{0,j})^2}, & k_0 \tau_{0,j} &= 0, & k_j \tau_{0,j} &= (\gamma_{0,j})^3, \\
 k_k \tau_{0,j} &= (\gamma_{0,k}) (\gamma_{0,j})^2, & k_0 (\gamma_{i,j})^2 &= 0, & \tau_I \tau_J &= (q_0 \tau_K I I_{i \in I \cap J} \omega_i) + \text{other terms}
 \end{aligned}$$

where $K = (I \cup J) - (I \cap J)$, $\tau_\phi = q_0$. Although we have not described $Ext_B(Z_2, Z_2)$ completely (there are even irreducible generators which have not been mentioned) the above information is sufficient to yield Table I which describes $Ext_B^{s,t}(Z_2, Z_2)$ for $t - s \leq 31$; actions of q_0 and k_0 are described by vertical and diagonal lines respectively.

TABLE I



In the case of 8-dimensional manifolds our calculations are easily exhibited. If M is an 8-dimensional Sp -manifold $S_1(e_p) T[M] = (1/6)S_{1^2}(p) [M]$ and $S_\phi(e_p) T[M] = (1/240)S_2(p) + (1/144)S_{1^2}(p) [M]$ where $S_2(p)$ and $S_{1^2}(p)$ generate $H^8(BSp; Z)$. The condition that $(1/6)S_{1^2}(p) [M]$ be even implies that $S_{1^2}(p) [M]$ is divisible by 4; this together with the requirement that $(1/240)S_2(p) [M] + (1/144) S_{1^2}(p) [M]$ be integral gives that $S_2(p) [M]$ and so also any integral cohomology characteristic number is divisible by 4.

The following has been proven by D. W. Anderson (unpublished).

PROPOSITION 3.2. *Let M be a connected spectrum, S the sphere spectrum, $u \in E_\infty^{s,t}(M)$ such that $h_0^n \cdot u \neq 0$ for all non-negative n (h_0 the generator of $E_\infty^{1,1}(S)$). Let $f: S \rightarrow M$ be represented by u . Then the image of $H^*(M, Z)$ in $H^*(S, Z)$ under f^* contains elements not divisible by 2^{s+1} . (We allow f to be a map of spectra which changes indexing.)*

The proof of Proposition 3.2 depends upon the fact that it is possible to map M into $K(Z)$ (the Eilenberg-MacLane spectrum) so that u is in the image of the induced map $E_\infty(K(Z)) \rightarrow E_\infty(M)$. The general result for M then follows from the case of $K(Z)$; the proposition is however obvious for $M = K(Z)$.

PROPOSITION 3.3.

- (i) $d_2 v_2 = \gamma_{0,1}$, (ii) $d_2 v_4 = \gamma_{0,2}$, (iii) $d_3 v_2^2 = k_1^3$,
- (iv) $d_2 v_5 = \gamma_{1,2}$, (v) $d_3 v_6 = k_1^2 k_2$, (vi) $d_2 v_8 = \gamma_{0,3}$,

Proof. Putting together Propositions 3.1 and 3.2 we have at once that $v_2, q_0 v_2, v_4, v_2^2, v_5, q_0 v_5$ and v_6 cannot persist to $E_\infty(MSp)$. This implies parts (i)–(v).

From the results of Anderson, Brown and Peterson [4] and analysis of the map of Adams spectral sequences induced by the standard map of MSp to MSU we may obtain (vi).

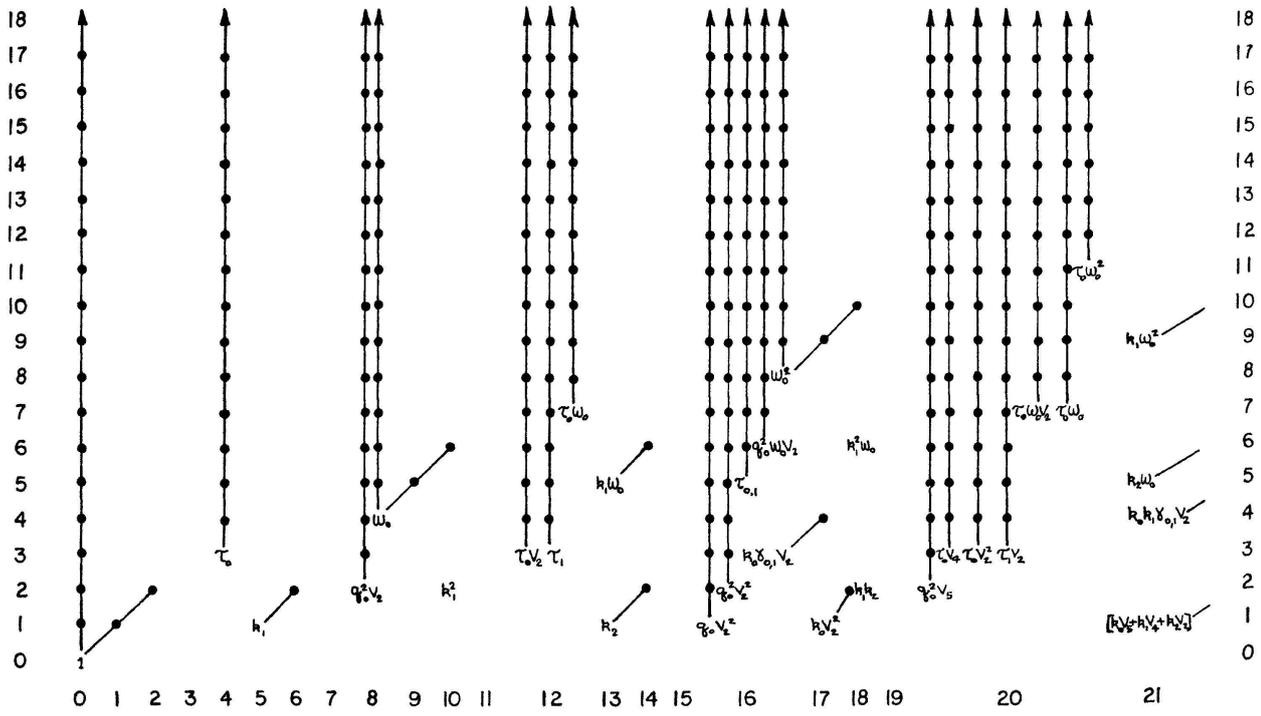
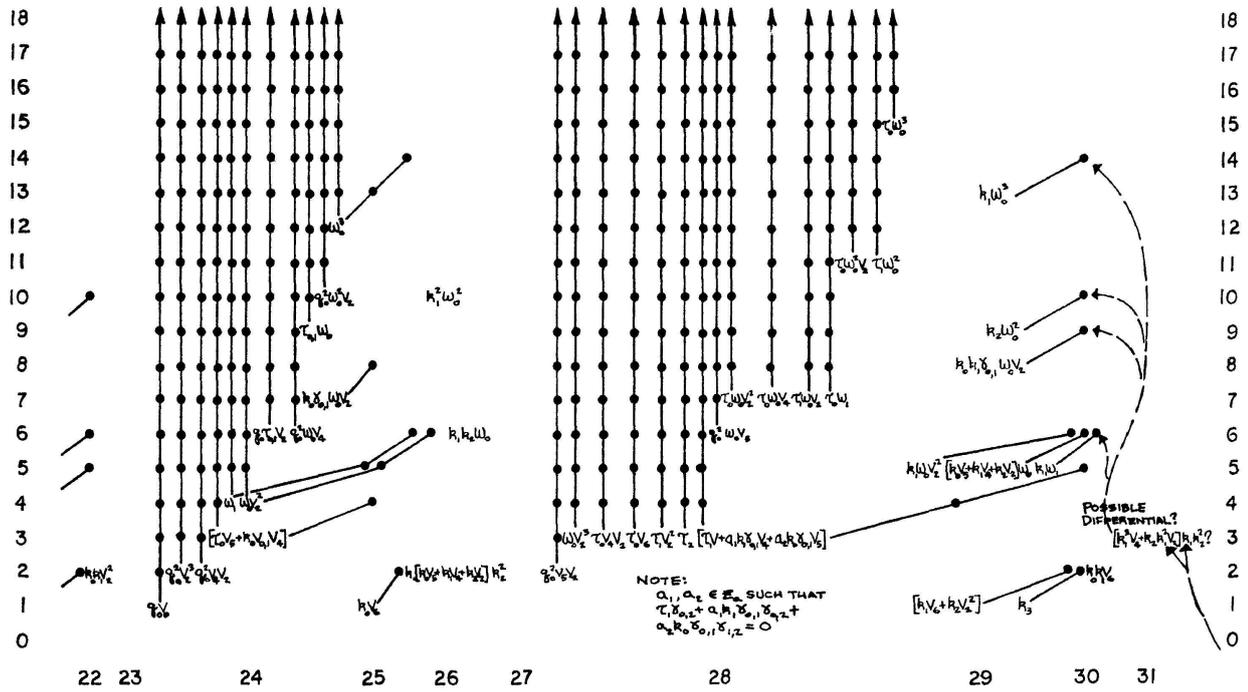
Putting Proposition 3.3 together with Proposition 2.1, Theorem 2.7 and Table I we are able to compute the Adams spectral sequence for MSp completely through the 31-stem save for two unsettled questions:

- i) The value of $d_3(v_4^2)$. (We conjecture that it is $k_1 k_2^2$.)
- ii) Is $k_1^3 v_4 + k_2 k_1^2 v_2$ an infinite cycle?

Our results concerning $E_\infty(MSp)$ are depicted in Table II. Since $d_2(v_2^4) = d_3(v_2^4) = 0$ it is now obvious that v_2^4 is an infinite cycle and Theorem 1.1 follows.

We now turn to the extension problems involved in passing from $E_\infty(MSp)$ to Ω_*^{Sp} . $\Omega_{10}^{Sp} = Z_2 \oplus Z_2$ where one Z_2 is the square of the Ω_5^{Sp} generator and the other is the product of the generators of Ω_1^{Sp} and Ω_9^{Sp} . Multiplication by the generator of Ω_1^{Sp} induces an isomorphism between Ω_{13}^{Sp} and Ω_{14}^{Sp} ; this implies $\Omega_{13}^{Sp} \simeq \Omega_{14}^{Sp} \simeq Z_2 \oplus Z_2$. In the 16-stem there is a non-trivial extension since the element $k_0 \gamma_{0,1} v_2$ goes into a non-zero element in $E_\infty(MSU)$ but Ω_{16}^{SU} is torsion free. (In fact careful analysis of

TABLE II



characteristic numbers shows that twice the class represented by $k_0\gamma_{0,1}v_2$ is represented by $\tau_{0,1}$.) Utilizing multiplication by k_1 as well as k_0 in $E_\infty(MSp)$ it is possible to show that all elements of the 17, 18, 21 and 22 stems of Ω_*^{Sp} are of order 2.

There are clearly no extension problems in the 20-stem while in the 24-stem we have a situation corresponding to that in the 16-stem (with $k_0\gamma_{0,1}\omega_0v_2$ and $\tau_{0,1}\omega_0$ replacing $k_0\gamma_{0,1}v_2$ and $\tau_{0,1}$ respectively). Beyond the 24-stem we have not succeeded in resolving the extension problems. Also, some very elementary questions on the ring structure are left open. For example we do not know if the product of the class represented by $k_0^2\gamma_{0,1}v_2$ by the class represented by k_0 is trivial or not. (If it is trivial then Ω_{25}^{Sp} is the sum of six copies of Z_2 .)

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