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An Application of Simplicial Profinite Groups

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In this paper we use simplicial pro-p-groups to give quite different proofs of the convergence theorems for the lower central series and p-lower central series spectral sequences of a simplicial group (CURTIS [2], RECTOR [7]). The proofs are, we believe, more conceptual than the delicate calculations with generators in free groups used in [2]. In addition in the case of the p-lower central series spectral sequence we obtain convergence for certain non-connected simplicial groups, in particular those corresponding to connected H-spaces with finitely generated homology in each dimension.

The idea of the proof is as follows. If G is a free simplicial group with finitely many generators in each dimension, then because inverse limits are exact for profinite groups the p-lower central series spectral sequence of G converges strongly to $\pi(\hat{G})$, where $\wedge$ denotes p-completion. So the weak convergence of the spectral sequence to $\pi(G)$ follows from the formula $(\pi G)^\wedge \simeq \pi(\hat{G})$, and our main theorem gives conditions under which this holds. The main theorem is proved by a modification of a method of ARTIN and MAZUR, who prove an analogous theorem for pro-p homotopy objects in their work on etale homotopy theory (to appear).

The paper is in three sections. In the first we give the statement of the main theorem and deduce the convergence theorems from it. Section 2 is devoted to generalizing to simplicial profinite groups standard properties of the cohomology of simplicial groups such as the Serre spectral sequence and the Whitehead theorems. In the third section we use these results to study the p-completion functor from simplicial groups to simplicial pro-p groups. Using ZEEMAN's comparison theorem in the way indicated by ARTIN and MAZUR we prove a result on the compatibility of completion and fibrations (Th. 3.6) from which the main theorem follows easily.

This paper resulted from putting ideas of ED CURTIS and MIKE ARTIN and BARRY MAZUR together. I wish to acknowledge the benefit of numerous conversations.

§ 1. Statement and Application of the Main Theorem

We assume familiarity with basic facts about profinite groups (SERRE [8]) and (semi-)simplicial groups (KAN [4]). A fundamental property of profinite groups is that the inverse limit functor is exact for filtered inverse systems. Consequently

$$\pi_q(\varprojlim \mathrm{G}_i) \simeq \varprojlim \pi_q \mathrm{G}_i \tag{1.1}$$

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if $\mathbf{G}_i$ $i \in I$ is an inverse system of simplicial profinite groups indexed by a direct set I.

If G is a simplicial group and M is a $\pi_0 G$ module, then M defines a local coefficient system on the "classifying space" simplicial set $\bar{W}G$, and we define the *homology* $H_q(G, M)$ and the *cohomology* $H^q(G, M)$ of G with values in M to be the homology and cohomology of this local coefficient system. We shall identify a group π with the *constant* simplicial group which is π in each dimension and has all face and degeneracy operators the identity, in which case $H_q(\pi, M)$ and $H^q(\pi, M)$ as just defined are the ordinary Eilenberg-MacLane homology and cohomology.

By a module M over a profinite group $\boldsymbol{\pi}$ we shall always mean one which is a topological $\boldsymbol{\pi}$ module when endowed with the discrete topology. If $\mathbf{G}$ is a simplicial profinite group and M is a $\pi_0 \mathbf{G}$ module, we define the *cohomology* of $\mathbf{G}$ with values in M to be

$$H^q(\mathbf{G}, M) = \varinjlim H^q(\mathbf{G}/\mathbf{U}, M^{\pi_0 \mathbf{U}}) \tag{1.2}$$

where $\mathbf{U}$ runs over the directed set of open normal simplicial subgroups of $\mathbf{G}$. When $\mathbf{G}$ is constant we obtain the definition of cohomology given in SERRE [8].

Let p be a prime number. If π is a group, let $\hat{\pi}$ be its *p-completion*, i.e. $\varprojlim \pi/\pi'$ where π' runs over the set of normal subgroups of index a power of p. If M is a π module which is also an abelian p-group, then the following conditions are equivalent: (i) M comes from a $\hat{\pi}$ module under the map $\pi \to \hat{\pi}$, (ii) M has a composition series with quotients of the form $\mathbf{Z}/p$ (integers mod p) with trivial π action, (iii) the subgroup of π consisting of the elements acting trivially on M is of index a power of p. If these conditions hold, we say that π acts *unipotently* on M.

Let $G \to \hat{G}$ be the dimension-wise extension of $\wedge$ to a functor from the category of simplicial groups to the category of simplicial pro-p groups. The normalized subgroups $N_q\mathbf{G}$ and the Moore homotopy groups $\pi_q\mathbf{G}$ of a simplicial pro-p group $\mathbf{G}$ are pro-p groups, hence the map $\pi_q G \to \pi_q \hat{G}$ extends uniquely to a canonical map

$$(\pi_q G)^\wedge \to \pi_q \hat{G}. \tag{1.3}$$

If M is a $\pi_0 \hat{G}$ module, then there is also a canonical map

$$H^q(\hat{G}, M) \to H^q(G, M) \tag{1.4}$$

and we say that G is *p-good* if this map is an isomorphism for q and for all M which are abelian p-groups. (It suffices by the five lemma and (ii) above that 1.4 be an isomorphism for $M = \mathbf{Z}/p$ and all q). This definition in the case of a constant G is an obvious extension of the one given in SERRE [8], p. 1–16. In Section 3 we shall shown that if each G_n is p-good so is G, and that free groups and finitely generated nilpotent groups are p-good.

Let $\tilde{G}$ = the kernel of the augmentation $G \to \pi_0 G$ be the 'universal covering' of

G. If M is a $\pi_0 G$ module, then G_0 acts by conjugation on the pair $(\tilde{G}, M)$, hence on $H_q(\tilde{G}, M)$ and this action induces an action of $\pi_0 G$ on $H_q(\tilde{G}, M)$.

We can now state our main result which will be proved in § 3.

THEOREM 1.5: *Let G be a simplicial group satisfying the following conditions:*

(i) *G is p-good*

(ii) *$\pi_0 G$ is p-good*

(iii) *$H_q(\tilde{G}, \mathbf{Z})$ is finitely generated for all q*

(iv) *$\pi_0 G$ acts unipotently on $H_q(\tilde{G}, \mathbf{Z}/p)$ for all q.*

Then the canonical map (1.3) *is an isomorphism for all q.*

Our first application of this theorem is to deduce CURTIS' connectivity theorem for simplicial groups [2] from the simpler one for simplicial Lie algebras [1]. Let $f(q)$ be any function of q (e.g. 2^q) such that $\pi_q(L_r X)=0$ if $r \geqslant f(q)$ and X is any connected free simplicial abelian group. Let $\Gamma_r \pi$ be the lower central series of a group π, and let $\operatorname{gr}\pi = \oplus \Gamma_r \pi / \Gamma_{r+1}\pi$ be the associated graded Lie algebra.

COROLLARY 1.6: *If G is a free simplicial group such that $\pi_0 G=0$, then $\pi_q \Gamma_r G=0$ for $r>f(q)$.*

Proof: We first reduce to the case where G is finitely generated in each dimension. As G is connected we may by the simplicial analogue of cell-attaching construct a weak equivalence $F \to G$, where F is free with all generators of dimension >0. As G is free it is homotopy equivalent to F, so we may assume $G_0=1$. But then G is the filtered inductive limit of those simplicial subgroups whose set of non-degenerate generators is a finite subset of the set of non-degenerate generators of G. Each of these subgroups is connected since $G_0=1$, and since π_* and Γ_r commute with filtered inductive limits we may assume G has finitely many non-degenerate cells and that $G_0=1$.

Consider the diagram

$$\begin{array}{ccc} (\pi_q G)^\wedge & \to & \varprojlim_r \pi_q (G/\Gamma_r G)^\wedge \\ \downarrow & & \downarrow \\ \pi_q \hat{G} & \to & \varprojlim_r \pi_q ((G/\Gamma_r G)^\wedge) \end{array} \tag{1.7}$$

The vertical arrows are isomorphisms by the main theorem. In effect $H_q(G, \mathbf{Z})= \pi_{q-1}(\operatorname{gr}_1 G)$ is finitely generated, $\pi_0 G=0$, and free simplicial groups are good, so the main theorem applies to G. Also $\operatorname{gr}_r G = L_r(\operatorname{gr}_1 G)$ is finitely generated in each dimension, so by induction on r and the homotopy long exact sequence associated to $1 \to \operatorname{gr}_r G \to G/\Gamma_{r+1} G \to G/\Gamma_r G \to 1$ one sees that $\pi_q(G/\Gamma_r G)$ is finitely generated. But $G/\Gamma_r G$ is trivial in dimension 0 hence is connected, so by SERRE $H_q(G/\Gamma_r G, \mathbf{Z})$ is finitely generated. Also $G/\Gamma_r G$ is good because it's a finitely generated nilpotent group in each dimension. Thus the main theorem applies to $G/\Gamma_r G$.

Now if U runs over the normal subgroups of index a power of p in any group G we have

$$\left.\begin{aligned} \varprojlim_r (G/\Gamma_r G)^\wedge &= \varprojlim_r \varprojlim_U (G/U)/\Gamma_r(G/U) \\ &= \varprojlim_U \varprojlim_r (G/U)/\Gamma_r(G/U) = \varprojlim_U G/U = \hat{G} \end{aligned}\right\} \tag{1.8}$$

where we have used the fact that p groups are nilpotent. Consequently by 1.1 the bottom row of 1.7 is an isomorphism, and hence all maps in 1.7 are isomorphisms.

By CURTIS' connectivity theorem for simplicial Lie algebras $\pi_q(\mathrm{gr}_r G) = \pi_q(L_r(\mathrm{gr}_1 G)) = 0$ for $r \geqslant f(q)$. Hence the inverse system $\pi_q(G/\Gamma_r G)$ stabilizes for $r > f(q)$ and we see that $\pi_q(G)^\wedge \to \pi_q(G/\Gamma_r G)^\wedge$ is an isomorphism for $r > f(q)$. But this is true when $\wedge$ is the p-completion for any prime p, hence as both groups are finitely generated abelian groups $\pi_q G \simeq \pi_q(G/\Gamma_r G)$ for $r > f(q)$ and the corollary follows.

Our second application of the main theorem is to generalize RECTOR's result [7] to a class of non-connected simplicial groups. If π is a group let $\Gamma_r^p \pi$ $r \geqslant 1$ be its p-lower central series and let $\mathrm{gr}^p \pi = \oplus \Gamma_r^p \pi / \Gamma_{r+1}^p \pi$ be the associated graded p-Lie algebra over $\mathbf{Z}/p$. If $L^p V = \oplus L_r^p V$ is the free p-Lie algebra generated by a $\mathbf{Z}/p$ module V, then the canonical map

$$L^p(\mathrm{gr}_1^p \pi) \to \mathrm{gr}^p \pi \tag{1.9}$$

of p-Lie algebras which extends the identity in degree 1 is always surjective and is an isomorphism when π is free. If G is a free simplicial group, then the decreasing filtration $\Gamma_r^p G$ gives rise to the p-lower central series spectral sequence

$$E_{n\,m}^1 = \pi_n L_m^p \,\mathrm{gr}_1^p G \qquad d_r : E_{n\,m}^r \to E_{n-1,\,m+r}^r \tag{1.10}$$

PROPOSITION 1.11: *If G is a free simplicial group such that $H_q(G, \mathbf{Z}/p) = \pi_{q-1}(\mathrm{gr}_1^p G)$ is finite for all q, then the spectral sequence* 1.10 *converges strongly to $\pi_n \hat{G}$.*

Proof: Define the p-lower central series for a pro-p group $\boldsymbol{\pi}$ by $\Gamma_r^p \boldsymbol{\pi} = \varprojlim \Gamma_r^p(\boldsymbol{\pi}/\boldsymbol{\pi}')$ where $\boldsymbol{\pi}'$ runs over the open normal subgroups of $\boldsymbol{\pi}$. As inverse limits are exact for profinite groups $\mathrm{gr}^p \boldsymbol{\pi} = \varprojlim \mathrm{gr}^p(\boldsymbol{\pi}/\boldsymbol{\pi}')$, and moreover

$$\left.\begin{aligned} \boldsymbol{\pi} = \varprojlim_{\boldsymbol{\pi}'} \boldsymbol{\pi}/\boldsymbol{\pi}' &= \varprojlim_{\boldsymbol{\pi}'} \varprojlim_r (\boldsymbol{\pi}/\boldsymbol{\pi}')/\Gamma_r^p(\boldsymbol{\pi}/\boldsymbol{\pi}') \\ &= \varprojlim_r \varprojlim_{\boldsymbol{\pi}'} (\boldsymbol{\pi}/\boldsymbol{\pi}')/\Gamma_r^p(\boldsymbol{\pi}/\boldsymbol{\pi}') = \varprojlim_r \boldsymbol{\pi}/\Gamma_r^p \boldsymbol{\pi}. \end{aligned}\right\} \tag{1.12}$$

If $\mathbf{V}$ is a profinite $\mathbf{Z}/p$ module, define the p-Lie algebra it generates by $L^p \mathbf{V} = \varprojlim L^p(\mathbf{V}/\mathbf{V}')$, so that by passage to the inverse limit in 1.9 one obtained a canonical map

$$L^p(\mathrm{gr}_1^p \boldsymbol{\pi}) \to \mathrm{gr}^p \boldsymbol{\pi}. \tag{1.13}$$

LEMMA 1.14: *The map* 1.13 *is an isomorphism iff* π *is a free pro-p group.*

A free pro-p group $\mathbf{F}$ is by definition (SERRE [8], p. 1–5) the completion of the free group FS generated by a set S with respect to the family of normal subgroups which are of index a power of p and which contain almost all members of S. In other words $\mathbf{F}=\lim\limits_{\overleftarrow{S'}}\lim\limits_{\overleftarrow{r}} FS'/\Gamma_r^p FS'$, where S' runs over the finite subsets of S and the map $FS_1'\to FS_2'$ when $S_2'\subset S_1'$ is the identity on S_2' and sends $S_1'-S_2'$ to zero.

Taking into account the definition of 1.13, one sees that it is an isomorphism for $\mathbf{F}$, since 1.9 is an isomorphism for FS' and hence an isomorphism in degrees $\leqslant r$ for $\pi=FS'/\Gamma_r^p FS'$. Conversely if 1.13 is an isomorphism, then choosing a map $u:\mathbf{F}\to\pi$ where $\mathbf{F}$ is free such that $\mathrm{gr}_1^p u$ is an isomorphism (SERRE [8], p. 1–36, prop. 24), we have that $\mathrm{gr}^p u$ and hence u is an isomorphism. This proves the lemma.

The map $G\to\hat{G}$ carries $\Gamma_r^p G$ into $\Gamma_r^p\hat{G}$, hence induces a map of spectral sequences

$$\pi_*(\mathrm{gr}^p G)\to\pi_*(\mathrm{gr}^p\hat{G}). \tag{1.14}$$

$\mathrm{gr}_1^p G$ is a simplicial $\mathbf{Z}/p$ module hence is homotopy equivalent to the simplicial $\mathbf{Z}/p$ module $N=\bigoplus\limits_q K(\pi_q\,\mathrm{gr}_1^p G, q)$ which is finite in each dimension by hypothesis. So $\mathrm{gr}_1^p\hat{G}=(\mathrm{gr}_1^p G)^\wedge$ is homotopy equivalent to $\hat{N}$ and hence $\mathrm{gr}^p\hat{G}$, which equals $L^p(\mathrm{gr}_1^p\hat{G})$ by the lemma and the fact that the completion of a free group is free (SERRE [8], p. 1–6, Remarque), is homotopy equivalent to $L^p\hat{N}$. But $N=\hat{N}$ so $L^p\hat{N}=L^p N$, which is homotopy equivalent to $\mathrm{gr}^p G$. Thus 1.14 is an isomorphism of spectral sequences. But the second spectral sequence converges strongly to $\pi_n\hat{G}$ by 1.1 and 1.12, so the proof of the proposition is complete.

Combining the main theorem and this proposition we obtain

THEOREM 1.15: *Let* G *be a free simplicial group such that*

(i) $H_q(\tilde{G},\mathbf{Z})$ *is finitely generated for all* q

(ii) $H_q(G,\mathbf{Z}/p)$ *is finite for all* q

(iii) $\pi_0 G$ *is p-good*

(iv) $\pi_0 G$ *acts unipotently on* $H_q(\tilde{G},\mathbf{Z}/p)$ *for all* q.

Then the p-lower central series spectral sequence 1.10 *converges weakly to* $\pi_n G$ *in the sense that the following hold:*

(a) $E_{nm}^\infty=\lim\limits_{\leftarrow} E_{nm}^r\ r>m$

(b) *the topology on* $\pi_n G$ *given by the filtration* $\mathrm{Ker}\,\{\pi_n G\to\pi_n(G/\Gamma_m^p G)\}$ *is the p-topology.*

COROLLARY 1.16: *Let* G *be a free simplicial group which is an "H-space" object of the homotopy category of simplicial groups. If* $H_q(G,\mathbf{Z})$ *is finitely generated for all* q, *then the p-lower central series spectral sequence of* G *converges weakly to* $\pi_n G$.

Proof: In this case $\pi_0 G$ is abelian and acts trivially on $H_*(\tilde{G},\mathbf{Z})$. So $\pi_0 G=$

$H_1(G, \mathbf{Z})$ is finitely generated and hence p-good. Moreover by a well-known argument of SERRE [9] applied to the homology spectral sequence of the fibration $\tilde{G} \to G \to \pi_0 G$ one finds that $H_q(\tilde{G}, \mathbf{Z}/p)$ is finitely generated for all q. So the corollary follows from 1.15.

Example: Let G be a simplicial group corresponding under KAN's theory [4] to real projective space $\mathbf{R}P^k$. If $p=2$ then the hypotheses of 1.15 are satisfied, but if p is odd, then $\pi_1 \mathbf{R}P^k = \mathbf{Z}/2$ acts unipotently (trivially in this case) on $H_*(S^k, \mathbf{Z}/p)$ exactly when k is odd. If k is even then the map $* \to \mathbf{R}P^k$ is a $\mathbf{Z}/p$ homology isomorphism, so the sequence can't converge or else $0 = \pi_n(\mathbf{R}P^k)^\wedge = \pi_n(S^k)^\wedge$ $(n>1)$, which is non-sense.

§ 2. Cohomology of simplicial profinite groups

We shall need the following properties of the cohomology of simplicial groups. A map of simplicial sets or groups is called a *weak equivalence* if it induces isomorphisms on homotopy groups.

PROPOSITION 2.1: *Let G, H be simplicial groups and let M be a $\pi_0 G$ module.*

(a) *If $f: H \to G$ is a weak equivalence, then $H^*(f, M): H^*(G, M) \to H^*(H, M)$ is an isomorphism.*

(b) *If f and g are homotopic maps of simplicial groups from H to G, then $H^*(f, M) = H^*(g, M)$.*

(c) *There is a canonical spectral sequence*

$$E_2^{pq} = \pi^p \mathscr{H}^q(G, M) \Rightarrow H^{p+q}(G, M).$$

where $\mathscr{H}^q(G, M)$ is the cosimplicial abelian group $n \mapsto H^q(G_n, M)$ and π^p is its "cohomotopy", i.e. homology with respect to the differential $\delta = \Sigma(-1)^i \delta_i$.

(d) *There are canonical isomorphisms*

$$H^0(G, M) = M^{\pi_0 G}$$
$$H^1(G, M) = \mathrm{Hom}_{(gps)}(\pi_0 G, M), \text{ if } \pi_0 G \text{ acts trivially on } M.$$

(e) *If $1 \to R \to G \to H \to 1$ is an exact sequence of simplicial groups, then there is a Serre spectral sequence*

$$E_2^{pq} = H^p(H, H^q(R, M)) \Rightarrow H^{p+q}(G, M).$$

(f) *$H^*(G, M)$ is a cohomological functor of M.*

Proof: All of these except (c) follow from the properties of the classifying space functor $\bar{W}$. Thus (e) is the Serre spectral sequence for the fibration $\bar{W}R \to \bar{W}G \to \bar{W}H$ (see [3], Appendix II), and the second half of (d) follows from POINCARE's theorem and the universal coefficient theorem. (b) is trivial because $\bar{W}f$ and $\bar{W}g$ are homotopic, and (a) follows from the fact that $\bar{W}f$ is a weak equivalence and hence a homotopy equivalence since $\bar{W}G$ and $\bar{W}H$ satisfy the extension condition.

To prove (c) let $\mathscr{W}(G)$ be the bisimplicial group given by $\mathscr{W}(G)_{pq}=W(G_p)_q$, where $W(\pi)$ is the classifying simplicial set of a group π. As $W(\pi)$ has trivial homotopy, $\mathscr{W}(G)$ has trivial vertical homotopy, so by the spectral sequence of a bisimplicial group [6], $\Delta\mathscr{W}(G)$ has trivial homotopy, where $[\Delta\mathscr{W}(G)]_n=\mathscr{W}(G)_{nn}$. Now π is contained in $W(\pi)$ as a simplicial subgroup, so G is contained in $\Delta\mathscr{W}(G)$ as a simplicial subgroup. Therefore $\Delta\mathscr{W}(G)$ is a principal contractible simplicial G set and hence is homotopy equivalent as a simplicial G set to $W(G)$. Thus

$$\pi^n \operatorname{Map}_G(\Delta\mathscr{W}(G), M) = \pi^n \operatorname{Map}_G(W(G), M) = H^n(G, M),$$

where Map_G denotes the set of morphisms in the category of G sets. One of the spectral sequences of the bi-cosimplicial abelian group $\operatorname{Map}_G(\mathscr{W}(G), M)^{pq} = \operatorname{Map}_{G_p}(W(G_p)_q, M)$ is

$$E_2^{pq} = \pi_h^p \pi_v^q \operatorname{Map}_G(\mathscr{W}(G), M) \Rightarrow \pi^{p+q}(\Delta \operatorname{Map}_G(\mathscr{W}(G), M))$$

which one easily sees is the desired spectral sequence (c). This completes the proof of the proposition.

REMARK: Corresponding properties for homology were derived in [5], Ch. II, § 6 as consequences of Kunneth spectral sequences pertaining to the derived tensor product $X\overset{L}{\otimes}_R Y$ of a left simplicial module X and a right simplicial module Y over a simplicial ring R. In an analogous way one may derive the above properties of cohomology from general spectral sequences pertaining to a derived Hom functor $\mathbf{R}\operatorname{Hom}_R(X, Y)$ where X is a left simplicial module and Y is a left *co*simplicial module over the simplicial ring R.

PROPOSITION 2.2: *Let* $\mathbf{G}, \mathbf{H}$ *be simplicial profinite groups and let* M *be a* $\pi_0\mathbf{G}$ *module.*

(b) *If f and g are homotopic maps of simplicial profinite groups from* $\mathbf{H}$ *to* $\mathbf{G}$ *then* $H^*(f, M)=\mathbf{H}^*(g, M)$.

(c) *There is a canonical spectral sequence*

$$E_2^{pq} = \pi^p \mathscr{H}^q(\mathbf{G}, M) \Rightarrow H^{p+q}(\mathbf{G}, M).$$

(d) *There are canonical isomorphisms*

$$H^0(\mathbf{G}, M) = M^{\pi_0 \mathbf{G}}$$
$$H^1(\mathbf{G}, M) = \operatorname{Homcont}(\pi_0 \mathbf{G}, M)$$

if $\pi_0\mathbf{G}$ *acts trivially on* M, *where* Homcont *is the set of homomorphisms which are continuous for the topology on* $\pi_0\mathbf{G}$ *and the discrete topology on* M.

(e) *If* $1 \to \mathbf{R} \to \mathbf{G} \to \mathbf{H} \to 1$ *is an exact sequence of simplicial profinite groups then there is a Serre spectral sequence*

$$E_2^{pq} = H^p(\mathbf{H}, H^q(\mathbf{R}, M)) \Rightarrow H^{p+q}(\mathbf{G}, M).$$

(f) $H^q(\mathbf{G}, M)$ *is a cohomological functor of* M.

Proof: (b). Let $\mathbf{G} \mapsto \mathbf{G}^{\Delta(1)}$ be the path functor on the category of simplicial profinite groups so that $\mathcal{H}om(\mathbf{H}, \mathbf{G}^{\Delta(1)}) = \mathcal{H}om(\mathbf{H}, \mathbf{G})^{\Delta(1)}$, where $\mathcal{H}om$ is the function complex of maps for the category of simplicial profinite groups ([5], Ch. II, 1.3). Thus the homotopy from f to g is represented by a map $h: \mathbf{H} \to \mathbf{G}^{\Delta(1)}$. The object $\mathbf{G}^{\Delta(1)}$ exists and is the usual function complex of maps from $\Delta(1)$ to $\mathbf{G}$ in the category of simplicial sets endowed with the structure of a profinite simplicial group induced from that of $\mathbf{G}$ (see [5], II, § 1, prop. 2 and cor.). Consequently if $\mathbf{G}$ is a simplicial finite group so is $\mathbf{G}^{\Delta(1)}$. If $\mathbf{V}$ is an open normal simplicial subgroup of $\mathbf{G}$, this means that $h^{-1}(\operatorname{Ker} \mathbf{G}^{\Delta(1)} \to (\mathbf{G}/\mathbf{V})^{\Delta(1)})$ is an open normal simplicial subgroup of $\mathbf{H}$. Hence if $\mathbf{U}$ is a smaller open normal simplicial subgroup of $\mathbf{H}$, the maps $f_{\mathbf{U},\mathbf{V}}$ and $g_{\mathbf{U},\mathbf{V}}$ from $\mathbf{H}/\mathbf{U}$ to $\mathbf{G}/\mathbf{V}$ induced by f and g are homotopic, so $H^*(f_{\mathbf{U},\mathbf{V}}, M^{\pi_0 \mathbf{V}}) = H^*(g_{\mathbf{U},\mathbf{V}}, M^{\pi_0 \mathbf{V}})$. Taking the direct limit as $\mathbf{U}$ and $\mathbf{V}$ run over all such open normal simplicial subgroups we see that $H^*(f, M) = H^*(g, M)$, which proves (b).

LEMMA 2.3: *If* $\mathbf{G}$ *is a simplicial profinite group, then for each* n, $\mathbf{G}_n = \varprojlim (\mathbf{G}/\mathbf{U})_n$ *where* $\mathbf{U}$ *runs over the directed set of open normal simplicial subgroups of* $\mathbf{G}$.

Given $\mathbf{V}$ open and normal in $\mathbf{G}_n$, set $\mathbf{U}_k = \cap (\varphi^*)^{-1}\mathbf{V}$ where φ runs over the finite set of monotone maps from $[n]$ to $[k]$. Clearly $\mathbf{U}$ is an open normal simplicial subgroup of $\mathbf{G}$ with $\mathbf{U}_n \subset \mathbf{V}$, so the lemma follows.

(c) follows by passage to the inductive limit in the spectral sequences 2.1 (c) for the simplicial groups $\mathbf{G}/\mathbf{U}$ using 2.3.

(d) follows from 2.1 (d) by passage to the limit.

LEMMA 2.4: *Let* $\mathbf{G}_i$ $i \in I$ *be an inverse system of simplicial groups and let* M_i $i \in I$ *be a directed system of abelian groups indexed by the same directed set* I. *Suppose each* M_i *has the structure of a* $\pi_0 \mathbf{G}_i$ *module such that for* $i \leqslant j$ *the map* $M_i \to M_j$ *is a homomorphism of* $\pi_0 \mathbf{G}_j$ *modules. Then*

$$\varinjlim H^q(\mathbf{G}_i, M_i) \simeq H^q(\varprojlim \mathbf{G}_i, \varinjlim M_i).$$

This may be reduced to the case of constant simplicial groups by 2.2 (c) where it is easy.

(e) As $\mathbf{U}$ runs over the directed set of open normal simplicial subgroups of $\mathbf{G}$, $\mathbf{U}_n$ runs over a neighborhood basis of e in $\mathbf{G}_n$ by the lemma. Hence if i and f are the maps $\mathbf{R} \to \mathbf{G}$ and $\mathbf{G} \to \mathbf{H}$, $i^{-1}\mathbf{U}_n$ and $f\,\mathbf{U}_n$ form a basis for e in $\mathbf{R}_n$ and $\mathbf{H}_n$ respectively.

Thus $\mathbf{R}=\varprojlim \mathbf{R}/i^{-1}\mathbf{U}$ and $\mathbf{H}=\varprojlim \mathbf{H}/f\mathbf{U}$, so using 2.4 we may obtain (e) by passing to the limit in the spectral sequences 2.1 (e) associated to the exact sequences $1\to \mathbf{R}/i^{-1}\mathbf{U}\to\mathbf{G}/\mathbf{U}\to\mathbf{H}/f\,\mathbf{U}\to 1$ and the module M^{π_0} , where $\mathbf{U}$ runs over the open normal simplicial subgroups of $\mathbf{G}$. This concludes the proof of 2.2.

By the method of SERRE [9] we may now use the Serre spectral sequence to prove

WHITEHEAD THEOREM 2.5: *Let $f\colon\mathbf{G}\to\mathbf{H}$ be a map of simplicial profinite groups and let n be $\geqslant 1$. The following conditions are equivalent:*

(i) *$\pi_q f$ is an isomorphism for $q<n$ and a surjection for $q=n$.*

(ii) *$\pi_0 f$ is an isomorphism and for every $\pi_0\mathbf{H}$ module M $H^q(f, M)$ is an isomorphism for $q\leqslant n$ and an injection for $q=n+1$.*

(ii)′ *same as* (ii) *but where M is any irreducible $\pi_0\mathbf{H}$ module* (such an M is necessarily finite dimesional over $\mathbf{Z}/p$ for some prime p).

COROLLARY 2.6: *Let $\mathbf{R}$ be a simplicial profinite group and let $n\geqslant 1$. Then $\pi_q\mathbf{R}=0$ for $q<n$ iff $\pi_0\mathbf{R}=0$ and $H^q(\mathbf{R}, \mathbf{Z}/p)=0$* for *$0<q<n$ and all primes p.*

Proof: Equivalence of (ii) and (ii)′: (ii)$\Rightarrow$(ii)′ is trivial. Assume (ii)′. By the five lemma, the family $\mathscr{C}$ of $\pi_0\mathbf{H}$ modules M for which $H^q(f, M)$ is an isomorphism for $q\leqslant n$ and a surjection for $q=n+1$ has the property that if $0\to M'\to M\to M''\to 0$ is an exact sequence, then $M', M''\in\mathscr{C}\Rightarrow M\in\mathscr{C}$ and $M, M''\in\mathscr{C}\Rightarrow M'\in\mathscr{C}$. Also $\mathscr{C}$ is closed under filtered inductive limits. Any finite $\pi_0\mathbf{H}$ module has a composition series hence is in $\mathscr{C}$, so any torsion $\pi_0\mathbf{H}$ module is in $\mathscr{C}$. If M is a vector space over $\mathbf{Q}$, then $M\in\mathbf{C}$ because then $H^q(\mathbf{G}, M)=0$ for $q<0$, as one sees by using the spectral sequence (2.2c) to reduce to the case of a single profinite group. If M is torsion-free there is an exact sequence $0\to M\to M\otimes\mathbf{Q}\to M\otimes(\mathbf{Q}/\mathbf{Z})\to 0$ where the second two belong to $\mathscr{C}$, hence $M\in\mathscr{C}$. Finally if M_t is the torsion subgroup of M, the exact sequence $0\to M_t\to M\to M/M_t\to 0$ together with what has been proved shows that $M\in\mathscr{C}$, so (ii) is proved.

Reduction of the theorem to the corollary: Factor the map f in the standard way $\mathbf{G}\xrightarrow{i}\mathbf{G}\times_{\mathbf{H}}\mathbf{H}^{\Delta(1)}\xrightarrow{p}\mathbf{H}$ dual to the mapping cylinder construction. i is a homotopy equivalence so induces an isomorphism on π_* and H^* (2.2(b)), while p is surjective since it's a fibration and $\pi_0 f$ is surjective. Thus we may assume f is surjective.

Let $\mathbf{R}=\operatorname{Ker} f$ and consider the Serre spectral sequence

$$E_2^{pq} = H^p(\mathbf{H}, H^q(\mathbf{R}, M)) \Rightarrow H^{p+q}(\mathbf{G}, M). \tag{2.7}$$

If (i) holds, then $\pi_q\mathbf{R}=0$ for $q<n$. If we take M to be an irreducible $\pi_0\mathbf{H}$ module, then $M\simeq(\mathbf{Z}/p)^k$ as $\pi_0\mathbf{R}$ modules for some k, so by the corollary $H^q(\mathbf{R}, M)=M$ for $q=0$ and 0 for $0<q<n$; thus the spectral sequence 2.7 yields (ii)′.

Now suppose (ii) holds and let s be the greatest integer $0<s<n$ such that $H^q(\mathbf{R}, A)=0$ for $0<q<s$ and all finite abelian groups A. Then from 2.7 we obtain an exact

sequence

$$H^s(\mathbf{H}, M) \underset{\simeq}{\to} H^s(\mathbf{G}, M) \to H^0(\mathbf{H}, H^s(\mathbf{R}, M)) \to H^{s+1}(\mathbf{H}, M) \underset{\subsetneq}{\to} H^{s+1}(\mathbf{G}, M)$$

which shows that $H^0(\mathbf{H}, H^s(\mathbf{R}, M))=0$ for all $\pi_0\mathbf{H}$ modules M. Given an abelian group A, let M be the $\pi_0\mathbf{H}$ module induced from A regarded as a module over the identity subgroup. Thus $M=\varinjlim_{\mathbf{U}}$ {set maps: $\pi_0\mathbf{H}/\mathbf{U}\to A$} where $\pi_0\mathbf{H}$ acts on $\pi_0\mathbf{H}/\mathbf{U}$" by right translation, so

$$\begin{aligned} H^0(\mathbf{H}, H^s(\mathbf{R}, M)) &= \varinjlim_{\mathbf{U}} H^0(\mathbf{H}, \{\text{set maps: } \pi_0\mathbf{H}/\mathbf{U} \to H^s(\mathbf{R}, A)\}) \\ &= \varinjlim_{\mathbf{U}} \{f : \pi_0\mathbf{H}/\mathbf{U} \to H^0(\mathbf{R}, A) \mid f \text{ set map such that} \\ &\qquad f(x\gamma) = \gamma^{-1} f(x)\} \quad \text{for all} \quad \gamma \in \pi_0\mathbf{H}\} \\ &= H^s(\mathbf{R}, A) \end{aligned}$$

Hence $H^q(\mathbf{R}, A)=0$ for $0<q\leqslant s$ and all abelian groups A, which shows that $s=n$. In particular by 2.2 (d)

$$H^1(\mathbf{R}, A) = \text{Homcont}(\pi_0\mathbf{R}, A) = 0 \tag{2.8}$$

hence $\pi_0\mathbf{R}=0$ because $\pi_0 f$ is an isomorphism and so $\pi_0\mathbf{R}=\text{Coker } \pi_1 f$ is abelian. Thus by the corollary $\pi_q\mathbf{R}=0$ for $q<n$ and so we obtain (i).

Proof of the corollary: As $\pi_0\mathbf{R}=0$, there is a canonical exact sequence (see [6] for formulas)

$$1 \to \Omega\mathbf{R} \to E\mathbf{R} \to \mathbf{R} \to 1$$

where $E\mathbf{R}$ is contractible. This gives rise to a spectral sequence

$$E_2^{pq} = H^p(\mathbf{R}, H^q(\Omega\mathbf{R}, A)) \Rightarrow H^{p+q}(1, A).$$

Using this, the fact that $\pi_0\,\Omega\mathbf{R}=\pi_1\mathbf{R}$ is abelian, and the formula (2.8) one establishes the corollary by induction on n.

For simplicial pro-p groups the Whitehead theorem may be strengthened as follows:

COROLLARY 2.9: *Let f:* $\mathbf{G}\to\mathbf{H}$ *be a map of simplicial pro-p groups and let* $n\geqslant 0$. *The following conditions are equivalent:*

(i) $\pi_q f$ *is an isomorphism for* $q<n$ *and a surjection for* $q=n$.

(ii) $H^q(f, \mathbf{Z}/p)$ *is an isomorphism for* $q\leqslant n$ *and an injection for* $q=n+1$.

Proof: Since $H^1(\mathbf{G}, \mathbf{Z}/p)=\text{Homcont}(\pi_0\mathbf{G}, \mathbf{Z}/p)=H^1(\pi_0\mathbf{G}, \mathbf{Z}/p)$, $H^1(f, \mathbf{Z}/p)$ injective implies that $\pi_0 f$ is surjective (SERRE [8], p. 1–35, prop. 23), which proves the corollary when $n=0$. If $n\geqslant 1$ then we can apply the Whitehead theorem once we know that (ii) implies that $\pi_0 f$ is an isomorphism. We know $\pi_0 f$ is surjective, so replacing f by a surjection as in the proof of the theorem we may assume f surjective. If $\mathbf{R} = \operatorname{Ker} f$, then there is a five term exact sequence

$$H^1(\mathbf{H}, \mathbf{Z}/p) \underset{\simeq}{\rightarrow} H^1(\mathbf{G}, \mathbf{Z}/p) \rightarrow H^0(\mathbf{H}, H^1(\mathbf{R}, \mathbf{Z}/p)) \rightarrow H^2(\mathbf{H}, \mathbf{Z}/p) \underset{\hookrightarrow}{\rightarrow} H^2(\mathbf{G}, \mathbf{Z}/p).$$

so $H^0(\mathbf{H}, H^1(\mathbf{R}, \mathbf{Z}/p)) = 0$. But the action of a pro-p group on a non-zero p primary module has non-zero invariants, so $H^1(\mathbf{R}, \mathbf{Z}/p) = 0$ and $\pi_0 \mathbf{R} = 0$. Therefore $\pi_0 f$ is an isomorphism, and the corollary follows from 2.5.

§ 3. Completion and cohomology

In this section we shall abbreviate $H^*(G, \mathbf{Z}/p)$ to $H^*(G)$, use "good" instead of "p-good", and call a subgroup of a group open if it contains a normal subgroup of index a power of p.

PROPOSITION 3.1: *If G is a simplicial group such that G_n is good for all n, then G is good.*

Proof: There is a map of spectral sequences 2.1 and 2.2 (c)

$$\begin{array}{ccc} \pi^p \mathscr{H}^q(\hat{G}) & \Rightarrow & H^{p+q}(\hat{G}) \\ \downarrow & & \downarrow \\ \pi^p \mathscr{H}^q(G) & \Rightarrow & H^{p+q}(G) \end{array}$$

which is an isomorphism on the E_2 terms, hence also on the abutment.

PROPOSITION 3.2: *If $G \rightarrow H$ is a weak equivalence of simplicial groups and G, H are good, then $\hat{G} \rightarrow \hat{H}$ is a weak equivalence.*

Proof: In the square

$$\begin{array}{ccc} H^q(\hat{H}) & \rightarrow & H^q(H) \\ \downarrow & & \downarrow \\ H^q(\hat{G}) & \rightarrow & H^q(G) \end{array}$$

the horizontal arrows are isomorphisms since G and H are good, and the left vertical arrow is an isomorphism by 2.1 (a). So the proposition follows from Whitehead theorem 2.9.

PROPOSITION 3.3: *Let $1 \rightarrow R \rightarrow G \rightarrow H \rightarrow 1$ be an exact sequence of groups such that*

(i) *H and R are good*

(ii) *$H^q(R)$ is finite for all q*

(iii) *H acts unipotently on $H^1(R)$.*

Then (a) *$1 \rightarrow \hat{R} \rightarrow \hat{G} \rightarrow \hat{H} \rightarrow 1$ is exact*

(b) *G is good.*

Proof: (b) follows from (a) since then there is a map of spectral sequences

$$\begin{array}{ccc} H^p(\hat{H}, H^q(\hat{R})) & \Rightarrow & H^{p+q}(\hat{G}) \\ \downarrow & & \downarrow \\ H^p(H, H^q(R)) & \Rightarrow & H^{p+q}(G) \end{array}$$

which will be an isomorphism on E_2 by hypotheses (i) and (ii).

To prove (a) we must show that $\hat{R}\to\hat{G}$ is injective, or equivalently that if V is an open subgroup of R, then $V\supset U\cap R$ for some open subgroup U of G. As $H^1(R)=\mathrm{Hom}(\mathrm{gr}_1^p R, \mathbf{Z}/p)$ is finite, $\mathrm{gr}_1^p R$ is finite, hence $\mathrm{gr}_q^p R$, which is a quotient of $L_q^p(\mathrm{gr}_1^p R)$ (1.9) is finite, so $R/\Gamma_r^p R$ is a p-group. As $V\supset {}_r^p\Gamma R$ for some r, we may by shrinking V assume it is invariant under the conjugation action of G. Notice also that G acts unipotently on $\mathrm{gr}_1^p R$ by hypothesis (iii) and hence acts unipotently on $\mathrm{gr}^p R$. Consequently there is a sequence of subgroups $R=V_0\supset V_1\supset\cdots\supset V_n=V$ normal in G such that $V_i/V_{i+1}\simeq\mathbf{Z}/p$ with trivial G action. We are going to construct inductively a sequence $G=U_0\supset U_1\supset\ldots$ of open subgroups of G such that $U_i\cap R=V_i$. For $i=1$ the extension

$$1\to V_0/V_1\to G/V_1\to H\to 1$$

is classified by an element $a\in H^2(H)$. As H is good $a\in\mathrm{Im}\{H^2(H/H_1)\to H^2(H)\}$ for some H_1 open and normal in H. In other words there is a diagram

$$\begin{array}{ccccccccc} 1 & \to & V_0/V_1 & \to & G/V_1 & \to & H & \to & 1 \\ & & \wr\! & & \downarrow & * & \downarrow & & \\ 1 & \to & \mathbf{Z}/p & \to & Q & \to & H/H_1 & \to & 1 \end{array}$$

where the square $*$ is cartesian. Hence over H_1 there is a section homomorphism s of $G/V_1\to H$, so $sH_1\subset G/V_1$ is an open subgroup with $sH_1\cap(V_0/V_1)=1$. If U_1 is the inverse image of sH_1 under the map $G\to G/V_1$, then U_1 is open in G and $U_1\cap R=V_1$. This takes care of the case $i=1$. Having found U_i one applies the same argument to the extension

$$1\to V_i/V_{i+1}\to U_i/V_{i+1}\to U_i/V_i\to 1$$

which is possible since $U_i/V_i\subsetneq H$ is an open subgroup, hence good by

LEMMA 3.4: *An open subgroup of a good group is good.*

Proof: If H_1 is open in H, then $\hat{H}_1$ clearly maps injectively into $\hat{H}$. Let $i: H_1\to H$ be the inclusion and let $i_*(\mathbf{Z}/p)$ be the H module induces by the trivial H_1 module $\mathbf{Z}/p$. Then $(\hat{\imath})_*(\mathbf{Z}/p)=i_*(\mathbf{Z}/p)$ as $\hat{H}$ modules so we have a square

$$\begin{array}{ccc} H^q(\hat{H}, i_*(\mathbf{Z}/p)) & \simeq & H^q(\hat{H}_1) \\ \downarrow & & \downarrow \\ H^q(H, i_*(\mathbf{Z}/p)) & \simeq & H^q(H_1) \end{array}$$

where the first vertical arrow is an isomorphism since H is good. This proves the lemma and completes the proof of 3.3.

COROLLARY 3.5: *Any finitely generated nilpotent group is good. Any free group is good.*

Proof: Assume G finitely generated and nilpotent. $\mathrm{gr}\, G = \oplus \Gamma_r G/\Gamma_{r+1} G$ is a Lie algebra over $\mathbf{Z}$ generated by $\mathrm{gr}_1 G$ which is a finitely generated abellian group. Hence $\mathrm{gr}_q G$ is a finitely generated abelian group for each q, so we may refine the lower central series of G to a sequence of normal subgroups $G = G_0 \supset \cdots \supset G_n = 1$ where G_i/G_{i+1} is either $\mathbf{Z}$ or $\mathbf{Z}/l$, l a prime number, with trivial G action. One may then use the proposition inductively to prove G/G_i is good once one knows that A is good and $H^q(A)$ is finite for each q where $A = \mathbf{Z}$ or $\mathbf{Z}/l$. In the latter case $\bar{W}(A)$ is a simplicial finite set hence $H^q(A)$ is finite. Also A is good if $l = p$ since $A = \hat{A}$, and A is good if $l \neq p$ because then $H^q(A) = 0$ for $q > 0$. If $A = \mathbf{Z}$ then A is free, and free groups are good because $H^q(\hat{G}) = H^q(G)$ for $q \leqslant 1$ for any group G and because $H^q(\hat{G}) = H^q(G) = 0$ for $q \geqslant 2$ if G is free. This completes the proof of 3.5.

THEOREM 3.6: *Let $1 \to R \to G \to H \to 1$ be an exact sequence of good simplicial groups. Suppose that for each q $H^q(R)$ is finite and that $\pi_0 H$ acts unipotently on it. Then $\hat{R} \to \mathrm{Ker}\{\hat{G} \to \hat{H}\}$ is a weak equivalence.*

LEMMA 3.7: *Let $\mathbf{H}$ be a simplicial pro-p group and let $u: A \to B$ be a map of p-primary $\pi_0 \mathbf{H}$ modules. Then*

(i) *$H^0(\mathbf{H}, u)$ injective $\Rightarrow$ u injective*

(ii) *$H^0(\mathbf{H}, u)$ bijective and $H^1(\mathbf{H}, u)$ injective $\Rightarrow u$ is an isomorphism.*

Proof: (i). If K is the kernel of u, then there is an exact sequence $0 \to H^0(\mathbf{H}, K) \to H^0(\mathbf{H}, A) \hookrightarrow H^0(\mathbf{H}, B)$ so $H^0(\mathbf{H}, K) = 0$ and so $K = 0$ since it is p-primary. (ii) We have that u is injective by (i), so letting $C = \mathrm{Coker}\, u$, there is an exact sequence $H^0(\mathbf{H}, A) \xrightarrow{\sim} H^0(\mathbf{H}, B) \to H^0(\mathbf{H}, C) \to H^1(\mathbf{H}, A) \hookrightarrow H^1(\mathbf{H}, B)$ whence by the same argument $C = 0$. This proves the lemma.

The following is the core of ZEEMAN's comparison theorems for spectral sequences [10].

LEMMA 3.8: *Let $f: \{E_2^{pq} \Rightarrow H^{p+q}\} \to \{\bar{E}_2^{pq} \Rightarrow \bar{H}^{p+q}\}$ be a map of first quadrant spectral sequences of cohomological type. If $H^n(f)$ is an isomorphism for all n and $E_2^{pq}(f)$ is an isomorphism for $q < s$, then*

(a) *$E_2^{0s}(f)$ is an isomorphism*

(b) *$E_2^{1s}(f)$ is injective.*

Proof: Let $\mathbf{Z}_r^{pq}$, $B_r^{pq} \subset E_2^{pq}$ be defined recursively by

$$Z_r^{pq} = \mathrm{Ker}\{Z_{r-1}^{pq} \twoheadrightarrow E_{r-1}^{pq} \xrightarrow{d_{r-1}} E_{r-1}^{p+r-1,q-r+2}\} \qquad Z_2^{pq} = E_2^{pq}$$

$$B_r^{pq}/B_{r-1}^{pq} = \mathrm{Im}\{E_{r-1}^{p-r+1,q+r-2} \xrightarrow{d_{r-1}} E_{r-1}^{pq}\} \qquad B_2^{pq} = 0$$

so that $E_r^{pq} \simeq \mathbf{Z}_r^{pq}/B_r^{pq}$. By induction on r one simultaneously establishes that

$$q + r - 1 \leqslant s \Rightarrow Z_r^{pq},\ B_r^{pq},\ E_r^{pq} \text{ are isomorphisms} \tag{1}$$

$$q < s \Rightarrow Z_r^{pq},\ E_r^{pq} \text{ are surjective}, \tag{2}$$

where we have abbreviated $E_2^{pq}(f)$ to E_2^{pq}, etc. Using induction on r one obtains

$$p + q \leqslant s \Rightarrow Z_r^{pq},\ B_r^{pq},\ E_r^{pq} \text{ are isomorphisms}. \tag{3}$$

Let F_pH^n be the filtration on the abutment, so that $E_\infty^{pq} = F_pH^{p+q}/F_{p+1}H^{p+q}$. Using (2) and induction on q one shows that

$$E_\infty^{pq},\ F_pH^{p+q} \text{ are isomorphisms for } q < s \text{ and injections for } q = s. \tag{4}$$

If $p<r$ there is an exact sequence

$$0 \to E_{r+1}^{pq} \to E_r^{pq} \xrightarrow{d_r} E_r^{p+r,\,q-r+1}$$

which by descending induction on r using (1), (2) and (4) may be used to prove

$$\text{If } p < r, \text{ then } E_r^{pq} \text{ is an isomorphisms for } q < s \text{ and an injection for } q = s. \tag{5}$$

In particular taking $p=1$, $r=2$ we have established part (b) of the lemma.

Note that $E_\infty^{0\,s} = H^s/F_1H^s$ is injective by (4) and surjective since H^s is an isomorphism. In virtue of the exact sequences

$$\begin{array}{l}
0 \to E_{r+1}^{0\,s} \to E_r^{0\,s} \xrightarrow{d_r} B_{r+1}^{r,\,s-r+1}/B_r^{r,\,s-r+1} \to 0 \\
0 \to B_{r+1}^{r,\,s-r+1}/B_r^{r,\,s-r+1} \to Z_{r+1}^{r,\,s-r+1}/B_r^{r,\,s-r+1} \to \underset{\simeq \text{ by } (5)}{E_{r+1}^{r,\,s-r+1}} \to 0 \\
0 \to Z_{r+1}^{r,\,s-r+1}/B_r^{r,\,s-r+1} \to \underset{\simeq \text{ by } (1)}{E_r^{r,\,s-r+1}} \xrightarrow{d_r} \underset{\simeq \text{ by } (1)}{E_r^{2r,\,s-2r+2}}
\end{array}$$

one sees by descending induction on r that

$$E_r^{0\,s} \text{ is an isomorphism for all } r \tag{6}$$

which proves part (a) and hence completes the proof of the lemma.

Proof of 3.6: Let $\mathbf{K} = \operatorname{Ker}\{\hat{G} \to \hat{H}\}$. If $\mathbf{U}$ is an open normal simplicial subgroup of $\mathbf{G}$, then the natural map of the exact sequence $1 \to R \to G \to H \to 1$ into $1 \to R/R \cap U \to G/U \to G/RU \to 1$ gives rise to a corresponding map of Serre spectral sequences, and hence on passage to the limit over U a map f of spectral sequences

$$\begin{array}{ccc}
H^p(\hat{H}, H^q(\mathbf{K})) & \Rightarrow & H^{p+q}(\hat{G}) \\
\downarrow & & \downarrow \\
H^p(H, H^q(R)) & \Rightarrow & H^{p+q}(G).
\end{array} \tag{1}$$

Assume that the canonical map $H^q(\mathbf{K}) \to H^q(R)$ is an isomorphism for $q<s$. As H is good $E_2^{pq}(f)$ is an isomorphism for $q<s$ and as G is good f induces an isomorphism on the abutment. Therefore by 3.8

$$\begin{array}{l}
H^0(\hat{H}, H^s(\mathbf{K})) \xrightarrow{\sim} H^0(H, H^s(R)) \\
H^1(\hat{H}, H^s(\mathbf{K})) \hookrightarrow H^1(H, H^s(R))
\end{array}$$

Now since H is good and $\pi_0 H$ acts unipotently on $H^s(R) = \operatorname{Hom}(H_s(R, \mathbf{Z}/p), \mathbf{Z}/p)$, H may be replaced by $\hat{H}$ in the right side of these maps, so by 3.7 $H^s(\mathbf{K}) \to H^s(R)$ is an isomorphism. Thus in the triangle

$$\begin{tikzcd}[column sep=small] H^q(\mathbf{K}) \arrow[rr] \arrow[dr] & & H^q(\hat{R}) \arrow[dl] \\ & H^q(R) & \end{tikzcd}$$

the left vertical arrow is an isomorphism by induction on q and the right one is an isomorphism since R is good. Thus $\hat{R} \to \mathbf{K}$ induces an isomorphism on cohomology so is a weak equivalence by 2.9. Q.E.D.

Proof of main theorem 1.5: We show that $\pi_q(G)^\wedge \xrightarrow{\sim} \pi_q(\hat{G})$ for all simplicial groups G satisfying the hypotheses by induction on q. For $q=0$ it is clear since both groups represent maps of G into a constant simplicial pro-p-group so we assume $q>0$. By 3.2 we may replace G by a free simplicial group. Consider the exact sequences

$$\begin{array}{ccccccccc} 1 & \to & \tilde{G} & \to & G & \to & \pi_0 G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \hat{G}^{\sim} & \to & \hat{G} & \to & \pi_0 \hat{G} & \to & 1\,. \end{array}$$

We apply 3.6 to the upper exact sequence. $\tilde{G}$ is a subgroup of a free simplicial group, hence is free; thus $\tilde{G}$, G, $\pi_0 G$ are all good. By hypotheses (iii) and iv) on G, $H^q(\tilde{G}) = \operatorname{Hom}(H_q(\tilde{G}, \mathbf{Z}/p), \mathbf{Z}/p)$ is finite and $\pi_0 G$ acts unipotently on it for each q. Thus by 3.6 $\tilde{G}^\wedge \to \hat{G}^{\sim}$ is a weak equivalence.

As $\pi_0 \tilde{G} = 0$ there is a surjection $F \to \tilde{G}$, where F is a free contractible simplicial group, whose kernel R is of the weak homotopy type of ΩG. Consider the exact sequences

$$\begin{array}{ccccccccc} 1 & \to & R & \to & F & \to & \tilde{G} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbf{K} & \to & F^\wedge & \to & \tilde{G}^\wedge & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbf{K}' & \to & F^\wedge & \to & \hat{G}^{\sim} & \to & 1 \end{array}$$

and apply 3.6 to the upper exact sequence. This is legitimate because $H^*(R) = H^*(\Omega \tilde{G})$ is finite by a well-known argument of Serre, because $\pi_0 \tilde{G} = 0$, and because R, F and $\tilde{G}$ are all free. Thus $\hat{R} \to \mathbf{K}$ and hence $\hat{R} \to \mathbf{K}'$ is a weak equivalence and we obtain the diagram

$$\begin{tikzcd} (\pi_q G)^\wedge \arrow[dd] & (\pi_q \tilde{G})^\wedge \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[dd] & (\pi_{q-1} R)^\wedge \arrow[d] \\ & & \pi_{q-1} \hat{R} \arrow[d, "\wr"] \\ \pi_q \hat{G} & \pi_q(\hat{G}^{\sim}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \pi_{q-1} \mathbf{K}'\,. \end{tikzcd}$$

But R satisfies the hypotheses of the theorem. In effect $H_q(R, \mathbf{Z})$ and $H_q(\tilde{R}, \mathbf{Z})$ are finitely generated by SERRE, so $\pi_0 R = H_1(R, \mathbf{Z})$ is good, and also $\pi_0 R$ acts trivially on $H_*(\tilde{R}, \mathbf{Z})$. Thus by induction $(\pi_{q-1} R)^\wedge \simeq \pi_{q-1}\hat{R}$ and the theorem follows.

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