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Imbeddings of Simplicial Complexes

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1. Introduction

The main aim of the present note is to show that certain n-dimensional simplicial complexes which are not imbeddable into the $(2n)$-dimensional Euclidean space E^{2n} are *minimal* with respect to that property, in the following strong sense: Every proper subcomplex of one of those complexes is even geometrically imbeddable in E^{2n}. (A simplicial complex is *geometrically imbedded* in E^k provided each of its simplices is a geometric, rectilinear simplex.) This result adds credibility to the following conjecture, established for $n=1$ by Wagner [14] (see also Fáry [4] and Stojaković [13]):

Conjecture. If an n-dimensional simplicial complex is topologically imbeddable in E^{2n} then it is even geometrically imbeddable in E^{2n}.

It has recently been established by Weber [15] that the weaker conjecture dealing with piecewise-linear (instead of geometric) imbeddings is true.

We shall start (in Section 2) by extending the class of known examples of n-complexes (that is finite, n-dimensional, simplicial complexes) not imbeddable in E^{2n}. The only examples of such complexes we found in the literature (van Kampen [8], Flores [5, 6], Rosen [11], Wu [16]) are:

(i) The *complete n-complex* $\mathscr{C}^n(k)$ with k vertices, where $k \geqslant 2n+3$; clearly, only the case $k=2n+3$ is interesting.

(ii) The *join* $\mathscr{C}^0(3) \vee \mathscr{C}^0(3) \vee \cdots \vee \mathscr{C}^0(3)$ of $n+1$ triplets of points.

For $n=1$ those examples reduce to the well-known graphs of Kuratowski [10], which may be used to characterize non-planar graphs.

In Section 3 we shall show that each subcomplex of each of the n-complexes constructed in Section 2 is geometrically imbeddable in E^{2n}. This generalizes recent results of Zaks [17], who has established for some of the complexes of Section 2 the possibility of geometrically imbedding each of their subcomplexes in E^{2n}, while establishing for the other cases only the possibility of a piecewise-linear imbedding (see the more detailed comments in Section 4).

The last Section is devoted to some additional remarks and problems.

I am indebted to the referee for a number of helpful remarks and references.

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2. Some n-Complexes Not Imbeddable in E^{2n}

We shall denote finite simplicial complexes by script capitals $\mathscr{C}$, $\mathscr{K}$, etc.; their faces (simplices) will be indicated by capitals F, V, etc., and also by enumerating the vertices; for example, $F^3=(V_0, V_1, V_2, V_3)$. Superscripts will denote dimension; S^n is the n-sphere (the unit sphere in E^{n+1} if metric considerations are involved), and T^n indicates the n-simplex. For a complex K imbedded in some Euclidean space we denote by set K the set of points underlying the complex. (We find this symbol more convenient and more indicative than the more usual "absolute value" notation.)

If $F'=(V_0,\ldots, V_n)$ and $F''=(W_0,\ldots, W_m)$ are disjoint (abstract) simplices, we shall denote their *join*[1]) by $F'\vee F''=(V_0,\ldots, V_n, W_0,\ldots, W_m)$. For disjoint (abstract) simplicial complexes $\mathscr{K}'$ and $\mathscr{K}''$ the *join* $\mathscr{K}'\vee\mathscr{K}''$ is defined by $\mathscr{K}'\vee\mathscr{K}''=\{F'\vee F'' \mid F'\in\mathscr{K}', F''\in\mathscr{K}''\}$. Note that this coincides with the usual definition, since we include the empty set $\emptyset$ as face in each complex.

If $\mathscr{K}'$ and $\mathscr{K}''$ are topological simplicial complexes contained in skew affine subspaces of a Euclidean space, their join $\mathscr{K}'\vee\mathscr{K}''$ is also a topological simplicial complex. Its faces $F'\vee F''$ may be represented by

$$F' \vee F'' = \{\lambda x' + (1-\lambda)\, x'' \mid x'\in F', x''\in F'', 0\leqslant\lambda\leqslant 1\}.$$

If $\mathscr{K}'$ and $\mathscr{K}''$ are geometric simplicial complexes in skew affine subspaces of a Euclidean space, then $\mathscr{K}'\vee\mathscr{K}''$ is also a geometric complex, and the above representation simplifies to $F'\vee F''=\operatorname{conv}(F'\cup F'')$, where $\operatorname{conv} A$ denotes the convex hull of the set A. (As is well known, the assumption that $\mathscr{K}'$ and $\mathscr{K}''$ are contained in skew affine spaces is not essential; the only condition required is that the convex combinations used do not introduce any unwanted intersections, or degenerate simplices. We shall assume this condition fulfilled whenever we use the symbol $\vee$.)

If A' and A'' are topological spaces, the *join* $A'\vee A''$ is the space obtained from the Cartesian product $A'\times A''\times[0, 1]$ by identifying $(a', A'', 0)$ with a' for each $a'\in A'$, and similarly identifying $(A', a'', 1)$ with a'' for each $a''\in A''$. If $j: A'\times A''\times[0, 1]\to A'\vee A''$ is the identification map, then $A'\vee A''$ may be topologized by defining $N\subset A'\vee A''$ open if and only if $j^{-1}(N)$ is open in $A'\times A''\times[0, 1]$.

The connection between the two notions of join is given by the easily established fact:

For topological simplicial complexes $\mathscr{K}'$ and $\mathscr{K}''$, there is a natural homeomorphism between $\operatorname{set}(\mathscr{K}'\vee\mathscr{K}'')$ and $(\operatorname{set}\mathscr{K}')\vee(\operatorname{set}\mathscr{K}'')$.

If $B=\{b\}$ is a one-pointed set, then $A\vee B$ is for obvious reasons called the *pyramid* over A with apex b; we shall denote it by $A^+(b)$, or simply A^+ if no confusion is

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[1]) Because of lattice-theoretic connotations we prefer to indicate the join-operation by the symbol $\vee$ instead of the frequently used $*$.

likely to arise. ($A^+(b)$ is frequently called the "cone" over A with vertex b; we avoid this term since it is used with a different meaning in other fields.)

We note the well known and easily established facts:

(1) $T^n \vee T^m$ is homeomorphic to T^{n+m+1}.

(2) $S^n \vee S^m$ is homeomorphic to S^{n+m+1}.

A selfhomeomorphism π of a topological space A is called *antipodal* provided π is an involution (that is, $\pi^2(a)=a$ for each $a\in A$) and has no fixed points. The unit n-sphere S^n has a *natural antipodality* π defined by $\pi(a)=-a$. If A' and A'' are topological subspaces of a Euclidean space, with antipodalities π' and π'', then there is a natural antipodality $\pi=\pi'\vee\pi''$ on $A'\vee A''$ defined by

$$\pi(\lambda a' + (1-\lambda)\, a'') = \lambda\pi'(a') + (1-\lambda)\,\pi''(a'').$$

The homeomorphism mentioned in (2) above may be chosen in such a manner as to preserve the natural antipodalities of the spheres involved. Indeed, let

$$S^n = \{x=(x_1,\ldots,x_{n+m+2})\in E^{n+m+2} \mid \|x\|=1,\, x_{n+2}=\cdots=x_{n+m+2}=0\}$$
and
$$S^m = \{x=(x_1,\ldots,x_{n+m+2})\in E^{n+m+2} \mid \|x\|=1,\, x_1=\cdots=x_{n+1}=0\};$$

then the mapping which sends the point $\lambda x+(1-\lambda)\,y$ of $S^n\vee S^m$ (where $x\in S^n$, $y\in S^m$, $0\leqslant\lambda\leqslant 1$) onto the point $\lambda^{1/2}x+(1-\lambda)^{1/2}\,y$ of S^{n+m+1} has this property.

Let now $\mathscr{K}$ be a topological simplicial n-complex, and let $\mathscr{K}^*=\vartheta(\mathscr{K})$ be a complex isomorphic to $\mathscr{K}$ under an isomorphism ϑ such that $\mathscr{K}$ and $\mathscr{K}^*$ are contained in skew affine spaces. We define a simplicial $(2n+1)$-complex $\mathscr{K}^\vee$ as the subcomplex of $\mathscr{K}\vee\mathscr{K}^*$ consisting of all simplices $F\vee F^*$ (where $F\in\mathscr{K}$, $F^*\in\mathscr{K}^*$, and $F\cap\vartheta^{-1}(F^*)=\emptyset$) and their faces. If $K=\operatorname{set}\mathscr{K}$, we shall use the notation $K^\vee=\operatorname{set}(\mathscr{K}^\vee)$. The set $K^\vee$ has a natural antipodality π defined by

$$\pi(\lambda x_1 + (1-\lambda)\,\vartheta(x_2)) = (1-\lambda)\,x_2 + \lambda\vartheta(x_1),$$

where x_i belongs to an n-simplex F_i of $\mathscr{K}$ and $F_1\cap F_2=\emptyset$.

We have the following *lemma*:

(3) *If $\mathscr{K}_i$ is a simplicial n_i-complex, $i=1,\cdots,p$, then there is a natural homeomorphism φ, which preserves the natural antipodalities, between $(\mathscr{K}_1\vee\mathscr{K}_2\vee\cdots\vee\mathscr{K}_p)^\vee$ and $K_1^\vee\vee K_2^\vee\vee\cdots\vee K_p^\vee$.*

Proof. It is clearly enough to consider the case $p=2$. Then a typical point of $(\mathscr{K}_1\vee\mathscr{K}_2)^\vee$ is of the form

$$x = \lambda(\mu' x_1' + (1-\mu')\,x_2') + (1-\lambda)\,\vartheta(\mu'' x_1'' + (1-\mu'')\,x_2''),$$

where $0\leqslant\lambda, \mu', \mu''\leqslant 1$, $x_i'\in F_i'$, $x_i''\in F_i''$, $F_i', F_i''\in\mathscr{K}_i$, and $(F_1'\vee F_2')\cap(F_1''\vee F_2'')=\emptyset$, that is, $F_1'\cap F_1''=\emptyset$ and $F_2'\cap F_2''=\emptyset$. On the other hand, the typical point of $K_1^\vee\vee K_2^\vee$ is given by

$$y = \beta(\alpha_1 y_1' + (1-\alpha_1)\,\vartheta(y_1'')) + (1-\beta)(\alpha_2 y_2' + (1-\alpha_2)\,\vartheta(y_2'')),$$

where $0 \leqslant \alpha_1, \alpha_2, \beta \leqslant 1$, $y_i' \in F_i'$, $y_i'' \in F_i''$, $F_i', F_i'' \in \mathscr{K}_i$, and $F_i' \cap F_i'' = \emptyset$ for $i=1,2$. Assuming without loss of generality that ϑ is affine, x may be made to correspond to y by taking $x_i' = y_i'$, $x_i'' = y_i''$, and

$$\beta = \lambda\mu' + (1-\lambda)\,\mu'',$$

$$\alpha_1 = \frac{\lambda\mu'}{\lambda\mu' + (1-\lambda)\,\mu''} \qquad \alpha_2 = \frac{\lambda(1-\mu')}{\lambda(1-\mu') + (1-\lambda)(1-\mu'')}.$$

The continuity of the mapping and the preservation of antipodality by it are easily checked, and the proof of lemma (3) is completed.

As a corollary of (3) and (2) we have:

(4) If $\mathscr{K}_i$ is a complex such that $K_i^\vee$ is homeomorphic to the n_i-sphere S^{n_i}, then $(\mathscr{K}_1 \vee \mathscr{K}_2 \vee \cdots \vee \mathscr{K}_p)^\vee$ is homeomorphic to the $(p-1+\sum_{i=1}^{p} n_i)$-sphere $S^{n_1} \vee S^{n_2} \vee \vee \cdots \vee S^{n_p}$. Moreover, the homeomorphism may be assumed to preserve antipodes.

Let $\mathscr{K}$ be a topological simplicial n-complex; we construct a new set $\hat{K}$ as follows. $\hat{K}$ is a subset of $K^+ \times K^+$, and consists of those pairs (a, b) of points of K^+ which satisfy:

(i) at least one of a, b belongs to K;

(ii) there exist disjoint n-simplices F_a and F_b of such that $a \in F_a^+$, $b \in F_b^+$.

$\hat{K}$ is clearly a compact metric space; it has a natural antipodality making points (a, b) and (b, a) correspond to each other. One of the properties of $\hat{K}$ which is of special interest to us is:

(5) For each n-complex $\mathscr{K}$, the set $\hat{K}$ is homeomorphic to the set $K^\vee$ by a homeomorphism φ which preserves antipodality.

Indeed, denoting by v the apex of K^+, each point of $\hat{K}$ is uniquely expressible in the form $(\lambda a + (1-\lambda)\,v, \mu b + (1-\mu)\,v)$, where a and b belong to disjoint n-simplices of $\mathscr{K}$, $0 \leqslant \lambda, \mu \leqslant 1$, and $\max\{\lambda, \mu\} = 1$. We define

$$\varphi(\lambda a + (1-\lambda)\,v, \mu b + (1-\mu)\,v) = \begin{cases} (1-\tfrac{1}{2}\mu)\,a + \tfrac{1}{2}\mu\vartheta(b) \in K^\vee & \text{if } \lambda = 1 \\ \tfrac{1}{2}\lambda a + (1-\tfrac{1}{2}\lambda)\,\vartheta(b) \in K^\vee & \text{if } \mu = 1. \end{cases}$$

It is trivial to check that φ has all the desired properties.

We need one more definition. Let $\mathscr{K}$ be a topological simplicial complex and let $K = \operatorname{set} \mathscr{K}$. A mapping f of K^+ shall be called a *K-homeomorphism* provided the restriction of f to K is a homeomorphism (between K and $f(K)$). We shall say that $\mathscr{K}$ is *n-entangled* (or absolutely knotted in E^n) if and only if

$$f(K) \cap f(K^+ \backslash K) \neq \emptyset$$

for every K-homeomorphism f of K^+ into E^n.

If K is homeomorphic to a subset of E^{n-1} then $\mathscr{K}$ is clearly not n-entangled (since in this case K^+ is homeomorphic to a subset of E^n). Hence, if we succeed in proving that some complex is n-entangled, then this complex is certainly not imbeddable in E^{n-1}.

Let now $\mathscr{K}$ be an n-complex and let f be a K-homeomorphism of K^+ into E^{2n+1}. Then we define a mapping $\hat{f}$ of $\hat{K}$ into E^{2n+1} by setting, for $(a, b) \in \hat{K}$,

$$\hat{f}(a, b) = f(a) - f(b).$$

Clearly, $\hat{f}$ is continuous and $\hat{f}(a, b) = -\hat{f}(b, a)$.

We shall prove:

(6) If $0 \in \hat{f}(\hat{K})$ for every K-homeomorphism f of K^+ into E^{2n+1}, then $\mathscr{K}$ is $(2n+1)$-entangled (and therefore not homeomorphic to any subset of E^{2n}).

Indeed, if $\mathscr{K}$ is not $(2n+1)$-entangled there exists a K-homeomorphism f of K^+ into E^{2n+1} such that $f(K) \cap f(K^+ \backslash K) = \emptyset$. From $0 \in \hat{f}(\hat{K})$ it follows that for suitable $(a, b) \in \hat{K}$ we have $0 = \hat{f}(a, b) = f(a) - f(b)$, that is, $f(a) = f(b)$. Since K contains at least one of a, b, and since f is a K-homeomorphism, it follows that $a = b$, contradicting condition (ii) of the definition of $\hat{K}$.

Combining lemma (6) with the above remark $\hat{f}(a, b) = -\hat{f}(b, a)$ we obtain at once:

(7) If for every K-homeomorphism f of K^+ into E^{2n+1} some pair of antipodal points of $\hat{K}$ is mapped by $\hat{f}$ onto the same point of E^{2n+1}, then $\hat{K}$ is $(2n+1)$-entangled.

In the cases we shall discuss we shall find the following situation: $\hat{K}$ is homeomorphic to $K^\vee$, and $K^\vee$ is homeomorphic to S^{2n+1}, both homeomorphisms preserving antipodality. By the Borsuk-Ulam theorem (see Borsuk [1]), *every* mapping of S^{2n+1} into E^{2n+1} maps some pair of antipodal points of S^{2n+1} onto the same point of E^{2n+1}; because of the antipodality-preserving homeomorphism between $\hat{K}$ and S^{2n+1} the same conclusion is valid for $\hat{K}$. Hence, by lemma (7), the complex $\mathscr{K}$ is $(2n+1)$-entangled and thus not imbeddable in E^{2n}.

Now we are ready for

THEOREM 1. *Let $n, p, n_1, \dots, n_p$ be non-negative integers such that $n = n_1 + n_2 + \dots + n_p + p - 1$. Then the n-complex*

$$\mathscr{C}^{n_1}(2n_1 + 3) \vee \mathscr{C}^{n_2}(2n_2 + 3) \vee \cdots \vee \mathscr{C}^{n_p}(2n_p + 3)$$

is not imbeddable in E^{2n}.

Proof. In view of the above remark and previous lemmas, it is obviously enough to show that

(8) For each positive integer k, the set $(\mathscr{C}^k(2k+3))^\vee$ is homeomorphic to S^{2k+1} under a mapping that preserves antipodes.

Let $\mathscr{C}^k(2k+3)$ be represented by the k-skeleton $\mathscr{K}$ of a $(2k+2)$-simplex $T^{2k+2}= =\operatorname{conv}\{x_0,\dots,x_{2k+2}\}\subset R^{2k+2}$ such that

$$\sum_{i=0}^{2k+2} x_i = 0, \tag{*}$$

but each proper subset of the x_i's is linearly independent. Then $K^\vee$ may be obtained by taking $\vartheta(\mathscr{K})=-\mathscr{K}=\{-F \mid F\in\mathscr{K}\}$; hence $K^\vee$ is the union of all sets of the form $\operatorname{conv}(F_i\cup(-F_j))$, where F_i and F_j are disjoint members of $\mathscr{K}$.

In order to show that $K^\vee$ is homeomorphic to S^{2k+1} it is obviously enough to show that for each unit vector u in E^{2k+2}, the ray $L=\{\lambda u \mid \lambda\geqslant 0\}$ intersects $K^\vee$ in precisely one point, different from the origin.

We first establish $L\cap K^\vee\neq\emptyset$. Let $\lambda u=\sum_{i=0}^{2k+2}\alpha_i x_i$, with $\lambda>0$, $\alpha_i\geqslant 0$, $\sum_i\alpha_i=1$. Without loss of generality we may assume that $\alpha_0\leqslant\alpha_1\leqslant\dots\leqslant\alpha_{2k+2}$. Then, using (*), we have

$$0\neq\lambda u=\lambda u-\alpha_{k+1}0=\sum_{i=0}^{2k+2}(\alpha_i-\alpha_{k+1})\,x_i=\sum_{i=0}^{2k+2}\beta_i x_i .$$

where $\beta_i\leqslant 0$ for $0\leqslant i\leqslant k$, $\beta_{k+1}=0$, $\beta_i\geqslant 0$ for $k+2\leqslant i\leqslant 2k+2$, and not all β_i are 0. Let $\beta'=-\sum_{i=0}^{k}\beta_i$, $\beta''=\sum_{i=k+2}^{2k+1}\beta_i$, and $\beta=\beta'+\beta''$; then

$$L\ni\frac{\lambda}{\beta}u=\frac{\beta'}{\beta}\sum_{i=0}^{k}\left(-\frac{\beta_i}{\beta'}\right)(-x_i)+\frac{\beta''}{\beta}\sum_{i=k+2}^{2k+2}\frac{\beta_i}{\beta''}x_i\in K^\vee .$$

as claimed. (If $\beta'=0$ or $\beta''=0$, the corresponding sum should be omitted.)

On the other hand we shall show that if $y\in K^\vee$ for $y\neq 0$, and if $\lambda y\in K^\vee$ for $\lambda>0$, then $\lambda=1$. Indeed, assuming without loss of generality that $y=\sum_{i=0}^{2k+2}\alpha_i x_i$, where

$$\left.\begin{aligned}&\alpha_i\leqslant 0 \quad\text{for}\quad 0\leqslant i\leqslant k,\\ &\alpha_i=0 \quad\text{for}\quad i=k+1,\\ &\alpha_i\geqslant 0 \quad\text{for}\quad k+2\leqslant i\leqslant 2k+2,\end{aligned}\qquad \sum_{i=0}^{2k+2}|\alpha_i|=1,\right\} \tag{**}$$

and $\lambda y=\sum_{i=0}^{2k+2}\beta_i x_i$, where

$$\left.\begin{aligned}&\sum_{i=0}^{2k+2}|\beta_i|=1,\text{ at most } k+1 \text{ of the } \beta_i\text{'s are negative and at most}\\ &k+1 \text{ of the } \beta_i\text{'s are positive}.\end{aligned}\right\} \tag{***}$$

Then

$$0=y-\frac{\lambda y}{\lambda}=\sum_{i=0}^{2k+2}\left(\alpha_i-\frac{\beta_i}{\lambda}\right)x_i .$$

By (*) it follows that $\alpha_i-\beta_i\lambda^{-1}=\gamma$ is a constant independent of i. In other words,

$\beta_i = \lambda(\alpha_i - \gamma)$. Therefore, if $\gamma = 0$, assumptions (**) and (***) would contradict each other. Thus $\gamma = 0$, and then $1 = \sum_i |\beta_i| = \sum_i |\lambda\alpha_i| = \lambda \sum_i |\alpha_i| = \lambda$, as claimed.

Finally, $0 \notin K^{\vee}$ follows at once from (*).

This completes the proof of lemma (8) and thus also the proof of Theorem 1.

3. Geometric Imbeddings in E^{2n}

In the present section we shall show that every proper subcomplex of each of the n-complexes of Theorem 1 is geometrically imbeddable in E^{2n}.

A few *lemmas* are needed in the proof; the first is a special case of the general theorem.

(1) *Let $\mathscr{C}_0^k(2k+3)$ be a complex obtained from $\mathscr{C}^k(2k+3)$ by deleting one k-face. Then $\mathscr{C}_0^k(2k+3)$ is geometrically imbeddable in E^{2k}.*

Proof. Let T_1 and T_2 be two k-simplices in E^{2k} such that $T_1 \cap T_2$ is a single point, relatively interior to both T_1 and T_2. Let $T_k^{2k} = \operatorname{conv}(T_1 \cup T_2)$, and denote by $\mathscr{T}_m$ the m-skeleton of T_k^{2k}. It is well known (see, for example, Grünbaum [7], where the terms and facts used in the sequel may be found) that $\mathscr{T}_k$ contains all the geometric k-simplices determined by the $2k+2$ vertices of T_k^{2k}, except T_1 and T_2, while $\mathscr{T}_{k-1}$ contains all the $(k-1)$-simplices determined by those vertices ([7, p. 98]). Taking, if necessary, a suitable projective image of T_k^{2k}, we may without loss of generality assume that there exists a point $V \in E^{2k}$ that is beyond all facets of T_k^{2k} except one. Then $\{T_1\} \cup \mathscr{T}_{k-1}^{+}(V) \cup \mathscr{T}_k$ is isomorphic to $\mathscr{C}_0^k(2k+3)$, and the proof of (1) is completed.

Figure 1 illustrates the steps of the above proof for $k=1$.

We shall say that an n-complex $\mathscr{K}$ is *nicely imbedded* in E^m provided $\mathscr{K}$ is geometrically imbedded in E^m and there exists a point (say the origin 0 of E^m) with the property:

For each unit vector $u \in E^m$ except one, u_0, the ray $L(u) = \{\lambda u \mid \lambda \geqslant 0\}$ intersects $\operatorname{set}\mathscr{K}$ in at most one point, while $L(u_0) \cap \operatorname{set}\mathscr{K}$ consists of two points, each in the relative interior of an n-face of $\mathscr{K}$. We call u_0 the *exceptional direction*, and the two n-faces $L(u_0)$ meet the *exceptional faces* of $\mathscr{K}$.

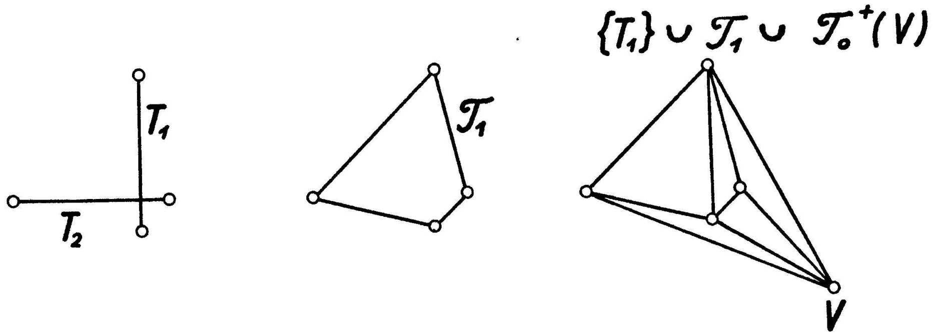

Figure 1

We need the following lemmas:

(2) $\mathscr{C}^k(2k+3)$ *is nicely imbeddable in* E^{2k+1}.

Proof. Let E_1^k be the n-dimensional affine subspace of E^{2k+1} defined by

$$E_1^k = \{(x_1, \ldots, x_{2k+1}) \in E^{2k+1} \mid x_{k+1} = \cdots = x_{2k} = 0,\ x_{2k+1} = 1/(2k+3)\}$$

and let

$$E_2^k = \{(x_1, \ldots, x_{2k+1}) \in E^{2k+1} \mid x_1 = \cdots = x_k = 0,\ x_{2k+1} = 1\}.$$

Let T_i^k be a regular simplex of edge-length 1 in E_i^k, $i=1,2$ having its centroid at $x_1 = \cdots = x_{2k} = 0$, and $x_{2k+1} = 1/(2k+3)$ respectively $x_{2k+1} = 1$. Let $\mathscr{C}^k(2k+3)$ have as vertices the $2k+2$ vertices of T_1^k and T_2^k, and the point $V=(-1, 0, \ldots, 0, -2)$. Then a trivial computation shows that this $\mathscr{C}^k(2k+3)$ is nicely imbedded in E^{2k+1}, with $u_0 = (0, \ldots, 0, 1)$ as the only exceptional direction. (See Figure 2 for an illustration of the case $k=1$.) This completes the proof of (2).

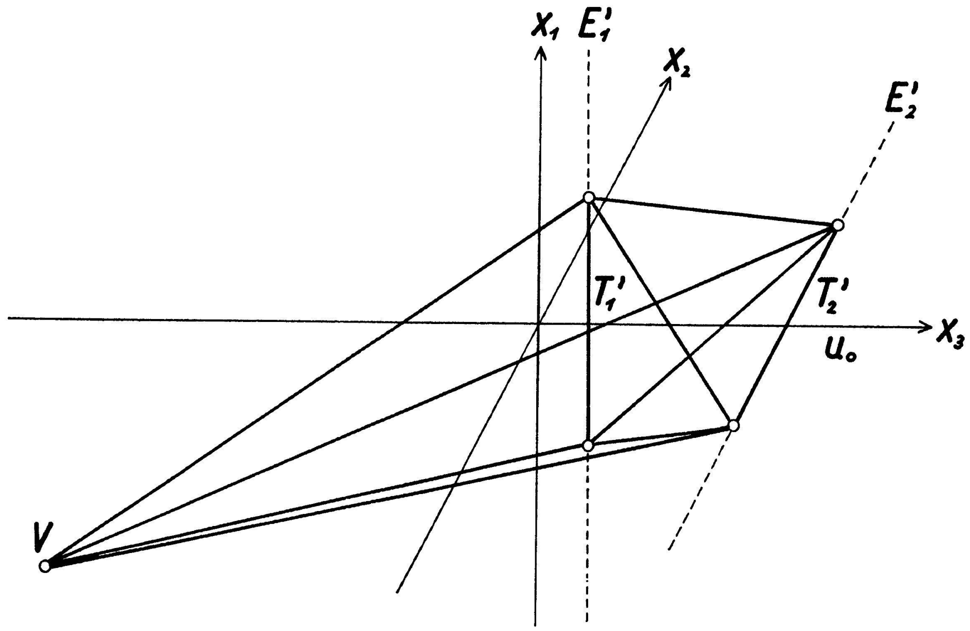

Figure 2

(3) *Let* $\mathscr{K}^{k_1}$ and $\mathscr{K}^{k_2}$ *be complexes nicely imbedded in spaces* E^{2k_1+1} *and* E^{2k_2+1}, *and let* $k = k_1 + k_2 + 1$. *Then* $\mathscr{K} = \mathscr{K}^{k_1} \vee \mathscr{K}^{k_2}$ *is a k-complex nicely imbeddable in* E^{2k+1}.

Proof. Let us imbed E^{2k_i+1} in E^{2k+1} by

$$E^{2k_1+1} = \{(x_1, \ldots, x_{2k+1}) \in E^{2k+1} \mid x_{2k_1+2} = \cdots = x_{2k} = x_{2k+1} = 0\},$$
$$E^{2k_2+1} = \{(x_1, \ldots, x_{2k+1}) \in E^{2k+1} \mid x_1 = \cdots = x_{2k_1+1} = 0,\ x_{2k+1} = 0\}.$$

We denote by v the vector $v = (0, 0, \ldots, 0, 1) \in E^{2k+1}$, and we consider the copy $\tilde{\mathscr{K}}^{k_1}$

of $\mathscr{K}^{k_1}$ imbedded in $E^{2k_1+1}+v$, and the copy $\tilde{\mathscr{K}}^{k_2}$ of $\mathscr{K}^{k_2}$ imbedded in $E^{2k_2+1}+2v$. Defining now $\mathscr{K}=\tilde{\mathscr{K}}^{k_1}\vee\tilde{\mathscr{K}}^{k_2}$, we shall show that $\mathscr{K}$ is nicely imbedded in E^{2k+1}. Clearly, $\mathscr{K}$ is a geometric complex in E^{2k+1}.

Let $u\in E^{2k+1}$ be a unit vector such that for some λ, μ with $0<\lambda<\mu$ we have $\lambda u\in\operatorname{set}\mathscr{K}$ and $\mu u\in\operatorname{set}\mathscr{K}$. That is,

$$\left.\begin{aligned}\lambda u &= \alpha y_1 + (1-\alpha)\, z_1 + (2-\alpha)\, v\\ \mu u &= \beta y_2 + (1-\beta)\, z_2 + (2-\beta)\, v,\end{aligned}\right\}\qquad (*)$$

where $0<\alpha$, $\beta<1$, $y_1, y_2\in\operatorname{set}\mathscr{K}^{k_1}$, and $z_1, z_2\in\operatorname{set}\mathscr{K}^{k_2}$.

Eliminating u from (*) and equating points in E^{2k_1+1}, E^{2k_2+1}, and multiples of v, we obtain

$$\left.\begin{aligned}\mu\alpha y_1 &= \lambda\beta y_2\\ \mu(1-\alpha)\, z_1 &= \lambda(1-\mu)\, z_2\\ \mu(2-\alpha) &= \lambda(2-\beta).\end{aligned}\right\}\qquad (**)$$

Clearly $y_1=y_2$ or $z_1=z_2$ would imply $\lambda=\mu$, contradicting the assumption. Hence $y_1\neq y_2$ and $z_1\neq z_2$, and thus (**) expresses the fact that

$$\begin{aligned} y_1 &= \gamma_1 u_1 \qquad & y_2 &= \gamma_2 u_1\\ z_1 &= \delta_1 u_2 \qquad & z_2 &= \delta_2 u_2,\end{aligned}$$

where u_1 and u_2 are the exceptional directions of $\mathscr{K}^{k_1}$ and $\mathscr{K}^{k_2}$, while $\gamma=\gamma_1/\gamma_2<1$ and $\delta=\delta_1/\delta_2>1$ are well-determined constants. Inserting those values into (**) we obtain

$$\alpha = 2\,\frac{\delta-1}{2\delta-\gamma-1} \quad\text{and}\quad \beta = 2\gamma\,\frac{\delta-1}{\delta\gamma+\delta-2\gamma}.$$

Substituting into (*) we see that u, λ and μ are uniquely determined. Hence the complex $\mathscr{K}$ is nicely imbedded in E^{2k+1} and the proof of (3) is completed.

The last lemma we shall need is

(4) *Let $\mathscr{K}^{k_1}$ and $\mathscr{K}^{k_2}$ be complexes nicely imbedded in E^{2k_1+1} respectively E^{2k_2+1}, let F^{k_1} be the exceptional face of $\mathscr{K}^{k_1}$ nearer to* 0, *and let F^{k_2} be the exceptional face of $\mathscr{K}^{k_2}$ further from* 0. *Then $(\mathscr{K}^{k_1}\vee\mathscr{K}^{k_1})\backslash(F^{k_1}\vee F^{k_2})$ is a k-complex, $k=k_1+k_2+1$, which is geometrically imbeddable in E^{2k}.*

Proof. Let $\mathscr{K}$ be the complex constructed in the proof of Lemma (3). Since $\mathscr{K}$ is nicely imbedded in E^{2k+1}, the radial projection of $\mathscr{K}\backslash\{(v+F^{k_1})\vee(2v+F^{k_2})\}$ into the $(2n)$-dimensional affine subspace $\{x\in E^{2k+1}\mid\langle x, v\rangle=3\}$ is clearly a geometric imbedding. This completes the proof of (4).

Now we are ready for our main result:

THEOREM 2. *Let $n, p, n_1, \dots, n_p$ be non-negative integers such that $n=n_1+\dots+n_p+p-1$. Then every proper subcomplex of the n-complex*

$$\mathscr{K}(n_1, \dots, n_p) = \mathscr{C}^{n_1}(2n_1+3) \vee \dots \vee \mathscr{C}^{n_p}(2n_p+3)$$

is geometrically imbeddable in E^{2n}.

Proof. It is clearly sufficient to prove the theorem for each complex $\mathscr{K}_0(n_1, \dots, n_p)$ obtained from $\mathscr{K}(n_1, \dots, n_p)$ by deleting one n-face F_0. Each such $\mathscr{K}_0(n_1, \dots, n_p)$ is obtained by singling out an n_i-face $F_0^{n_i}$ of $\mathscr{C}^{n_i}(2n_i+3)$ and setting $F_0=F_0^{n_1} \vee \dots \vee F_0^{n_p}$.

We distinguish two cases:

(i) $p=1$. Then the assertion of Theorem 2 reduces to that of lemma (1) above.

(ii) $p>1$. Using lemmas (2) and (3) we find a nice imbedding of $\mathscr{K}_1=\mathscr{C}^{n_1}(2n_1+3)$ in E^{2n_1+1}, and a nice imbedding of $\mathscr{K}_2=\mathscr{C}^{n_2}(2n_2+3) \vee \dots \vee \mathscr{C}^{n_p}(2n_p+3)$ in E^{2m+1}, where $m=n_2+\dots+n_p+p-2$, such that $F_0^{n_1}$ is the exceptional face of $\mathscr{K}_1$ nearer 0 while $F_0^{n_2} \vee \dots \vee F_0^{n_p}$ is the exceptional face of $\mathscr{K}_2$ further from 0. An application of lemma (4) to the complexes $\mathscr{K}_1$ and $\mathscr{K}_2$ completes the proof of Theorem 2.

4. Remarks

(i) The method used in the proof of Theorem 1 is an elaboration of Flores' [6] proof, extending the similar proofs in Rosen [11] and Grünbaum [7, p. 210]. By avoiding the more powerful – but also more unmanageable – "imbedding classes" of Wu [16, p. 114], it is possible to give a quite elementary proof of the non-imbeddability of the complexes of Theorem 1. By standard manipulations (van Kampen [8], Chrislock [3]) it is easy to extend Theorem 1 to the following

THEOREM 3. *Let n_i, m_i, p be non-negative integers such that $n_i+3 \leqslant m_i \leqslant 2n_i+3$ for $i=1, \dots, p$. Then the $(n_1+\dots+n_p+p-1)$-complex*

$$\mathscr{C}^{n_1}(m_1) \vee \dots \vee \mathscr{C}^{n_p}(m_p)$$

is not imbeddable in the Euclidean $(m_1+\dots+m_p-p-2)$-space, but it is even geometrically imbeddable in Euclidean $(m_1+\dots+m_p-p-1)$-space.

Theorem 3 may easily be modified to allow the inclusion of complexes $\mathscr{C}^{n_i}(n_i+1)$ or $\mathscr{C}^{n_i}(n_i+2)$. (For some special cases see Wu [16, p. 118].)

The significant difference between Theorems 1 and 3 is the observation that if $m_i<2n_i+3$ then the complex is not minimal with respect to the property of being non-imbeddable in the appropriate space. For example ($p=1, n_1=2, m_1=6$) the 2-complex $\mathscr{C}^2(6)$ is by Theorem 3 not imbeddable in E^3; however, even the complex obtained from $\mathscr{C}^2(6)$ by deleting the ten 2-faces incident with one vertex is not imbeddable in E^3. Hence there is no hope that the complexes of Theorem 3 satisfy an analogue of Theorem 2.

(ii) Zaks [17] has established Theorem 2 if either $p=1$ (i.e., in the case covered by Lemma (1) of Section 3), or else if all n_i's with at most one exception are equal to 0. His method does not seem to extend to the general case. On the other hand, Zaks proved: If all the proper subcomplexes of the arbitrary simplicial complex K_i are piecewise linearly imbeddable in E^{k_i}, $i=1, 2$, then each proper subcomplex of $K_1 \vee K_2$ is piecewise linearly imbeddable in $E^{k_1+k_2+2}$.

(iii) It is well known that a k-complex imbeddable in E^n is not necessarily geometrically imbeddable in E^n, if $n<2k$ (Cairns [2], van Kampen [9], Grünbaum [7, p. 202]). However, the published examples deal only with the case $n=3$; it would be of some interest to find analogous examples for all k and n with $k \leqslant n \leqslant 2k-1$.

Probably more interesting is the

Conjecture. Each simplicial (= triangulated) manifold imbeddable in a Euclidean space is even geometrically imbeddable in the same space.

This conjecture is open even for triangulations of the torus (in E^3), as well as for triangulations of S^k for $k \geqslant 3$. For triangulations of S^2 an affirmative answer results from a more general theorem of Steinitz concerning convex 3-polytopes (see Steinitz-Rademacher [12, p. 192], Grünbaum [7, p. 235]).

(iv) Considering simplicial complexes imbedded in the n-sphere S^n one may distinguish (as in the case of complexes imbedded in E^n) between topological and geometric imbeddings. While it is easy to show that a simplicial complex geometrically imbeddable in E^n is also geometrically imbeddable in S^n, the following converse seems to be still unsettled:

Conjecture. If $\mathscr{C}$ is a simplicial complex geometrically imbeddable in S^n and if set $\mathscr{C} \neq S^n$, then $\mathscr{C}$ is geometrically imbeddable in E^n.

(v) For $n=1$, the two 1-complexes (= graphs) given by Theorem 1 characterize graphs not imbeddable in the plane as follows (Kuratowski [10]): A graph $\mathscr{G}$ is not imbeddable in E^2 if and only if $\mathscr{G}$ contains a subgraph homeomorphic to one of the graphs of Theorem 1. However, the analogous statement is false for $n \geqslant 2$. As shown by Zaks [18], for every $n \geqslant 2$ there exist infinitely many n-complexes, none homeomorphic to a subcomplex of another, with the property of not being imbeddable in E^{2n}, though each proper subcomplex is piecewise-linearly imbeddable in E^{2n}.

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