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Imbeddings of Simplicial Complexes

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1. Introduction

The main aim of the present note is to show that certain n -dimensional simplicial complexes which are not imbeddable into the $(2n)$ -dimensional Euclidean space E^{2n} are *minimal* with respect to that property, in the following strong sense: Every proper subcomplex of one of those complexes is even geometrically imbeddable in E^{2n} . (A simplicial complex is *geometrically imbedded* in E^k provided each of its simplices is a geometric, rectilinear simplex.) This result adds credibility to the following conjecture, established for $n=1$ by Wagner [14] (see also Fáry [4] and Stojaković [13]):

Conjecture. If an n -dimensional simplicial complex is topologically imbeddable in E^{2n} then it is even geometrically imbeddable in E^{2n} .

It has recently been established by Weber [15] that the weaker conjecture dealing with piecewise-linear (instead of geometric) imbeddings is true.

We shall start (in Section 2) by extending the class of known examples of n -complexes (that is finite, n -dimensional, simplicial complexes) not imbeddable in E^{2n} . The only examples of such complexes we found in the literature (van Kampen [8], Flores [5, 6], Rosen [11], Wu [16]) are:

(i) The *complete n -complex* $\mathcal{C}^n(k)$ with k vertices, where $k \geq 2n+3$; clearly, only the case $k=2n+3$ is interesting.

(ii) The *join* $\mathcal{C}^0(3) \vee \mathcal{C}^0(3) \vee \dots \vee \mathcal{C}^0(3)$ of $n+1$ triplets of points.

For $n=1$ those examples reduce to the well-known graphs of Kuratowski [10], which may be used to characterize non-planar graphs.

In Section 3 we shall show that each subcomplex of each of the n -complexes constructed in Section 2 is geometrically imbeddable in E^{2n} . This generalizes recent results of Zaks [17], who has established for some of the complexes of Section 2 the possibility of geometrically imbedding each of their subcomplexes in E^{2n} , while establishing for the other cases only the possibility of a piecewise-linear imbedding (see the more detailed comments in Section 4).

The last Section is devoted to some additional remarks and problems.

I am indebted to the referee for a number of helpful remarks and references.

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2. Some n -Complexes Not Imbeddable in E^{2n}

We shall denote finite simplicial complexes by script capitals \mathcal{C}, \mathcal{K} , etc.; their faces (simplices) will be indicated by capitals F, V , etc., and also by enumerating the vertices; for example, $F^3 = (V_0, V_1, V_2, V_3)$. Superscripts will denote dimension; S^n is the n -sphere (the unit sphere in E^{n+1} if metric considerations are involved), and T^n indicates the n -simplex. For a complex K imbedded in some Euclidean space we denote by set K the set of points underlying the complex. (We find this symbol more convenient and more indicative than the more usual "absolute value" notation.)

If $F' = (V_0, \dots, V_n)$ and $F'' = (W_0, \dots, W_m)$ are disjoint (abstract) simplices, we shall denote their *join*¹⁾ by $F' \vee F'' = (V_0, \dots, V_n, W_0, \dots, W_m)$. For disjoint (abstract) simplicial complexes \mathcal{K}' and \mathcal{K}'' the *join* $\mathcal{K}' \vee \mathcal{K}''$ is defined by $\mathcal{K}' \vee \mathcal{K}'' = \{F' \vee F'' \mid F' \in \mathcal{K}', F'' \in \mathcal{K}''\}$. Note that this coincides with the usual definition, since we include the empty set \emptyset as face in each complex.

If \mathcal{K}' and \mathcal{K}'' are topological simplicial complexes contained in skew affine subspaces of a Euclidean space, their join $\mathcal{K}' \vee \mathcal{K}''$ is also a topological simplicial complex. Its faces $F' \vee F''$ may be represented by

$$F' \vee F'' = \{\lambda x' + (1 - \lambda) x'' \mid x' \in F', x'' \in F'', 0 \leq \lambda \leq 1\}.$$

If \mathcal{K}' and \mathcal{K}'' are geometric simplicial complexes in skew affine subspaces of a Euclidean space, then $\mathcal{K}' \vee \mathcal{K}''$ is also a geometric complex, and the above representation simplifies to $F' \vee F'' = \text{conv}(F' \cup F'')$, where $\text{conv} A$ denotes the convex hull of the set A . (As is well known, the assumption that \mathcal{K}' and \mathcal{K}'' are contained in skew affine spaces is not essential; the only condition required is that the convex combinations used do not introduce any unwanted intersections, or degenerate simplices. We shall assume this condition fulfilled whenever we use the symbol \vee .)

If A' and A'' are topological spaces, the *join* $A' \vee A''$ is the space obtained from the Cartesian product $A' \times A'' \times [0, 1]$ by identifying $(a', A'', 0)$ with a' for each $a' \in A'$, and similarly identifying $(A', a'', 1)$ with a'' for each $a'' \in A''$. If $j: A' \times A'' \times [0, 1] \rightarrow A' \vee A''$ is the identification map, then $A' \vee A''$ may be topologized by defining $N \subset A' \vee A''$ open if and only if $j^{-1}(N)$ is open in $A' \times A'' \times [0, 1]$.

The connection between the two notions of join is given by the easily established fact:

For topological simplicial complexes \mathcal{K}' and \mathcal{K}'' , there is a natural homeomorphism between set $(\mathcal{K}' \vee \mathcal{K}'')$ and $(\text{set } \mathcal{K}') \vee (\text{set } \mathcal{K}'')$.

If $B = \{b\}$ is a one-pointed set, then $A \vee B$ is for obvious reasons called the *pyramid* over A with apex b ; we shall denote it by $A^+(b)$, or simply A^+ if no confusion is

¹⁾ Because of lattice-theoretic connotations we prefer to indicate the join-operation by the symbol \vee instead of the frequently used $*$.

likely to arise. ($A^+(b)$ is frequently called the "cone" over A with vertex b ; we avoid this term since it is used with a different meaning in other fields.)

We note the well known and easily established facts:

- (1) $T^n \vee T^m$ is homeomorphic to T^{n+m+1} .
- (2) $S^n \vee S^m$ is homeomorphic to S^{n+m+1} .

A selfhomeomorphism π of a topological space A is called *antipodal* provided π is an involution (that is, $\pi^2(a) = a$ for each $a \in A$) and has no fixed points. The unit n -sphere S^n has a *natural antipodality* π defined by $\pi(a) = -a$. If A' and A'' are topological subspaces of a Euclidean space, with antipodalities π' and π'' , then there is a natural antipodality $\pi = \pi' \vee \pi''$ on $A' \vee A''$ defined by

$$\pi(\lambda a' + (1 - \lambda) a'') = \lambda \pi'(a') + (1 - \lambda) \pi''(a'').$$

The homeomorphism mentioned in (2) above may be chosen in such a manner as to preserve the natural antipodalities of the spheres involved. Indeed, let

$$S^n = \{x = (x_1, \dots, x_{n+m+2}) \in E^{n+m+2} \mid \|x\| = 1, x_{n+2} = \dots = x_{n+m+2} = 0\}$$

and

$$S^m = \{x = (x_1, \dots, x_{n+m+2}) \in E^{n+m+2} \mid \|x\| = 1, x_1 = \dots = x_{n+1} = 0\};$$

then the mapping which sends the point $\lambda x + (1 - \lambda) y$ of $S^n \vee S^m$ (where $x \in S^n, y \in S^m, 0 \leq \lambda \leq 1$) onto the point $\lambda^{1/2} x + (1 - \lambda)^{1/2} y$ of S^{n+m+1} has this property.

Let now \mathcal{K} be a topological simplicial n -complex, and let $\mathcal{K}^* = \mathcal{G}(\mathcal{K})$ be a complex isomorphic to \mathcal{K} under an isomorphism \mathcal{G} such that \mathcal{K} and \mathcal{K}^* are contained in skew affine spaces. We define a simplicial $(2n+1)$ -complex \mathcal{K}^\vee as the subcomplex of $\mathcal{K} \vee \mathcal{K}^*$ consisting of all simplices $F \vee F^*$ (where $F \in \mathcal{K}, F^* \in \mathcal{K}^*$, and $F \cap \mathcal{G}^{-1}(F^*) = \emptyset$) and their faces. If $K = \text{set } \mathcal{K}$, we shall use the notation $K^\vee = \text{set } (\mathcal{K}^\vee)$. The set K^\vee has a natural antipodality π defined by

$$\pi(\lambda x_1 + (1 - \lambda) \mathcal{G}(x_2)) = (1 - \lambda) x_2 + \lambda \mathcal{G}(x_1),$$

where x_i belongs to an n -simplex F_i of \mathcal{K} and $F_1 \cap F_2 = \emptyset$.

We have the following *lemma*:

(3) *If \mathcal{K}_i is a simplicial n_i -complex, $i = 1, \dots, p$, then there is a natural homeomorphism φ , which preserves the natural antipodalities, between $(\mathcal{K}_1 \vee \mathcal{K}_2 \vee \dots \vee \mathcal{K}_p)^\vee$ and $K_1^\vee \vee K_2^\vee \vee \dots \vee K_p^\vee$.*

Proof. It is clearly enough to consider the case $p=2$. Then a typical point of $(\mathcal{K}_1 \vee \mathcal{K}_2)^\vee$ is of the form

$$x = \lambda(\mu' x'_1 + (1 - \mu') x'_2) + (1 - \lambda) \mathcal{G}(\mu'' x''_1 + (1 - \mu'') x''_2),$$

where $0 \leq \lambda, \mu', \mu'' \leq 1, x'_i \in F'_i, x''_i \in F''_i, F'_i, F''_i \in \mathcal{K}_i$, and $(F'_1 \vee F'_2) \cap (F''_1 \vee F''_2) = \emptyset$, that is, $F'_1 \cap F''_1 = \emptyset$ and $F'_2 \cap F''_2 = \emptyset$. On the other hand, the typical point of $K_1^\vee \vee K_2^\vee$ is

given by

$$y = \beta(\alpha_1 y'_1 + (1 - \alpha_1) \vartheta(y''_1)) + (1 - \beta)(\alpha_2 y'_2 + (1 - \alpha_2) \vartheta(y''_2)),$$

where $0 \leq \alpha_1, \alpha_2, \beta \leq 1$, $y'_i \in F'_i$, $y''_i \in F''_i$, $F'_i, F''_i \in \mathcal{K}_i$, and $F'_i \cap F''_i = \emptyset$ for $i=1, 2$. Assuming without loss of generality that ϑ is affine, x may be made to correspond to y by taking $x'_i = y'_i$, $x''_i = y''_i$, and

$$\beta = \lambda \mu' + (1 - \lambda) \mu'',$$

$$\alpha_1 = \frac{\lambda \mu'}{\lambda \mu' + (1 - \lambda) \mu''} \qquad \alpha_2 = \frac{\lambda(1 - \mu')}{\lambda(1 - \mu') + (1 - \lambda)(1 - \mu'')}.$$

The continuity of the mapping and the preservation of antipodality by it are easily checked, and the proof of lemma (3) is completed.

As a corollary of (3) and (2) we have:

(4) If \mathcal{K}_i is a complex such that K_i^\vee is homeomorphic to the n_i -sphere S^{n_i} , then $(\mathcal{K}_1 \vee \mathcal{K}_2 \vee \dots \vee \mathcal{K}_p)^\vee$ is homeomorphic to the $(p-1 + \sum_{i=1}^p n_i)$ -sphere $S^{n_1} \vee S^{n_2} \vee \dots \vee S^{n_p}$. Moreover, the homeomorphism may be assumed to preserve antipodes.

Let \mathcal{K} be a topological simplicial n -complex; we construct a new set \hat{K} as follows. \hat{K} is a subset of $K^+ \times K^+$, and consists of those pairs (a, b) of points of K^+ which satisfy:

- (i) at least one of a, b belongs to K ;
- (ii) there exist disjoint n -simplices F_a and F_b of such that $a \in F_a^+, b \in F_b^+$.

\hat{K} is clearly a compact metric space; it has a natural antipodality making points (a, b) and (b, a) correspond to each other. One of the properties of \hat{K} which is of special interest to us is:

(5) For each n -complex \mathcal{K} , the set \hat{K} is homeomorphic to the set K^\vee by a homeomorphism φ which preserves antipodality.

Indeed, denoting by v the apex of K^+ , each point of \hat{K} is uniquely expressible in the form $(\lambda a + (1 - \lambda) v, \mu b + (1 - \mu) v)$, where a and b belong to disjoint n -simplices of \mathcal{K} , $0 \leq \lambda, \mu \leq 1$, and $\max\{\lambda, \mu\} = 1$. We define

$$\varphi(\lambda a + (1 - \lambda) v, \mu b + (1 - \mu) v) = \begin{cases} (1 - \frac{1}{2}\mu) a + \frac{1}{2}\mu \vartheta(b) \in K^\vee & \text{if } \lambda = 1 \\ \frac{1}{2}\lambda a + (1 - \frac{1}{2}\lambda) \vartheta(b) \in K^\vee & \text{if } \mu = 1. \end{cases}$$

It is trivial to check that φ has all the desired properties.

We need one more definition. Let \mathcal{K} be a topological simplicial complex and let $K = \text{set } \mathcal{K}$. A mapping f of K^+ shall be called a K -homeomorphism provided the restriction of f to K is a homeomorphism (between K and $f(K)$). We shall say that \mathcal{K} is n -entangled (or absolutely knotted in E^n) if and only if

$$f(K) \cap f(K^+ \setminus K) \neq \emptyset$$

for every K -homeomorphism f of K^+ into E^n .

If K is homeomorphic to a subset of E^{n-1} then \mathcal{K} is clearly not n -entangled (since in this case K^+ is homeomorphic to a subset of E^n). Hence, if we succeed in proving that some complex is n -entangled, then this complex is certainly not imbeddable in E^{n-1} .

Let now \mathcal{K} be an n -complex and let f be a K -homeomorphism of K^+ into E^{2n+1} . Then we define a mapping \hat{f} of \hat{K} into E^{2n+1} by setting, for $(a, b) \in \hat{K}$,

$$\hat{f}(a, b) = f(a) - f(b).$$

Clearly, \hat{f} is continuous and $\hat{f}(a, b) = -\hat{f}(b, a)$.

We shall prove:

(6) If $0 \in \hat{f}(\hat{K})$ for every K -homeomorphism f of K^+ into E^{2n+1} , then \mathcal{K} is $(2n+1)$ -entangled (and therefore not homeomorphic to any subset of E^{2n}).

Indeed, if \mathcal{K} is not $(2n+1)$ -entangled there exists a K -homeomorphism f of K^+ into E^{2n+1} such that $f(K) \cap f(K^+ \setminus K) = \emptyset$. From $0 \in \hat{f}(\hat{K})$ it follows that for suitable $(a, b) \in \hat{K}$ we have $0 = \hat{f}(a, b) = f(a) - f(b)$, that is, $f(a) = f(b)$. Since K contains at least one of a, b , and since f is a K -homeomorphism, it follows that $a = b$, contradicting condition (ii) of the definition of \hat{K} .

Combining lemma (6) with the above remark $\hat{f}(a, b) = -\hat{f}(b, a)$ we obtain at once:

(7) If for every K -homeomorphism f of K^+ into E^{2n+1} some pair of antipodal points of \hat{K} is mapped by \hat{f} onto the same point of E^{2n+1} , then \hat{K} is $(2n+1)$ -entangled.

In the cases we shall discuss we shall find the following situation: \hat{K} is homeomorphic to K^\vee , and K^\vee is homeomorphic to S^{2n+1} , both homeomorphisms preserving antipodality. By the Borsuk-Ulam theorem (see Borsuk [1]), every mapping of S^{2n+1} into E^{2n+1} maps some pair of antipodal points of S^{2n+1} onto the same point of E^{2n+1} ; because of the antipodality-preserving homeomorphism between \hat{K} and S^{2n+1} the same conclusion is valid for \hat{K} . Hence, by lemma (7), the complex \mathcal{K} is $(2n+1)$ -entangled and thus not imbeddable in E^{2n} .

Now we are ready for

THEOREM 1. *Let n, p, n_1, \dots, n_p be non-negative integers such that $n = n_1 + n_2 + \dots + n_p + p - 1$. Then the n -complex*

$$\mathcal{C}^{n_1}(2n_1 + 3) \vee \mathcal{C}^{n_2}(2n_2 + 3) \vee \dots \vee \mathcal{C}^{n_p}(2n_p + 3)$$

is not imbeddable in E^{2n} .

Proof. In view of the above remark and previous lemmas, it is obviously enough to show that

(8) For each positive integer k , the set $(\mathcal{C}^k(2k+3))^\vee$ is homeomorphic to S^{2k+1} under a mapping that preserves antipodes.

Let $\mathcal{C}^k(2k+3)$ be represented by the k -skeleton \mathcal{K} of a $(2k+2)$ -simplex $T^{2k+2} = \text{conv}\{x_0, \dots, x_{2k+2}\} \subset R^{2k+2}$ such that

$$\sum_{i=0}^{2k+2} x_i = 0, \tag{*}$$

but each proper subset of the x_i 's is linearly independent. Then K^\vee may be obtained by taking $\mathcal{D}(\mathcal{K}) = -\mathcal{K} = \{-F \mid F \in \mathcal{K}\}$; hence K^\vee is the union of all sets of the form $\text{conv}(F_i \cup (-F_j))$, where F_i and F_j are disjoint members of \mathcal{K} .

In order to show that K^\vee is homeomorphic to S^{2k+1} it is obviously enough to show that for each unit vector u in E^{2k+2} , the ray $L = \{\lambda u \mid \lambda \geq 0\}$ intersects K^\vee in precisely one point, different from the origin.

We first establish $L \cap K^\vee \neq \emptyset$. Let $\lambda u = \sum_{i=0}^{2k+2} \alpha_i x_i$, with $\lambda > 0$, $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$. Without loss of generality we may assume that $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_{2k+2}$. Then, using (*), we have

$$0 \neq \lambda u = \lambda u - \alpha_{k+1} 0 = \sum_{i=0}^{2k+2} (\alpha_i - \alpha_{k+1}) x_i = \sum_{i=0}^{2k+2} \beta_i x_i.$$

where $\beta_i \leq 0$ for $0 \leq i \leq k$, $\beta_{k+1} = 0$, $\beta_i \geq 0$ for $k+2 \leq i \leq 2k+2$, and not all β_i are 0. Let $\beta' = -\sum_{i=0}^k \beta_i$, $\beta'' = \sum_{i=k+2}^{2k+2} \beta_i$, and $\beta = \beta' + \beta''$; then

$$L \ni \frac{\lambda}{\beta} u = \frac{\beta'}{\beta} \sum_{i=0}^k \left(-\frac{\beta_i}{\beta'}\right) (-x_i) + \frac{\beta''}{\beta} \sum_{i=k+2}^{2k+2} \frac{\beta_i}{\beta''} x_i \in K^\vee.$$

as claimed. (If $\beta' = 0$ or $\beta'' = 0$, the corresponding sum should be omitted.)

On the other hand we shall show that if $y \in K^\vee$ for $y \neq 0$, and if $\lambda y \in K^\vee$ for $\lambda > 0$, then $\lambda = 1$. Indeed, assuming without loss of generality that $y = \sum_{i=0}^{2k+2} \alpha_i x_i$, where

$$\left. \begin{aligned} \alpha_i &\leq 0 & \text{for } 0 \leq i \leq k, \\ \alpha_i &= 0 & \text{for } i = k + 1, \\ \alpha_i &\geq 0 & \text{for } k + 2 \leq i \leq 2k + 2, \end{aligned} \right\} \sum_{i=0}^{2k+2} |\alpha_i| = 1, \tag{**}$$

and $\lambda y = \sum_{i=0}^{2k+2} \beta_i x_i$, where

$$\left. \begin{aligned} \sum_{i=0}^{2k+2} |\beta_i| &= 1, \text{ at most } k + 1 \text{ of the } \beta_i\text{'s are negative and at most} \\ &k + 1 \text{ of the } \beta_i\text{'s are positive.} \end{aligned} \right\} \tag{***}$$

Then

$$0 = y - \frac{\lambda y}{\lambda} = \sum_{i=0}^{2k+2} \left(\alpha_i - \frac{\beta_i}{\lambda}\right) x_i.$$

By (*) it follows that $\alpha_i - \beta_i \lambda^{-1} = \gamma$ is a constant independent of i . In other words,

$\beta_i = \lambda(\alpha_i - \gamma)$. Therefore, if $\gamma = 0$, assumptions (**) and (***) would contradict each other. Thus $\gamma = 0$, and then $1 = \sum_i |\beta_i| = \sum_i |\lambda\alpha_i| = \lambda \sum_i |\alpha_i| = \lambda$, as claimed.

Finally, $0 \notin K^\vee$ follows at once from (*).

This completes the proof of lemma (8) and thus also the proof of Theorem 1.

3. Geometric Imbeddings in E^{2n}

In the present section we shall show that every proper subcomplex of each of the n -complexes of Theorem 1 is geometrically imbeddable in E^{2n} .

A few *lemmas* are needed in the proof; the first is a special case of the general theorem.

(1) Let $\mathcal{C}_0^k(2k+3)$ be a complex obtained from $\mathcal{C}^k(2k+3)$ by deleting one k -face. Then $\mathcal{C}_0^k(2k+3)$ is geometrically imbeddable in E^{2k} .

Proof. Let T_1 and T_2 be two k -simplices in E^{2k} such that $T_1 \cap T_2$ is a single point, relatively interior to both T_1 and T_2 . Let $T_k^{2k} = \text{conv}(T_1 \cup T_2)$, and denote by \mathcal{F}_m the m -skeleton of T_k^{2k} . It is well known (see, for example, Grünbaum [7], where the terms and facts used in the sequel may be found) that \mathcal{F}_k contains all the geometric k -simplices determined by the $2k+2$ vertices of T_k^{2k} , except T_1 and T_2 , while \mathcal{F}_{k-1} contains all the $(k-1)$ -simplices determined by those vertices ([7, p. 98]). Taking, if necessary, a suitable projective image of T_k^{2k} , we may without loss of generality assume that there exists a point $V \in E^{2k}$ that is beyond all facets of T_k^{2k} except one. Then $\{T_1\} \cup \mathcal{F}_{k-1}^+(V) \cup \mathcal{F}_k$ is isomorphic to $\mathcal{C}_0^k(2k+3)$, and the proof of (1) is completed.

Figure 1 illustrates the steps of the above proof for $k=1$.

We shall say that an n -complex \mathcal{K} is *nicely imbedded* in E^m provided \mathcal{K} is geometrically imbedded in E^m and there exists a point (say the origin 0 of E^m) with the property:

For each unit vector $u \in E^m$ except one, u_0 , the ray $L(u) = \{\lambda u \mid \lambda \geq 0\}$ intersects set \mathcal{K} in at most one point, while $L(u_0) \cap \text{set } \mathcal{K}$ consists of two points, each in the relative interior of an n -face of \mathcal{K} . We call u_0 the *exceptional direction*, and the two n -faces $L(u_0)$ meet the *exceptional faces* of \mathcal{K} .

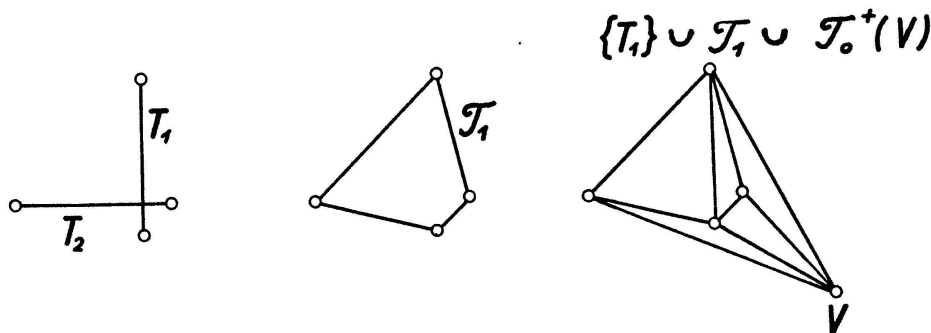


Figure 1

We need the following lemmas:

(2) $\mathcal{C}^k(2k+3)$ is nicely imbeddable in E^{2k+1} .

Proof. Let E_1^k be the n -dimensional affine subspace of E^{2k+1} defined by

$$E_1^k = \{(x_1, \dots, x_{2k+1}) \in E^{2k+1} \mid x_{k+1} = \dots = x_{2k} = 0, x_{2k+1} = 1/(2k+3)\}$$

and let

$$E_2^k = \{(x_1, \dots, x_{2k+1}) \in E^{2k+1} \mid x_1 = \dots = x_k = 0, x_{2k+1} = 1\}.$$

Let T_i^k be a regular simplex of edge-length 1 in E_i^k , $i=1, 2$ having its centroid at $x_1 = \dots = x_{2k} = 0$, and $x_{2k+1} = 1/(2k+3)$ respectively $x_{2k+1} = 1$. Let $\mathcal{C}^k(2k+3)$ have as vertices the $2k+2$ vertices of T_1^k and T_2^k , and the point $V = (-1, 0, \dots, 0, -2)$. Then a trivial computation shows that this $\mathcal{C}^k(2k+3)$ is nicely imbedded in E^{2k+1} , with $u_0 = (0, \dots, 0, 1)$ as the only exceptional direction. (See Figure 2 for an illustration of the case $k=1$.) This completes the proof of (2).

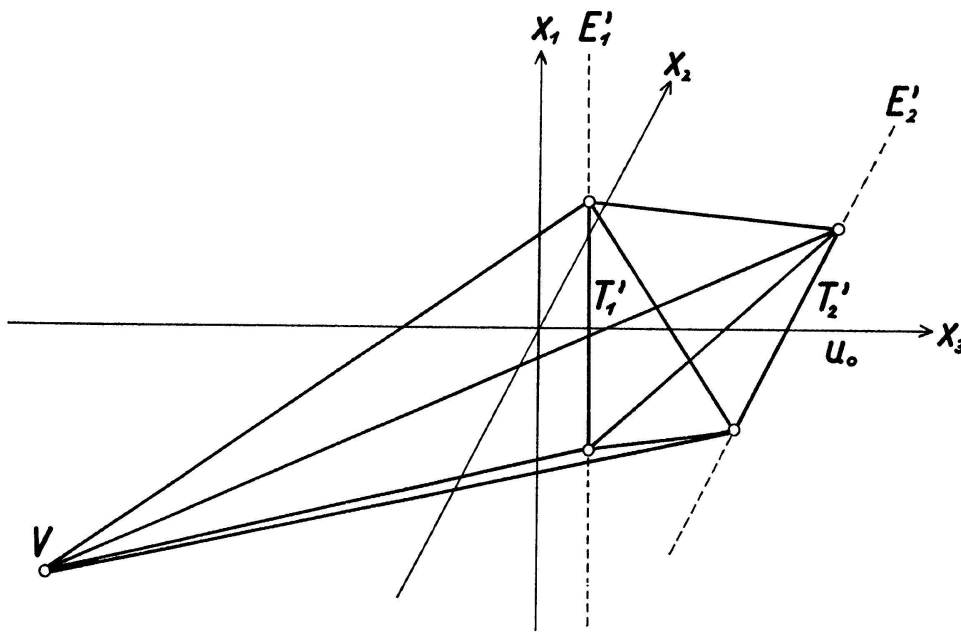


Figure 2

(3) Let \mathcal{K}^{k_1} and \mathcal{K}^{k_2} be complexes nicely imbedded in spaces E^{2k_1+1} and E^{2k_2+1} , and let $k = k_1 + k_2 + 1$. Then $\mathcal{K} = \mathcal{K}^{k_1} \vee \mathcal{K}^{k_2}$ is a k -complex nicely imbeddable in E^{2k+1} .

Proof. Let us imbed E^{2k_i+1} in E^{2k+1} by

$$E^{2k_1+1} = \{(x_1, \dots, x_{2k+1}) \in E^{2k+1} \mid x_{2k_1+2} = \dots = x_{2k} = x_{2k+1} = 0\},$$

$$E^{2k_2+1} = \{(x_1, \dots, x_{2k+1}) \in E^{2k+1} \mid x_1 = \dots = x_{2k_1+1} = 0, x_{2k+1} = 0\}.$$

We denote by v the vector $v = (0, 0, \dots, 0, 1) \in E^{2k+1}$, and we consider the copy $\tilde{\mathcal{K}}^{k_1}$

of \mathcal{K}^{k_1} imbedded in $E^{2k_1+1} + v$, and the copy $\tilde{\mathcal{K}}^{k_2}$ of \mathcal{K}^{k_2} imbedded in $E^{2k_2+1} + 2v$. Defining now $\mathcal{K} = \tilde{\mathcal{K}}^{k_1} \vee \tilde{\mathcal{K}}^{k_2}$, we shall show that \mathcal{K} is nicely imbedded in E^{2k+1} . Clearly, \mathcal{K} is a geometric complex in E^{2k+1} .

Let $u \in E^{2k+1}$ be a unit vector such that for some λ, μ with $0 < \lambda < \mu$ we have $\lambda u \in \text{set } \mathcal{K}$ and $\mu u \in \text{set } \mathcal{K}$. That is,

$$\left. \begin{aligned} \lambda u &= \alpha y_1 + (1 - \alpha) z_1 + (2 - \alpha) v \\ \mu u &= \beta y_2 + (1 - \beta) z_2 + (2 - \beta) v, \end{aligned} \right\} (*)$$

where $0 < \alpha, \beta < 1$, $y_1, y_2 \in \text{set } \mathcal{K}^{k_1}$, and $z_1, z_2 \in \text{set } \mathcal{K}^{k_2}$.

Eliminating u from (*) and equating points in E^{2k_1+1} , E^{2k_2+1} , and multiples of v , we obtain

$$\left. \begin{aligned} \mu \alpha y_1 &= \lambda \beta y_2 \\ \mu(1 - \alpha) z_1 &= \lambda(1 - \mu) z_2 \\ \mu(2 - \alpha) &= \lambda(2 - \beta). \end{aligned} \right\} (**)$$

Clearly $y_1 = y_2$ or $z_1 = z_2$ would imply $\lambda = \mu$, contradicting the assumption. Hence $y_1 \neq y_2$ and $z_1 \neq z_2$, and thus (**) expresses the fact that

$$\begin{aligned} y_1 &= \gamma_1 u_1 & y_2 &= \gamma_2 u_1 \\ z_1 &= \delta_1 u_2 & z_2 &= \delta_2 u_2, \end{aligned}$$

where u_1 and u_2 are the exceptional directions of \mathcal{K}^{k_1} and \mathcal{K}^{k_2} , while $\gamma = \gamma_1/\gamma_2 < 1$ and $\delta = \delta_1/\delta_2 > 1$ are well-determined constants. Inserting those values into (**) we obtain

$$\alpha = 2 \frac{\delta - 1}{2\delta - \gamma - 1} \quad \text{and} \quad \beta = 2\gamma \frac{\delta - 1}{\delta\gamma + \delta - 2\gamma}.$$

Substituting into (*) we see that u , λ and μ are uniquely determined. Hence the complex \mathcal{K} is nicely imbedded in E^{2k+1} and the proof of (3) is completed.

The last lemma we shall need is

(4) *Let \mathcal{K}^{k_1} and \mathcal{K}^{k_2} be complexes nicely imbedded in E^{2k_1+1} respectively E^{2k_2+1} , let F^{k_1} be the exceptional face of \mathcal{K}^{k_1} nearer to 0, and let F^{k_2} be the exceptional face of \mathcal{K}^{k_2} further from 0. Then $(\mathcal{K}^{k_1} \vee \mathcal{K}^{k_2}) \setminus (F^{k_1} \vee F^{k_2})$ is a k -complex, $k = k_1 + k_2 + 1$, which is geometrically imbeddable in E^{2k} .*

Proof. Let \mathcal{K} be the complex constructed in the proof of Lemma (3). Since \mathcal{K} is nicely imbedded in E^{2k+1} , the radial projection of $\mathcal{K} \setminus \{(v + F^{k_1}) \vee (2v + F^{k_2})\}$ into the $(2n)$ -dimensional affine subspace $\{x \in E^{2k+1} \mid \langle x, v \rangle = 3\}$ is clearly a geometric imbedding. This completes the proof of (4).

Now we are ready for our main result:

THEOREM 2. *Let n, p, n_1, \dots, n_p be non-negative integers such that $n = n_1 + \dots + n_p + p - 1$. Then every proper subcomplex of the n -complex*

$$\mathcal{K}(n_1, \dots, n_p) = \mathcal{C}^{n_1}(2n_1 + 3) \vee \dots \vee \mathcal{C}^{n_p}(2n_p + 3)$$

is geometrically imbeddable in E^{2n} .

Proof. It is clearly sufficient to prove the theorem for each complex $\mathcal{K}_0(n_1, \dots, n_p)$ obtained from $\mathcal{K}(n_1, \dots, n_p)$ by deleting one n -face F_0 . Each such $\mathcal{K}_0(n_1, \dots, n_p)$ is obtained by singling out an n_i -face $F_0^{n_i}$ of $\mathcal{C}^{n_i}(2n_i + 3)$ and setting $F_0 = F_0^{n_1} \vee \dots \vee F_0^{n_p}$.

We distinguish two cases:

(i) $p = 1$. Then the assertion of Theorem 2 reduces to that of lemma (1) above.

(ii) $p > 1$. Using lemmas (2) and (3) we find a nice imbedding of $\mathcal{K}_1 = \mathcal{C}^{n_1}(2n_1 + 3)$ in E^{2n_1+1} , and a nice imbedding of $\mathcal{K}_2 = \mathcal{C}^{n_2}(2n_2 + 3) \vee \dots \vee \mathcal{C}^{n_p}(2n_p + 3)$ in E^{2m+1} , where $m = n_2 + \dots + n_p + p - 2$, such that $F_0^{n_1}$ is the exceptional face of \mathcal{K}_1 nearer 0 while $F_0^{n_2} \vee \dots \vee F_0^{n_p}$ is the exceptional face of \mathcal{K}_2 further from 0. An application of lemma (4) to the complexes \mathcal{K}_1 and \mathcal{K}_2 completes the proof of Theorem 2.

4. Remarks

(i) The method used in the proof of Theorem 1 is an elaboration of Flores' [6] proof, extending the similar proofs in Rosen [11] and Grünbaum [7, p. 210]. By avoiding the more powerful – but also more unmanageable – “imbedding classes” of Wu [16, p. 114], it is possible to give a quite elementary proof of the non-imbeddability of the complexes of Theorem 1. By standard manipulations (van Kampen [8], Chrislock [3]) it is easy to extend Theorem 1 to the following

THEOREM 3. *Let n_i, m_i, p be non-negative integers such that $n_i + 3 \leq m_i \leq 2n_i + 3$ for $i = 1, \dots, p$. Then the $(n_1 + \dots + n_p + p - 1)$ -complex*

$$\mathcal{C}^{n_1}(m_1) \vee \dots \vee \mathcal{C}^{n_p}(m_p)$$

is not imbeddable in the Euclidean $(m_1 + \dots + m_p - p - 2)$ -space, but it is even geometrically imbeddable in Euclidean $(m_1 + \dots + m_p - p - 1)$ -space.

Theorem 3 may easily be modified to allow the inclusion of complexes $\mathcal{C}^{n_i}(n_i + 1)$ or $\mathcal{C}^{n_i}(n_i + 2)$. (For some special cases see Wu [16, p. 118].)

The significant difference between Theorems 1 and 3 is the observation that if $m_i < 2n_i + 3$ then the complex is not minimal with respect to the property of being non-imbeddable in the appropriate space. For example ($p = 1, n_1 = 2, m_1 = 6$) the 2-complex $\mathcal{C}^2(6)$ is by Theorem 3 not imbeddable in E^3 ; however, even the complex obtained from $\mathcal{C}^2(6)$ by deleting the ten 2-faces incident with one vertex is not imbeddable in E^3 . Hence there is no hope that the complexes of Theorem 3 satisfy an analogue of Theorem 2.

(ii) Zaks [17] has established Theorem 2 if either $p=1$ (i.e., in the case covered by Lemma (1) of Section 3), or else if all n_i 's with at most one exception are equal to 0. His method does not seem to extend to the general case. On the other hand, Zaks proved: If all the proper subcomplexes of the arbitrary simplicial complex K_i are piecewise linearly imbeddable in E^{k_i} , $i=1, 2$, then each proper subcomplex of $K_1 \vee K_2$ is piecewise linearly imbeddable in $E^{k_1+k_2+2}$.

(iii) It is well known that a k -complex imbeddable in E^n is not necessarily geometrically imbeddable in E^n , if $n < 2k$ (Cairns [2], van Kampen [9], Grünbaum [7, p. 202]). However, the published examples deal only with the case $n=3$; it would be of some interest to find analogous examples for all k and n with $k \leq n \leq 2k-1$.

Probably more interesting is the

Conjecture. Each simplicial (= triangulated) manifold imbeddable in a Euclidean space is even geometrically imbeddable in the same space.

This conjecture is open even for triangulations of the torus (in E^3), as well as for triangulations of S^k for $k \geq 3$. For triangulations of S^2 an affirmative answer results from a more general theorem of Steinitz concerning convex 3-polytopes (see Steinitz-Rademacher [12, p. 192], Grünbaum [7, p. 235]).

(iv) Considering simplicial complexes imbedded in the n -sphere S^n one may distinguish (as in the case of complexes imbedded in E^n) between topological and geometric imbeddings. While it is easy to show that a simplicial complex geometrically imbeddable in E^n is also geometrically imbeddable in S^n , the following converse seems to be still unsettled:

Conjecture. If \mathcal{C} is a simplicial complex geometrically imbeddable in S^n and if $\text{set } \mathcal{C} \neq S^n$, then \mathcal{C} is geometrically imbeddable in E^n .

(v) For $n=1$, the two 1-complexes (= graphs) given by Theorem 1 characterize graphs not imbeddable in the plane as follows (Kuratowski [10]): A graph \mathcal{G} is not imbeddable in E^2 if and only if \mathcal{G} contains a subgraph homeomorphic to one of the graphs of Theorem 1. However, the analogous statement is false for $n \geq 2$. As shown by Zaks [18], for every $n \geq 2$ there exist infinitely many n -complexes, none homeomorphic to a subcomplex of another, with the property of not being imbeddable in E^{2n} , though each proper subcomplex is piecewise-linearly imbeddable in E^{2n} .

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