

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 44 (1969)  
  
**Artikel:** Representative Functions on Topological Groups.  
**Autor:** Derighetti, Antoine  
**DOI:** <https://doi.org/10.5169/seals-33791>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 20.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# Representative Functions on Topological Groups

ANTOINE DERIGHETTI

## 1. Introduction

In this paper we shall study the relations existing between the topological properties of a completely regular topological group  $G$  and the algebraic properties of the space of all representative functions  $R(G)$  over  $G$ .

In the first part we give some results which generalize those of S. Kakutani ([4] pp. 430–431) concerning compactifications of locally compact abelian groups.

For a compact group  $G$  the Tannaka duality theorem shows that the algebraic properties of  $R(G)$  characterize completely those of  $G$ . Using [2], we find algebraic characterizations of the connectedness, local connectedness and arcwise connectedness of  $G$ . Similarly, we attempt to generalize, in a certain sense, the well-known result of Pontrjagin ([10] p. 32) about the covering dimension of a compact abelian group. Using these results we obtain some applications to more general topological groups.

I want to express my warmest thanks to Professor K. Chandrasekharan for his guidance in the preparation of this paper, to Professor R. Sridharan for many informative conversations and to Professor G. Hochschild for suggestions and corrections. Finally I want to acknowledge the generous support of the Forschungsinstitut für Mathematik without which it would have been impossible to carry out this work.

## 2. Compactifications and related questions

Let  $\gamma$  be the map of  $R(G)$  into  $R(G) \otimes_{\mathbb{C}} R(G)$  induced by the product in  $G$ . Following ([6]), one can say that, with the coproduct  $\gamma$  and the pointwise product,  $R(G)$  is a Hopf algebra. We consider, as in [2], only Hopf subalgebras of  $R(G)$  which are stable under complex conjugation.

Let  $\mathcal{H}$  be a Hopf subalgebra of  $R(G)$ . We denote by  $S(\mathcal{H})$  the set of all  $\mathbb{C}$ -algebra homomorphisms of  $\mathcal{H}$  onto  $\mathbb{C}$  which commute with complex conjugation. With the finite open topology  $S(\mathcal{H})$  is a compact space ([6] p. 28). Let  $\Gamma$  be a non empty subset of  $R(G)$ ; we denote by  $\mathcal{H}(\Gamma)$  the least Hopf subalgebra containing  $\Gamma$ . It follows from ([6] p. 29–30) that  $S(\mathcal{H}(\Gamma))$  is a compact group and the evaluation map  $\varphi_{\Gamma}$  of  $G$  into  $S(\mathcal{H}(\Gamma))$  is a continuous homomorphism.

**PROPOSITION 1.** *The group  $\varphi_{\Gamma}(G)$  is dense in  $S(\mathcal{H}(\Gamma))$  for every  $\Gamma \subset R(G)$ .*

*Proof.* Consider  $f \in R(S(\mathcal{H}(\Gamma)))$  with  $f=0$  on  $\varphi_{\Gamma}(G)$ . By the Tannaka duality

theorem ([6] p. 30) there exists  $h \in \mathcal{H}(\Gamma)$  such that  $s(h) = f(s)$  for every  $s \in S(\mathcal{H}(\Gamma))$ . In particular  $\varphi_\Gamma(x)(h) = h(x) = 0$  for every  $x \in G$ . This implies that  $h = 0$  and therefore  $f = 0$ . Using ([7] Lemma 5.2.) we obtain  $\overline{\varphi_\Gamma(G)} = S(\mathcal{H}(\Gamma))$ .

**COROLLARY 1.** Let  $\mathcal{H}$  be any Hopf subalgebra of  $R(G)$ . Let  $\tau$  be any element of  $S(\mathcal{H})$ , let  $f_1, \dots, f_n$  be a finite subset of  $\mathcal{H}$  and let  $\varepsilon$  be any positive number. Then there is a point  $x \in G$  such that  $|\tau(f_j) - f_j(x)| < \varepsilon$  ( $1 \leq j \leq n$ ).

*Proof.* By definition of the topology of  $S(\mathcal{H})$  the set  $\{\tau' \in S(\mathcal{H}) \mid |\tau'(f_j) - \tau(f_j)| < \varepsilon, 1 \leq j \leq n\}$  is an open neighborhood  $U$  of  $\tau$ . From prop. 1 the existence of  $x \in G$  then follows with the required properties.

*Remark.* This result is proved for characters over a topological group in ([5]). At the end of the same paper, the authors indicate the possibility of generalization.

**COROLLARY 2.** Let  $G$  be an infinite maximally almost periodic group and let  $f_1, \dots, f_n \in R(G)$  and  $\varepsilon > 0$ . Then there is an element  $x \in G$  such that  $x \neq e$  and  $|f_j(x) - f_j(e)| < \varepsilon$  ( $1 \leq j \leq n$ ).

The proof is analogous (using prop. 1) to that in the locally compact abelian case ([4] p. 431).

**PROPOSITION 2.** Let  $G$  be a topological group. Let  $H$  be a compact group. Then the following assertions are equivalent:

- (i) There is a continuous homomorphism  $\varphi$  of  $G$  into  $H$  such that  $\overline{\varphi(G)} = H$ .
- (ii)  $H$  is isomorphic to the compact group  $S(\Gamma)$  for some Hopf subalgebra  $\Gamma$  of  $R(G)$ .
- (iii) There is a Hopf algebra monomorphism  $\psi$  of  $R(H)$  into  $R(G)$ .

*Proof.* It is clear that (i) implies (iii) and that (ii) implies (i). Suppose that (iii) holds. The map  $\psi^*$  of  $S(R(G))$  into  $S(R(H))$  defined by  $\psi^*(s) = s \circ \psi$  is a continuous group homomorphism. There exists a continuous group homomorphism  $\psi'$  of  $G$  into  $H$  defined by the commutativity of

$$\begin{array}{ccc} S(R(G)) & \xrightarrow{\psi^*} & S(R(H)) \\ \varphi_{R(G)} \uparrow & & \uparrow \varphi_{R(H)} \\ G & \xrightarrow{\psi'} & H \end{array}$$

The relation  $\overline{\psi'(G)} \neq H$  implies the existence of  $f \in R(H)$  with  $f \neq 0$  and  $f(\psi'(x)) = 0$  for any  $x \in G$ . This contradicts the equality  $f \circ \psi' = \psi(f)$ . Therefore (iii) implies (i). It remains to prove that (i) implies (ii). Consider the Hopf algebra monomorphism  $\varphi^*$  of  $R(H)$  into  $R(G)$  defined by  $\varphi^*(f) = f \circ \varphi$  and set  $\Gamma = \varphi^*(R(H))$ . To every  $f \in R(H)$  there corresponds a function on  $S(\Gamma)$  defined by  $s(\varphi^*(f))$  for every

$s \in S(\Gamma)$ . This map is a Hopf algebra isomorphism of  $R(H)$  onto  $R(S(\Gamma))$  and therefore  $H$  and  $S(\Gamma)$  are isomorphic.

*Remark.* From the approximation theorem it follows that  $S(R(G))$  is isomorphic to the almost periodic compactification of  $G$  ([8] p. 168).

### 3. Some results concerning compact groups

For a compact group  $G$  we have  $\varphi(G) = S(R(G))$  (we set  $\varphi_{R(G)} = \varphi$ ). This equality permits us to characterize the topological properties of  $G$  (as in the abelian case) using the “algebraic” properties of  $R(G)$ .

First we introduce some notations. If  $\mathcal{H}$  is a Hopf subalgebra of  $R(G)$ , let  $\mathcal{H}^\perp$  denote the closed normal subgroup of  $G$  defined by  $\{h \in G \mid {}_h f = f \text{ for every } f \in \mathcal{H}\}$ . Conversely, if  $H$  is a closed normal subgroup of  $G$ , let  $H^\perp$  be the Hopf subalgebra of  $R(G)$  defined by  $\{f \in R(G) \mid {}_h f = f \text{ for every } h \in H\}$ . In [2] the following result was proved:

**THEOREM 1.** *For every compact group  $G$ ,  $G_0^\perp = \{f \in R(G) \mid f \text{ is an algebraic element of the C-algebra } R(G)\}$ , where  $G_0$  denotes the connected component of the identity in  $G$ .*

*Proof.* We prove at first that the above conditions are sufficient to insure the local connectedness of a compact group  $G$ .

**THEOREM 2.** *A compact group  $G$  is locally connected if and only if every finite set of representative functions on  $G$  is contained in a finitely generated Hopf subalgebra  $\mathcal{H}$  of  $R(G)$  such that every non constant element of  $R(\mathcal{H}^\perp)$  is not algebraic.*

*Proof.* We prove at first that the above conditions are sufficient to insure the local connectedness of  $G$ . For every open neighborhood  $U$  of  $e$  in  $G$  there exists an  $\varepsilon > 0$  and there exists a sequence  $\{f_j\}_{j=1}^n \subset R(G)$  such that the set  $\{x \in G \mid |f_j(x) - f_j(e)| < \varepsilon \text{ } 1 \leq j \leq n\}$  is contained in  $U$ . This implies that  $\mathcal{H}(f_1, \dots, f_n)^\perp \subset U$ . By hypothesis there exists a finitely generated Hopf subalgebra  $\mathcal{E}$  of  $R(G)$  with  $\mathcal{E} \supset \mathcal{H}(f_1, \dots, f_n)$  and  $\mathcal{E}^\perp$  connected. Let  $\pi$  be the canonical map of  $G$  onto  $G/\mathcal{E}^\perp$ . The factor group  $G/\mathcal{E}^\perp$  is a Lie group, since  $R(G/\mathcal{E}^\perp)$  and  $\mathcal{E}$  are isomorphic. Let  $\Sigma$  be a fundamental system of open connected neighborhoods of  $\pi(e)$  in  $G/\mathcal{E}^\perp$ . It is easy to demonstrate the existence of a subset  $O \in \Sigma$  with  $\pi^{-1}(O) \subset U$ . It suffices to prove that  $\pi^{-1}(O)$  is connected. Suppose the contrary. There exist open subsets of  $G$   $V_1, V_2$  such that  $V_1, V_2 \neq \emptyset$ ,  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = \pi^{-1}(O)$ . The existence of  $x \in G$  with  $\pi(x) \in \pi(V_1) \cap \pi(V_2)$  contradicts the connectedness of  $x\mathcal{E}^\perp$ . We therefore have  $\pi(V_1) \cap \pi(V_2) = \emptyset$  and this implies that  $O$  is not connected.

For the second part of the proof, we suppose that  $R(G)$  does not satisfy the above



conditions, and show that  $G$  is not locally connected. In this case there exists an  $M \subset R(G)$  with  $|M| < \infty$ , such that every Hopf subalgebra  $\mathcal{E}$  of  $R(G)$  with  $\mathcal{E} \supset M$  and  $\mathcal{E}^\perp$  connected is not finitely generated. Let  $\mathcal{H}$  be the Hopf subalgebra of  $R(G)$  with the property that  $\mathcal{H}^\perp$  is the connected component of the unit element in the subgroup  $\mathcal{H}(M)^\perp$  (the connected component of a normal closed subgroup is itself a normal subgroup). Denoting by  $\alpha$  the canonical map of  $G/\mathcal{H}^\perp$  onto  $G/\mathcal{H}(M)^\perp$ , we have  $\text{Ker } \alpha = \mathcal{H}(M)^\perp / \mathcal{H}^\perp$ . By a generalization of a wellknown theorem of Hurewicz ([9] theorem 4),  $\dim \text{Ker } \alpha = 0$  implies  $\dim G/\mathcal{H}^\perp \leq \dim G/\mathcal{H}(M)^\perp$ , and then  $\dim S(\mathcal{H}) \leq \dim S(\mathcal{H}(M))$ . It follows that  $\dim S(\mathcal{H})$  is finite, because  $S(\mathcal{H}(M))$  is a compact Lie group. By hypothesis  $\mathcal{H}$  is not finitely generated. This fact implies that  $S(\mathcal{H})$  is not locally connected, and therefore (since the natural map of  $G$  onto  $G/\mathcal{H}^\perp$  is open) that  $G$  itself is not locally connected.

*Remarks.*

1) In this proof we have used the two following results:  $\alpha$ ) A compact group  $G$  is a Lie group if and only if the  $\mathbf{C}$ -algebra  $R(G)$  is finitely generated;  $\beta$ ) Every compact (or locally compact) locally connected group with a finite dimension is a Lie group.

2) The corresponding classical result ([10] p. 33) for compact abelian groups is:  $G$  is locally connected if and only if every finite number of continuous characters over  $G$  is contained in a finitely generated subgroup  $H$  of  $\hat{G}$  (group of all continuous characters over  $G$ ) such that  $\hat{G}/H$  is torsion-free.

We denote by  $\mathcal{D}(G)$  the set of all  $\mathbf{C}$ -derivations of the  $\mathbf{C}$ -algebra  $R(G)$  which commute with complex conjugation and every left translation. Let  $D \in \mathcal{D}(G)$ . For every  $f \in R(G)$  consider the finite dimensional  $G$ -module  $R(f) = [\{f_x \mid x \in G\}]$ . By ([7] prop. 2.5)  $R(f)$  is stable under  $D$ . This implies that  $\sum_{n=1}^{\infty} D^n f / n!$  defines an element  $\exp Df$  of  $R(f)$  and therefore of  $R(G)$ .

**PROPOSITION 3.** *For every  $D \in \mathcal{D}(G)$  the map  $t \mapsto \varphi^{-1}(\varphi(e)\exp tD)$  is a one-parameter subgroup of  $G$ . Conversely every one-parameter subgroup admits such a unique representation.*

*Proof.* Let  $D \in \mathcal{D}(G)$  and  $t \in \mathbf{R}$ . It is easy to prove that  $\exp tD(fg) = \exp tD(f) \exp tD(g)$  for every  $f, g \in R(G)$ . It follows that  $\exp tD$  is a  $\mathbf{C}$ -algebra endomorphism of  $R(G)$ . From the fact that  $\exp tD$  commutes with complex conjugation it follows that  $\varphi(e)\exp tD \in S(R(G))$ . We have therefore that  $t \mapsto \varphi^{-1}(\varphi(e)\exp tD)$  is a one-parameter subgroup of  $G$ .

Let  $\lambda \in \text{Hom}_{\text{cont}}(\mathbf{R}, G)$ . For every  $f \in R(G)$  and  $t \in \mathbf{R}$  set  $U_t f = f_{\lambda(t)}$ . The operator  $U_t$  is unitary under the scalar-product of  $R(G)$  defined by the normalized Haar measure of  $G$ . We denote by  $U'_t$  the extension of  $U_t$  to  $L^2(G)$ . There exists an operator  $D$  of  $L^2(G)$  with  $iD$  selfadjoint and such that  $\lim_{t \rightarrow 0} \|(U'_t f - f)t^{-1} - Df\|_2 = 0$  for

every  $f \in R(G)$ . The operator  $-iD$  has the spectral representation  $\int_{-\infty}^{+\infty} \mu dE_\mu$  and  $U_t$  is equal to  $\int_{-\infty}^{+\infty} e^{i\mu t} dE_\mu$ . For every  $f$  in  $R(G)$  and  $t \neq 0$  we have  $(U_t f - f) t^{-1} \in R(f)$  and therefore  $Df \in R(f)$ , i.e.  $D(R(G)) \subset R(G)$ . It is easy to verify that the restriction of  $D$  to  $R(G)$  is contained in  $\mathcal{D}(G)$ . As above we can define  $\exp tD$ . It is clear that the  $\mathbf{C}$ -algebra endomorphism  $\exp tD$  commutes with complex conjugation and left translations and invoking ([7] Lemma 5.4) we obtain that  $\exp tD$  is a unitary operator of  $R(G)$ . For every  $f$  of  $R(G)$  we have  $\lim_{t \rightarrow 0} \|(\exp tD f - f) t^{-1} - Df\|_2 = 0$ . Let  $U_t''$  be the extension of  $\exp tD$  to  $L^2(G)$ . As above there exists an operator  $D'$  of  $L^2(G)$  with  $iD'$  self-adjoint and  $\lim_{t \rightarrow 0} \|(U_t'' h - h) t^{-1} - D'h\|_2 = 0$  for every  $h \in R(G)$ . We have therefore  $D = D'$  and  $U_t'' = U_t'$  i.e.  $\exp tD f = f_{\lambda(t)}$  for every  $f \in R(G)$ .

**COROLLARY.** For a compact Lie group  $G$ , the Lie algebra  $\mathfrak{g}$  of  $G$  is isomorphic to  $\mathcal{D}(G)$ .

*Remarks.*

1) Proposition 3 gives a characterisation of the Lie algebra of a compact group. The corollary has been already proved for more general Lie groups than compact Lie groups ([7] Theorem 11.1).

2) For the second part of the proof of proposition 3 Professor G. Hochschild has suggested a method which avoids the use of operator theory in  $L^2(G)$ . If  $V$  is any finite dimensional right-submodule of  $R(G)$  the map  $t \mapsto U_t$  (where  $U_t f = f_{\lambda(t)}$ ) defines a continuous homomorphism of  $\mathbf{R}$  into the full linear group of  $V$ . This homomorphism is therefore of the form  $t \mapsto \exp tD_V$ , where  $D_V$  is some linear endomorphism of  $V$ . Since  $R(G)$  is the union of such  $V$ 's, the  $D_V$ 's match up to give a linear endomorphism  $D$  of  $R(G)$  with the required properties.

We set for  $\Gamma \subset R(G)$  and  $M \subset \mathcal{D}(G)$ :

- (i)  $\text{Ann}(\Gamma) = \{D \in \mathcal{D}(G) \mid Df = 0 \text{ for every } f \in \Gamma\}$ ,
- (ii)  $\mathcal{H}_l(\Gamma) =$  the least subalgebra of  $R(G)$  invariant under the left-translations and the complex conjugation containing  $\Gamma$ .
- (iii)  $\text{Ann}(M) = \{f \in R(G) \mid Df = 0 \text{ for every } D \in M\}$ .

It is easy to see that  $\text{Ann}(\Gamma)$  is a Lie subalgebra of  $\mathcal{D}(G)$ , and that  $\text{Ann}(M) = \mathcal{H}_l(\text{Ann}(M))$ .

**PROPOSITION 4.** For every subset  $\Gamma$  of  $R(G)$ , we have  $\mathcal{H}_l(\Gamma \cup \mathcal{A}) = \text{Ann}(\text{Ann}(\Gamma))$ , where  $\mathcal{A}$  is the subset of all algebraic elements of  $R(G)$ .

*Proof.* Denote by  $\lambda(D)$  the element of  $\text{Hom}_{\text{cont}}(\mathbf{R}, G)$  corresponding to  $D \in \mathcal{D}(G)$ . From  $f \in \lambda(D)(\mathbf{R})_r^{\perp 1}$  it follows that  $\exp tD f = f$  for every  $t \in \mathbf{R}$  i.e.  $f \in \text{Ker } D$  and

<sup>1)</sup> For every subset  $H$  of  $G$ ,  $H_r^{\perp}$  denotes the set  $\{f \in R(G) \mid f_x = f \text{ for every } x \in H\}$  and for any subalgebra  $\Gamma$  of  $R(G)$  with  $\mathcal{H}_l(\Gamma) = \Gamma \Gamma_r^{\perp}$  is the closed subgroup  $\{x \in G \mid f_x = f \text{ for every } f \in \Gamma\}$ .

conversely, we have therefore  $\lambda(D)(\mathbf{R})_r^\perp = \text{Ker } D$ . Using the fact that every one-parameter subgroup is contained in  $G_0$  we obtain  $\text{Ker } D \supset \mathcal{A}$  and in particular  $\text{Ann}(\text{Ann}(\Gamma)) \supset \mathcal{H}_1(\Gamma \cup \mathcal{A})$ . It is easy to verify that  $\text{Ann}(\Gamma) = \{D \in \mathcal{D}(G) \mid \lambda(D)(\mathbf{R}) \subset \mathcal{H}_1(\Gamma \cup \mathcal{A})_r^\perp\}$ . Since the closed subgroup  $\mathcal{H}_1(\Gamma \cup \mathcal{A})_r^\perp$  is connected, we have  $\mathcal{H}_1(\Gamma \cup \mathcal{A}) = \{\lambda(D)(\mathbf{R}) \mid D \in \mathcal{D}(G), \lambda(D)(\mathbf{R}) \subset \mathcal{H}_1(\Gamma \cup \mathcal{A})_r^\perp\}_r^\perp$  and therefore  $\mathcal{H}_1(\Gamma \cup \mathcal{A}) = \{\lambda(D)(\mathbf{R}) \mid D \in \text{Ann}(\Gamma)\}_r^\perp = \bigcap \{\text{Ker } D \mid D \in \text{Ann}(\Gamma)\} = \text{Ann}(\text{Ann}(\Gamma))$ .

*Remarks.*

1) For  $\Gamma = \emptyset$  we obtain  $\mathcal{A} = \text{Ann}(\mathcal{D}(G))$  which gives another characterisation of the set of all algebraic elements of  $R(G)$ .

2) The group  $G$  is solenoidal if and only if there is  $D \in \mathcal{D}(G)$  with  $\text{Ker } D = \mathbf{C} \cdot 1_G$ .

3) There is a bijective map between the closed subgroups of  $G_0$  and the Lie subalgebras  $M$  of  $\mathcal{D}(G)$  such that  $M = \text{Ann}(\text{Ann}(M))$ . That is, to every closed subgroup  $H$  of  $G_0$  we associate  $M = \text{Ann}(H_r^\perp)$ . The subgroup  $H$  is normal in  $G$  if and only if  $M$  is an ideal of  $\mathcal{D}(G)$ .

**THEOREM 3.** *A compact group  $G$  is arcwise connected if and only if for every  $x \in G$  there is an element  $D$  of  $\mathcal{D}(G)$  such that the following diagram commutes:*

$$\begin{array}{ccc} R(G) & \xrightarrow{\varphi(x)} & \mathbf{C} \\ \exp D \searrow & & \nearrow \varphi(e) \\ & R(G) & \end{array}$$

**LEMMA.** *A compact group is arcwise connected if and only if it is the union of all one-parameter subgroups.*

*Proof.* By ([11] Theorem 1) every arc beginning at the unit element is homotopic to the restriction to  $[0, 1]$  of a one-parameter subgroup.

*Proof of theorem 3.* Suppose first that  $G$  is arcwise connected. In this case for every  $x \in G$  there exists  $\lambda \in \text{Hom}_{\text{cont}}(\mathbf{R}, G)$  and  $a \in \mathbf{R}$  with  $\lambda(a) = x$ . There exists  $D \in \mathcal{D}(G)$  such that  $\lambda(at) = \varphi^{-1}(\varphi(e) \exp t D)$  and therefore  $\varphi(e) \exp D = \varphi(x)$ .

Conversely suppose that for every  $x \in G$  there exists  $D \in \mathcal{D}(G)$  such that  $\varphi(x) = \varphi(e) \exp D$ . If we set  $\lambda(t) = \varphi^{-1}(\varphi(e) \exp t D)$  we obtain  $\lambda \in \text{Hom}_{\text{cont}}(\mathbf{R}, G)$  and  $\lambda(1) = x$ .

*Remarks.*

1) The classical result for compact abelian groups ([3]) is:  $G$  is arcwise connected if and only if for every  $x \in G$  there exists  $\lambda \in \text{Hom}(\hat{G}, \mathbf{R})$  such that

$$\begin{array}{ccc} \hat{G} & \xrightarrow{x} & S^1 \\ \lambda \searrow & & \uparrow e^{2\pi i} \\ & \mathbf{R} & \end{array}$$

commutes.

2) It is not necessary to give conditions which imply the local arcwise connectedness of  $G$  because a compact connected group is locally arcwise connected if and only if it is arcwise connected ([11]).

The dimension of a compact abelian group is equal by ([10] p. 32) to the rank of its character group. The next theorem is to be considered as a possible generalization to the non abelian case.

**THEOREM 4.** *The dimension of a compact group  $G$  is equal to the dimension of the real vector space  $\mathcal{D}(G)$ .*

*Proof.* There exists an inverse system  $(G_\alpha, u_{\alpha\beta})$  consisting of compact Lie groups  $G_\alpha$  and continuous epimorphisms  $u_{\beta\alpha}: G_\beta \rightarrow G_\alpha$  ( $\alpha < \beta$ ) such that  $G \cong \varprojlim (G_\alpha, u_{\alpha\beta})$ . We denote by  $\pi_\alpha$  the projection of  $G$  onto  $G_\alpha$ ; by  $R_\alpha$ , the Hopf subalgebra of  $R(G)$ ,  $(\text{Ker } \pi_\alpha)^\perp$ ; by  $\mathcal{D}_\alpha$  the set of all  $\mathbb{C}$ -derivations of  $R_\alpha$  which commute with complex conjugation and all left translations and finally by  $i_{\alpha\beta}$  ( $\alpha < \beta$ ) the natural injection of  $R_\alpha$  into  $R_\beta$ . It follows from ([2]) that  $R(G)$  and  $\varinjlim (R_\alpha, i_{\alpha\beta})$  are isomorphic. The restriction  $R_{\beta\alpha}$  ( $\alpha < \beta$ ) of an element of  $\mathcal{D}_\beta$  to  $R_\alpha$  belongs to  $\mathcal{D}_\alpha$ . The differential  $u_{\beta\alpha}^*$  of  $u_{\beta\alpha}$  is a linear map of the Lie algebra  $\mathfrak{g}_\beta$  of  $G_\beta$  onto  $\mathfrak{g}_\alpha$ . It is easy to verify that the projective systems  $(\mathcal{D}(G), \text{id})$ ,  $(\mathcal{D}_\alpha, \text{Res}_{\alpha\beta})$  and  $(\mathfrak{g}_\alpha, u_{\alpha\beta}^*)$  are isomorphic. From  $\dim \mathcal{D}_\alpha = \dim G_\alpha$  (corollary of prop. 3),  $\dim G = \sup_\alpha \dim G_\alpha$  and  $\dim \mathcal{D}(G) = \sup_\alpha \dim \mathcal{D}_\alpha$  the theorem follows.

#### 4. Applications

For non-compact groups the relations between the properties of  $G$  and those of  $R(G)$  are more complicated.

If the  $\mathbb{C}$ -algebra  $R(G)$  of a locally compact maximally almost periodic group  $G$  is finitely generated, then  $G$  is a Lie group. The condition is not necessary. However, if  $G$  is a Lie group such that  $G/G_0$  is finite then  $R(G)$  is finitely generated if and only if the factor group of  $G$  modulo the closure of the commutator of  $G_0$  is compact ([7] theorem 11.1).

**PROPOSITION 5.** *If a topological group  $G$  is connected, then every non constant representative function over  $G$  is non algebraic. If every representative function over a maximally almost periodic group is algebraic then the group is totally disconnected.*

*Proof.* The connectedness of  $G$  implies the same property for  $S(R(G))$ . From Theorem 1 the first part of proposition 5 follows. The proof of the second part is completely analogous.

**THEOREM 5.** *Every locally countably compact torsion group with a maximally almost periodic connected component of the identity is totally disconnected.*

*Proof.* Suppose that  $G$  is a compact torsion group. For every  $f \in R(G)$  consider  $R(f)$  and the corresponding continuous finite dimensional representation  $\varrho_f$ ;  $\varrho_f(G)$  is a compact torsion Lie group and therefore is a finite group. It follows  $\text{Ker } \varrho_f \supset G_0$  i.e.  $f \in G_0^\perp$ . Using theorem 1, we have that  $G$  is totally disconnected. For the general case consider, the continuous map  $\alpha_n: G \rightarrow G$  defined by  $\alpha_n(x) = x^n$  for every positive integer  $n$ . By assumption we have  $G = \bigcup_{n=1}^{\infty} \text{Ker } \alpha_n$ , the category theorem of Baire implies the existence of  $n_0$  such that  $\text{Ker } \alpha_{n_0}$  is open and therefore  $\text{Ker } \alpha_{n_0} \supset G_0$ . From this it follows that  $S(R(G_0))$  is a torsion group. Using the first part of the proof, theorem 1 and proposition 5 we have the desired result.

*Remark.* This theorem generalizes a result proved by Braconnier ([1] p. 51) for the case of a locally compact abelian group.

#### BIBLIOGRAPHY

- [1] BRACONNIER, J., *Sur les groupes topologiques localement compacts*, J. Math. Pures et Appl. [N. S.] 27 (1948), 1–85.
- [2] DERIGHETTI, A., *Über die Zusammenhangskomponente des Neutralelements einer kompakten Gruppe*, Math. Zeit. (to appear).
- [3] DIXMIER, J., *Quelques propriétés des groupes abéliens localement compacts*, Bull. Sci. Math. 81 (1957), 38–48.
- [4] HEWITT, E. and ROSS, K. A., *Abstract Harmonic Analysis*, Vol. I (Springer, 1963).
- [5] HEWITT, E. and ZUCKERMAN, H. S., *A Group-Theoretic Method in Approximation Theory*, Ann. of Math. (2) 52 (1950), 557–567.
- [6] HOCHSCHILD, G., *Structure of Lie groups* (Holden-Day, San Francisco 1965).
- [7] HOCHSCHILD, G. and MOSTOW, G. D., *Representations and Representative Functions of Lie Groups*, Ann. of Math. 66 (1957), 495–542.
- [8] LOOMIS, L. H., *An Introduction to Abstract Harmonic Analysis* (Van Nostrand, Princeton 1953).
- [9] MORITA, K., *On Closed Mappings and Dimension*, Proc. Japan Academy 32 (1956), 161–165.
- [10] PONTRJAGIN, L. S., *Topologische Gruppen*, Bd. II (Teubner, Leipzig 1958).
- [11] RICKERT, N. W., *Arcs in locally compact groups*, Math. Annalen 172 (1967), 222–228.

*Forschungsinstitut für Mathematik,  
E.T.H., Zürich, Switzerland*

Eingegangen den 18.2.69