

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 44 (1969)

**Artikel:** Hill Equations with Coexisting Periodic Solutions, II.  
**Autor:** Guggenheimer, H.  
**DOI:** <https://doi.org/10.5169/seals-33782>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 14.10.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Hill Equations with Coexisting Periodic Solutions, II.

by H. GUGGENHEIMER<sup>1)</sup>

There is a certain interest in finding all Hill equations with coexisting periodic solutions, i.e., Hill equations all whose solutions are periodic, cf. [4], Chap. VII. Recently, I gave a differential geometric method for the construction of second order linear differential equations with coexisting periodic solutions [2]. The same problem has been solved by F. Neuman with the tools of the theory of dispersions [5]. In the present paper, we construct directly all Sturm-Liouville equations  $x'' + Qx = 0$  with continuous, periodic coefficients and coexisting periodic solutions. The formulae were found by an interpretation of the theory of dispersions [1] in differential geometry. However, in the present formulation we need only the most elementary tools of calculus and analytic geometry. On the way, we give a geometric derivation of some results of H. A. Schwarz that contain the solution of our problem.

1. Polar coordinates in an  $(x_1, x_2)$ -plane are defined by

$$x_1 = r \cos \alpha, \quad x_2 = r \sin \alpha.$$

For a continuous curve  $x(t) = (x_1(t), x_2(t))$ , we determine  $\alpha(t)$  as a continuous function by an appropriate choice of the branch of  $\arctan x_2/x_1$ . The determinant of two vectors  $a, b$  in the plane is denoted by  $[a, b]$ .

Let  $Q(t)$  be a continuous function for  $-\infty < t < +\infty$ . We consider the differential equation

$$x'' + Q(t)x = 0 \tag{1}$$

on the real number line. We choose two linearly independent solutions  $x_1(t), x_2(t)$  of unit Wronskian. For the vector  $x(t)$  this means that  $[x, x'] = 1$ . The parameter  $t$  is twice the area  $A$  covered by the vector  $x$  since

$$2A = \int_{t_0}^t [x, dx] = \int_{t_0}^t dt = t - t_0.$$

This means that  $t$  is connected with the polar coordinates of  $x(t)$  by

$$t - t_0 = \int_{t_0}^t r^2(t) d\alpha(t),$$

---

<sup>1)</sup> Research supported partially by NSF Grant GP-8176.

i.e.,  $\frac{d\alpha}{dt} = \frac{1}{r^2}$ , (2)

or

$$r = (\alpha')^{-1/2} \quad (2a)$$

and  $\alpha$  is a strictly monotone function of  $t$ . In fact,  $r(t) > 0$  since the Wronskian never vanishes.

We introduce the two unit vectors

$$c(\alpha) = (\cos \alpha, \sin \alpha), \quad n(\alpha) = (-\sin \alpha, \cos \alpha).$$

Then  $x(t) = r(t) c(\alpha)$ ,  $x'(t) = r' c(\alpha) + r^{-1} n(\alpha)$  and

$$x''(t) = (r'' - r^{-3}) c(\alpha) = -r^{-4}(1 - r''r^3)x(t).$$

A comparison with (1) shows that (see also [1], (5) p. 32)

$$Q(t) = \frac{1}{r^4} \left( 1 - r^3 \frac{d^2 r}{dt^2} \right). \quad (3)$$

We see that for any continuous function  $Q(t)$  there exist nonzero  $C^2$ -functions  $r(t)$  such that (3) holds. Before we show that (3) solves our problem, we note that, by (2a), (3) is equivalent to ([1], p. 35)

$$Q(t) = \frac{1}{2} \frac{\alpha'''}{\alpha'} - \frac{3}{4} \left( \frac{\alpha''}{\alpha'} \right)^2 + \alpha'^2. \quad (4)$$

The Schwarzian derivative of a function  $s(t)$  is

$$\{s, t\} = \frac{s'''}{s'} - \frac{3}{2} \left( \frac{s''}{s'} \right)^2.$$

An easy verification shows that (4) is equivalent to Schwarz's formula [6]

$$Q(t) = \frac{1}{2} \{\tan \alpha, t\} = \frac{1}{2} \{x_2/x_1, t\}.$$

Schwarz also noted that the knowledge of a solution  $s = \tan \alpha$  of  $\{s, t\} = 2Q(t)$  is sufficient for the determination of a pair of solutions of (1) with unit Wronskian since  $s' = \alpha' \cos^{-2} \alpha = x_1^{-2}$ . Hence,

$$x_1 = s'^{-1/2}, \quad x_2 = s'^{-1/2}s, \quad (5)$$

and

**LEMMA 1:** *The function  $\alpha(t)$  uniquely determines  $x(t)$ .*

Similarly,  $r(t)$  and  $\alpha(0)$  determine  $x(t)$  via

$$\alpha(t) = \alpha(0) + \int_0^t \frac{dt}{r^2}. \quad (6)$$

2. We assume now that  $Q(t)$  is a periodic function of period  $\omega$ .

**LEMMA 2:** *All solutions of (1) are periodic ( $x(t+\omega)=x(t)$ ) or semi-periodic ( $x(t+\omega)=-x(t)$ ) if and only if*

$$\tan \alpha(t + \omega) = \tan \alpha(t),$$

i.e.

$$\alpha(t + \omega) = \alpha(t) + k\pi, \quad k \text{ integer.} \quad (7)$$

The necessity follows from the definition of  $\alpha$  and the sufficiency from lemma 1.

If (7) holds,  $r(t)$  is periodic of period  $\omega$  and

$$\int_0^\omega r^{-2} dt = k\pi.$$

The solution of our problem is immediate:

**THEOREM:** *All solutions of (1) are periodic or semi-periodic if and only if*

$$Q(t) = r^{-4}(1 - r''r^3)$$

where

$$r \in C^2, r(t) > 0, r(t + \omega) = r(t),$$

and

$$\int_0^\omega r^{-2}(t) dt = k\pi.$$

*The solutions are periodic for  $k$  even, semi-periodic for  $k$  odd.*

We note that the condition (7) implies that the problem

$$x'' + Qx = \lambda x, \quad x(\omega) = \pm x(0)$$

has a collapsing  $k$ -th interval of instability at  $\lambda=0$  ([4], Chap. VII). In fact, every  $x_i(t)$  ( $i=1, 2$ ) vanishes  $k$  times in an interval of periodicity since the radius vector

covers  $k$  straight angles in monotone motion. But the number of zeros in the interval of definition is the index of the eigenvalue (see, e.g., [3], p. 148).

If we want to insure  $Q(t) > 0$ , we have to ask in addition to the conditions of the theorem that  $r''(t) < r^3(t)$ . In that case, the curve  $x(t)$  is without inflection points.

Using (4) instead of (3), one may construct  $Q(t)$  starting from  $f(t) = \alpha'(t)$  with  $\int_0^\omega f(t) dt = k\pi$ . We obtain:

*All solutions of (1) are periodic ( $k=2i$ ) or semi-periodic ( $k=2i+1$ ) if and only if*

$$Q(t) = \frac{1}{2} \frac{f''(t)}{f(t)} - \frac{3}{4} \left( \frac{f'(t)}{f(t)} \right)^2 + f^2(t),$$

where

$$f \in C^2, f(t) > 0, f(t + \omega) = f(t)$$

and

$$\int_0^\omega f(t) dt = k\pi.$$

## BIBLIOGRAPHY

- [1] O. BORŮVKA, *Lineare Differentialtransformationen 2. Ordnung* (Deutscher Verlag der Wissenschaften, Berlin 1967).
- [2] H. GUGGENHEIMER, *Some Geometric Remarks about Dispersions*, to appear, Archivum Mathematicum, vol. 4.
- [3] H. HOCHSTADT, *Differential Equations* (Holt, Rinehart and Winston, New York 1964).
- [4] W. MAGNUS and S. WINKLER, *Hill's Equation* (Interscience, New York 1966).
- [5] F. NEUMAN, *Criterion of Periodicity of Solutions of a Certain Differential Equation with a Periodic Coefficient*, Ann. di Mat. pura appl. (4) 75 (1967), 385–396.
- [6] H. A. SCHWARZ, *Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt*, J. reine angew. Math. 75 (1872), 292–335.

Polytechnic Institute of Brooklyn

Received January 27, 1969