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Hill Equations with Coexisting Periodic Solutions, II.

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There is a certain interest in finding all Hill equations with coexisting periodic solutions, i.e., Hill equations all whose solutions are periodic, cf. [4], Chap. VII. Recently, I gave a differential geometric method for the construction of second order linear differential equations with coexisting periodic solutions [2]. The same problem has been solved by F. Neuman with the tools of the theory of dispersions [5]. In the present paper, we construct directly all Sturm–Liouville equations $x''+Qx=0$ with continuous, periodic coefficients and coexisting periodic solutions. The formulae were found by an interpretation of the theory of dispersions [1] in differential geometry. However, in the present formulation we need only the most elementary tools of calculus and analytic geometry. On the way, we give a geometric derivation of some results of H. A. Schwarz that contain the solution of our problem.

1. Polar coordinates in an (x_1, x_2)-plane are defined by

$$x_1 = r\cos\alpha, \qquad x_2 = r\sin\alpha.$$

For a continuous curve $x(t)=(x_1(t), x_2(t))$, we determine $\alpha(t)$ as a continuous function by an appropriate choice of the branch of arctan x_2/x_1. The determinant of two vectors a, b in the plane is denoted by $[a, b]$.

Let $Q(t)$ be a continuous function for $-\infty < t < +\infty$. We consider the differential equation

$$x'' + Q(t)\,x = 0 \tag{1}$$

on the real number line. We choose two linearly independent solutions $x_1(t)$, $x_2(t)$ of unit Wronskian. For the vector $x(t)$ this means that $[x, x']=1$. The parameter t is twice the area A covered by the vector x since

$$2A = \int_{t_0}^{t} [x, dx] = \int_{t_0}^{t} dt = t - t_0.$$

This means that t is connected with the polar coordinates of $x(t)$ by

$$t - t_0 = \int_{t_0}^{t} r^2(t)\,d\alpha(t),$$

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i.e.,

$$\frac{d\alpha}{dt} = \frac{1}{r^2}, \tag{2}$$

or

$$r = (\alpha')^{-1/2} \tag{2a}$$

and α is a strictly monotone function of t. In fact, $r(t) > 0$ since the Wronskian never vanishes.

We introduce the two unit vectors

$$c(\alpha) = (\cos\alpha, \sin\alpha), \qquad n(\alpha) = (-\sin\alpha, \cos\alpha).$$

Then $x(t) = r(t)\, c(\alpha)$, $x'(t) = r' c(\alpha) + r^{-1} n(\alpha)$ and

$$x''(t) = (r'' - r^{-3})\, c(\alpha) = -r^{-4}(1 - r'' r^3)\, x(t).$$

A comparison with (1) shows that (see also [1], (5) p. 32)

$$Q(t) = \frac{1}{r^4}\left(1 - r^3 \frac{d^2 r}{dt^2}\right). \tag{3}$$

We see that for any continuous function $Q(t)$ there exist nonzero C^2-functions $r(t)$ such that (3) holds. Before we show that (3) solves our problem, we note that, by (2a), (3) is equivalent to ([1], p. 35)

$$Q(t) = \frac{1}{2}\frac{\alpha'''}{\alpha'} - \frac{3}{4}\left(\frac{\alpha''}{\alpha'}\right)^2 + \alpha'^2. \tag{4}$$

The Schwarzian derivative of a function $s(t)$ is

$$\{s, t\} = \frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2.$$

An easy verification shows that (4) is equivalent to Schwarz's formula [6]

$$Q(t) = \tfrac{1}{2}\{\tan\alpha, t\} = \tfrac{1}{2}\{x_2/x_1, t\}.$$

Schwarz also noted that the knowledge of a solution $s = \tan\alpha$ of $\{s, t\} = 2Q(t)$ is sufficient for the determination of a pair of solutions of (1) with unit Wronskian since $s' = \alpha' \cos^{-2}\alpha = x_1^{-2}$. Hence,

$$x_1 = s'^{-1/2}, \qquad x_2 = s'^{-1/2} s, \tag{5}$$

and

LEMMA 1: *The function $\alpha(t)$ uniquely determines $x(t)$.*

Similarly, $r(t)$ and $\alpha(0)$ determine $x(t)$ via

$$\alpha(t) = \alpha(0) + \int_0^t \frac{dt}{r^2}. \tag{6}$$

2. We assume now that $Q(t)$ is a periodic function of period ω.

LEMMA 2: *All solutions of* (1) *are periodic* $(x(t+\omega)=x(t))$ *or semi-periodic* $(x(t+\omega)=-x(t))$ *if and only if*

$$\tan\alpha(t+\omega) = \tan\alpha(t),$$

i.e.

$$\alpha(t+\omega) = \alpha(t) + k\pi, \quad k \textit{ integer}. \tag{7}$$

The necessity follows from the definition of α and the sufficiency from lemma 1. If (7) holds, $r(t)$ is periodic of period ω and

$$\int_0^\omega r^{-2}\,dt = k\pi.$$

The solution of our problem is immediate:

THEOREM: *All solutions of* (1) *are periodic or semi-periodic if and only if*

$$Q(t) = r^{-4}(1 - r''r^3)$$

where

$$r \in C^2,\ r(t) > 0,\ r(t+\omega) = r(t),$$

and

$$\int_0^\omega r^{-2}(t)\,dt = k\pi.$$

The solutions are periodic for k even, semi-periodic for k odd.

We note that the condition (7) implies that the problem

$$x'' + Qx = \lambda x, \qquad x(\omega) = \pm x(0)$$

has a collapsing k-th interval of instability at $\lambda=0$ ([4], Chap. VII). In fact, every $x_i(t)\,(i=1,2)$ vanishes k times in an interval of periodicity since the radius vector covers k straight angles in monotone motion. But the number of zeros in the interval of definition is the index of the eigenvalue (see, e.g., [3], p. 148).

If we want to insure $Q(t)>0$, we have to ask in addition to the conditions of the theorem that $r''(t)<r^3(t)$. In that case, the curve $x(t)$ is without inflexion points.

Using (4) instead of (3), one may construct $Q(t)$ starting from $f(t)=\alpha'(t)$ with $\int_0^\omega f(t)dt=k\pi$. We obtain:

All solutions of (1) *are periodic* ($k=2i$) *or semi-periodic* ($k=2i+1$) *if and only if*

$$Q(t)=\frac{1}{2}\frac{f''(t)}{f(t)}-\frac{3}{4}\left(\frac{f'(t)}{f(t)}\right)^2+f^2(t),$$

where

$$f\in C^2, f(t)>0, f(t+\omega)=f(t)$$

and

$$\int_0^\omega f(t)\,dt=k\pi.$$

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