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## Hill Equations with Coexisting Periodic Solutions, II.

by H. GUGGENHEIMER<sup>1)</sup>

There is a certain interest in finding all Hill equations with coexisting periodic solutions, i.e., Hill equations all whose solutions are periodic, cf. [4], Chap. VII. Recently, I gave a differential geometric method for the construction of second order linear differential equations with coexisting periodic solutions [2]. The same problem has been solved by F. Neuman with the tools of the theory of dispersions [5]. In the present paper, we construct directly all Sturm–Liouville equations  $x'' + Qx = 0$  with continuous, periodic coefficients and coexisting periodic solutions. The formulae were found by an interpretation of the theory of dispersions [1] in differential geometry. However, in the present formulation we need only the most elementary tools of calculus and analytic geometry. On the way, we give a geometric derivation of some results of H. A. Schwarz that contain the solution of our problem.

1. Polar coordinates in an  $(x_1, x_2)$ -plane are defined by

$$x_1 = r \cos \alpha, \quad x_2 = r \sin \alpha.$$

For a continuous curve  $x(t) = (x_1(t), x_2(t))$ , we determine  $\alpha(t)$  as a continuous function by an appropriate choice of the branch of  $\arctan x_2/x_1$ . The determinant of two vectors  $a, b$  in the plane is denoted by  $[a, b]$ .

Let  $Q(t)$  be a continuous function for  $-\infty < t < +\infty$ . We consider the differential equation

$$x'' + Q(t)x = 0 \tag{1}$$

on the real number line. We choose two linearly independent solutions  $x_1(t), x_2(t)$  of unit Wronskian. For the vector  $x(t)$  this means that  $[x, x'] = 1$ . The parameter  $t$  is twice the area  $A$  covered by the vector  $x$  since

$$2A = \int_{t_0}^t [x, dx] = \int_{t_0}^t dt = t - t_0.$$

This means that  $t$  is connected with the polar coordinates of  $x(t)$  by

$$t - t_0 = \int_{t_0}^t r^2(t) d\alpha(t),$$

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i.e., 
$$\frac{d\alpha}{dt} = \frac{1}{r^2}, \quad (2)$$

or

$$r = (\alpha')^{-1/2} \quad (2a)$$

and  $\alpha$  is a strictly monotone function of  $t$ . In fact,  $r(t) > 0$  since the Wronskian never vanishes.

We introduce the two unit vectors

$$c(\alpha) = (\cos \alpha, \sin \alpha), \quad n(\alpha) = (-\sin \alpha, \cos \alpha).$$

Then  $x(t) = r(t) c(\alpha)$ ,  $x'(t) = r' c(\alpha) + r^{-1} n(\alpha)$  and

$$x''(t) = (r'' - r^{-3}) c(\alpha) = -r^{-4} (1 - r'' r^3) x(t).$$

A comparison with (1) shows that (see also [1], (5) p. 32)

$$Q(t) = \frac{1}{r^4} \left( 1 - r^3 \frac{d^2 r}{dt^2} \right). \quad (3)$$

We see that for any continuous function  $Q(t)$  there exist nonzero  $C^2$ -functions  $r(t)$  such that (3) holds. Before we show that (3) solves our problem, we note that, by (2a), (3) is equivalent to ([1], p. 35)

$$Q(t) = \frac{1}{2} \frac{\alpha'''}{\alpha'} - \frac{3}{4} \left( \frac{\alpha''}{\alpha'} \right)^2 + \alpha'^2. \quad (4)$$

The Schwarzian derivative of a function  $s(t)$  is

$$\{s, t\} = \frac{s'''}{s'} - \frac{3}{2} \left( \frac{s''}{s'} \right)^2.$$

An easy verification shows that (4) is equivalent to Schwarz's formula [6]

$$Q(t) = \frac{1}{2} \{\tan \alpha, t\} = \frac{1}{2} \{x_2/x_1, t\}.$$

Schwarz also noted that the knowledge of a solution  $s = \tan \alpha$  of  $\{s, t\} = 2Q(t)$  is sufficient for the determination of a pair of solutions of (1) with unit Wronskian since  $s' = \alpha' \cos^{-2} \alpha = x_1^{-2}$ . Hence,

$$x_1 = s'^{-1/2}, \quad x_2 = s'^{-1/2} s, \quad (5)$$

and

**LEMMA 1:** *The function  $\alpha(t)$  uniquely determines  $x(t)$ .*

Similarly,  $r(t)$  and  $\alpha(0)$  determine  $x(t)$  via

$$\alpha(t) = \alpha(0) + \int_0^t \frac{dt}{r^2}. \quad (6)$$

2. We assume now that  $Q(t)$  is a periodic function of period  $\omega$ .

LEMMA 2: *All solutions of (1) are periodic ( $x(t+\omega)=x(t)$ ) or semi-periodic ( $x(t+\omega)=-x(t)$ ) if and only if*

$$\tan \alpha(t + \omega) = \tan \alpha(t),$$

*i.e.*

$$\alpha(t + \omega) = \alpha(t) + k\pi, \quad k \text{ integer}. \quad (7)$$

The necessity follows from the definition of  $\alpha$  and the sufficiency from lemma 1. If (7) holds,  $r(t)$  is periodic of period  $\omega$  and

$$\int_0^\omega r^{-2} dt = k\pi.$$

The solution of our problem is immediate:

THEOREM: *All solutions of (1) are periodic or semi-periodic if and only if*

$$Q(t) = r^{-4}(1 - r''r^3)$$

where

$$r \in C^2, r(t) > 0, r(t + \omega) = r(t),$$

and

$$\int_0^\omega r^{-2}(t) dt = k\pi.$$

*The solutions are periodic for  $k$  even, semi-periodic for  $k$  odd.*

We note that the condition (7) implies that the problem

$$x'' + Qx = \lambda x, \quad x(\omega) = \pm x(0)$$

has a collapsing  $k$ -th interval of instability at  $\lambda=0$  ([4], Chap. VII). In fact, every  $x_i(t)$  ( $i=1, 2$ ) vanishes  $k$  times in an interval of periodicity since the radius vector

covers  $k$  straight angles in monotone motion. But the number of zeros in the interval of definition is the index of the eigenvalue (see, e.g., [3], p. 148).

If we want to insure  $Q(t) > 0$ , we have to ask in addition to the conditions of the theorem that  $r''(t) < r^3(t)$ . In that case, the curve  $x(t)$  is without inflexion points.

Using (4) instead of (3), one may construct  $Q(t)$  starting from  $f(t) = \alpha'(t)$  with  $\int_0^\omega f(t) dt = k\pi$ . We obtain:

*All solutions of (1) are periodic ( $k=2i$ ) or semi-periodic ( $k=2i+1$ ) if and only if*

$$Q(t) = \frac{1}{2} \frac{f''(t)}{f(t)} - \frac{3}{4} \left( \frac{f'(t)}{f(t)} \right)^2 + f^2(t),$$

where

$$f \in C^2, f(t) > 0, f(t + \omega) = f(t)$$

and

$$\int_0^\omega f(t) dt = k\pi.$$

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