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Foliations on Open Manifolds, II

by ANTHONY PHILLIPS

1. Introduction

Consider an n -dimensional smooth Riemannian manifold M which is open (i.e. has no compact components) and on M a field σ of tangent k -planes. This note gives a sufficient condition for σ to be homotopic to an integrable field. The condition is stated in terms of the complementary q -plane field σ^\perp ($q = n - k$), which we may consider as a q -dimensional subbundle of TM , the tangent bundle of M .

THEOREM: *If the structural group of σ^\perp can be reduced to a discrete group, then σ is homotopic to an integrable field.*

REMARKS: This theorem was suggested by the following result of Ehresmann ([1, p. 38], [2, p. 364]): Let $N \subset M$ be an embedded submanifold; then N is a leaf of a foliation of a neighborhood of N in M if and only if the structural group of the normal bundle of N in M can be reduced to a discrete group. It gives a partial answer to a question posed by Reeb [7], Haefliger [2] and Thomas [8]. Since the one-dimensional orthogonal group is discrete, this gives a simpler proof of [6], Theorem 1.2: every $(n-1)$ -plane field on M is homotopic to an integrable field. The restriction to open manifolds allows the use of submersion theory.

The proof of this theorem is given in the next two sections; the last section contains an example.

Early drafts of this note contain a much more restricted theorem. I am very grateful to André Haefliger for pointing out this generalisation.

2. Proof of Theorem

Let \tilde{M} be the universal cover of M , with $p: \tilde{M} \rightarrow M$ the projection, and denote by $\alpha: \tilde{M} \rightarrow \tilde{M}$ the covering transformation corresponding to the element α of the fundamental group π of M . We will use the notation $\alpha_*: T\tilde{M} \rightarrow T\tilde{M}$, etc., for the differential of $\alpha: \tilde{M} \rightarrow \tilde{M}$, etc. A field τ of tangent planes on \tilde{M} satisfying

$$\alpha_* \tau = \tau \quad \text{for } \alpha \in \pi \tag{*}$$

projects on M to a field $p_* \tau$ which is integrable if and only if τ is. The fields σ , σ^\perp are lifted up by p to fields $p^* \sigma$, $p^* \sigma^\perp$ on \tilde{M} satisfying (*). Let us give \tilde{M} the Riemannian metric pulled back from M by p ; then $p^* \sigma^\perp = (p^* \sigma)^\perp$.

The universal covering $p: \tilde{M} \rightarrow M$ is a principal bundle over M with group π . The bundle σ^\perp , having discrete structural group, is isomorphic to $\tilde{M} \times_{\pi} R^q$, the q -plane bundle associated to $p: \tilde{M} \rightarrow M$ by a representation $r: \pi \rightarrow O(q)$. (Compare [3], Lemma 1.) To simplify notation, let $r_\alpha = r(\alpha)$. As usual, we construct $\tilde{M} \times_{\pi} R^q$ by dividing $\tilde{M} \times R^q$ by the relation $(x, y) \sim (\alpha(x), r_\alpha(y))$, for $\alpha \in \pi$. Let $\varphi: \sigma^\perp \rightarrow \tilde{M} \times_{\pi} R^q$ be the isomorphism, and $\tilde{p}: \tilde{M} \times R^q \rightarrow \tilde{M} \times_{\pi} R^q$ the canonical projection.

There exists a unique trivialisation $\Phi: p^* \sigma^\perp \rightarrow \tilde{M} \times R^q$ making the following square of bundle maps commute, as can easily be verified.

$$\begin{array}{ccc} p^* \sigma^\perp & \xrightarrow{\Phi} & \tilde{M} \times R^q \\ p_* \downarrow & & \downarrow \tilde{p} \\ \sigma^\perp & \xrightarrow{\varphi} & \tilde{M} \times_{\pi} R^q \end{array}$$

The reason for defining Φ is to obtain the tangent bundle map $H: T\tilde{M} \rightarrow TR^q$ by the composition

$$T\tilde{M} \xrightarrow{\Phi} p^* \sigma^\perp \xrightarrow{\varphi} \tilde{M} \times R^q \xrightarrow{\text{proj.}} R^q = TR_0^q \subset TR^q$$

(the first map is orthogonal projection, and TR_0^q denotes the tangent space at the origin.) This map has kernel $\ker H = p^* \sigma$; furthermore it is easy to check that H is a π -equivariant epimorphism in the sense that $H \circ \alpha_* = (r_\alpha)_* \circ H$, for $\alpha \in \pi$, and that $\text{rank}(H) = q$.

LEMMA: *The map H is homotopic through π -equivariant epimorphisms to the differential f_* of a submersion $f: \tilde{M} \rightarrow R^q$.*

This lemma is proved in the next section. It yields the proof of the theorem, as follows. If H_t is the homotopy, $0 \leq t \leq 1$, then $\ker H_t$ defines a homotopy between $p^* \sigma$ and the k -plane field $\ker(f_*)$, which is tangent to the foliation given by the manifolds $\{f = \text{constant}\}$. By equivariance each $\ker H_t$ satisfies (*), so $p_* \ker H_t$, $0 \leq t \leq 1$, gives a homotopy between σ and the integrable field $p_* \ker(f_*)$.

3. Proof of Lemma

First, following [5], Corollary 1.2, realize M as $M = \bigcup_{i=1}^{\infty} U_i$, an expanding union of compact manifolds with boundary, such that U_0 is an n -disc and either a) U_{i+1} retracts into U_i through embeddings which leave U_{i-1} fixed, or b) U_{i+1} is U_i with a handle of index $\leq n-1$ attached. We may assume that the $p^{-1} U_i$ give a similar decomposition for \tilde{M} ; of course now in case b) $p^{-1} U_{i+1}$ will be $p^{-1} U_i$ with one handle attached for each element of π . This can be guaranteed, for instance, by taking neighborhoods of the simplexes of an $(n-1)$ -dimensional spine of M (which exists by [9], Theorem 3.2) in a sufficiently fine triangulation.

For convenience in indexing, we will parametrize the homotopy by $[0, \infty]$. Let U_0^* be one component of $p^{-1}U_0$; then U_0^* is an n -disc and, by Lemma 2.1 of [5], $H|_{U_0^*}$ is homotopic through bundle epimorphisms to the differential f'_* of a submersion $f': U_0^* \rightarrow \mathbb{R}^q$. Let V_0 be an open set containing U_0^* and such that $V_0 \cap \alpha(V_0) = \emptyset$ for $\alpha \in \pi$, and let H'_t , $t \in [0, 1]$, be a homotopy, fixed outside V_0 , between H and a bundle epimorphism H'_1 equal to f'_* on U_0^* . Such a homotopy, through bundle epimorphisms, exists by [5], Lemma 5.1. A π -equivariant homotopy $H_t: T\tilde{M} \rightarrow T\mathbb{R}^q$, $0 \leq t \leq 1$, is now defined by $\tilde{H}_t = H$ outside $p^{-1}p(V_0)$, and $H_t|_{\alpha(x)} = (r_\alpha)_* \circ H'_t|_x \circ \alpha_*^{-1}$ for $x \in V_0$, and $\alpha \in \pi$. Also define $f_0: p^{-1}U_0 \rightarrow \mathbb{R}^q$ by $f_0 \circ \alpha(x) = r_\alpha \circ f'(x)$, for $x \in U_0^*$, $\alpha \in \pi$.

Induction hypothesis: between $t=0$ and $t=k$ we have deformed H through π -equivariant epimorphisms to a map H_k which over $p^{-1}U_k$ is the differential of a submersion f_k . Observe that the deformation between k and $k+1$ will leave this map fixed on $p^{-1}U_{k-2}$. The induction step will thus imply the existence of a well-defined homotopy, since for any $x \in p^{-1}U_k$, $H_t|_x = H_{k+2}|_x$ for $t \geq k+2$.

Proof of induction step: cases a) and b) must be distinguished. In case a) there is a homotopy h_t , $t \in [0, 1]$, of embeddings of U_{k+1} in itself, joining the identity map of U_{k+1} to $h_1: U_{k+1} \rightarrow U_k$, and such that each h_t is the identity on U_{k-2} . Covering this homotopy defines a unique similar homotopy \tilde{h}_t on $p^{-1}U_{k+1}$. Let $f_{k+1} = f_k \circ \tilde{h}_1$. Extend the homotopy $(h_t)_*: TU_{k+1} \rightarrow TU_{k+1}$ to a homotopy of bundle epimorphisms $L_t: TM \rightarrow TM$ (see [5], Lemma 5.1); cover this homotopy to define $\tilde{L}_t: T\tilde{M} \rightarrow T\tilde{M}$, and finally let $H_{k+1} = H_k \circ \tilde{L}_t$, for $t \in [0, 1]$.

In case b), let U_{k+1}^* be one of the components of $p^{-1}U_{k+1} - p^{-1}U_k$. It follows from the proof of [5], Lemma 6.2 that $H_k|_{U_{k+1}^*}$ is homotopic through epimorphisms fixed near $p^{-1}U_k$ to f'_* , where $f': U_{k+1}^* \rightarrow \mathbb{R}^q$ is a submersion extending f_k . Proceed as in the case $k=0$ to define f_{k+1} and a homotopy from H_k to H_{k+1} satisfying the induction hypothesis.

4. An Example

Consider the punctured projective space $P^6 - x$. This manifold is doubly covered by $S^5 \times \mathbb{R}$. On $S^5 \times \mathbb{R}$ the 2-plane field spanned by v (the pullback of a non-zero vectorfield on P^5 via the projections $S^5 \times \mathbb{R} \rightarrow S^5 \rightarrow P^5$) and $w = \partial/\partial t$ is invariant under the antipodal map and so defines a 2-plane field on $P^6 - x$. This bundle clearly has discrete structural group; it now follows that $P^6 - x$ has a 4-dimensional foliation.

The existence of such a foliation cannot be directly deduced from submersion theory, for the following argument shows there can be no submersion from $P^6 - x$ to a 2-dimensional manifold M . Such a submersion f would split $T(P^6 - x)$ as $\eta^2 \oplus \xi^4$, where $\eta = f^*TM$. Note first that for $i \leq 5$ the i^{th} Stiefel-Whitney class [4] of $P^6 - x$ is $w_i(P^6 - x) = a^i$, where a generates $H^1(P^6 - x; \mathbb{Z}_2) = \mathbb{Z}_2$. The equation $w_5(P^6 - x) = w_1\eta w_4\xi + w_2\eta w_3\xi$ shows that $w_1\eta$ and $w_2\eta$ cannot both be 0. Suppose $w_1\eta \neq 0$, that

is $w_1\eta = a$. This would imply $a^3 = w_1(f^*TM)^3 = f^*(w_1TM)^3 = 0$. Similarly $w_2\eta \neq 0$ is also impossible.

Finally, note that the 2-plane field spanned by v and w on $S^5 \times R$ is integrable, and thus projects to give an integrable 2-plane field on $P^6 - x$. This manifold therefore carries foliations of dimensions 1 (integrate a non-zero vectorfield), 2, 4 and 5 (see the Remarks in § 1). On the other hand, a straightforward argument with Stiefel-Whitney classes shows that there can exist no 3-plane field on $P^6 - x$.

REFERENCES

- [1] C. EHRESMANN, *Les connexions infinitesimales dans un espace fibré différentiable*, Colloque de Topologie du CBRM, Liège (1950), pp. 29–55.
- [2] A. HAEFLIGER, *Variétés feuilletées*, Annali della Scuola Normale Superiore, Pisa (3) 16 (1962), 367–397.
- [3] J. MILNOR, *On the existence of a connection with curvature 0*, Commentarii Math. Helv. 32 (1957), 215–223.
- [4] J. MILNOR, *Lectures on Characteristic Classes* (mimeo. notes) (Princeton 1958).
- [5] A. PHILLIPS, *Submersions of open manifolds*, Topology 6 (1967), 171–206.
- [6] A. PHILLIPS, *Foliations on open manifolds*, I, Commentarii Math. Helv. 43 (1968), 204–211.
- [7] G. REEB, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Scientifiques et Industrielles (Hermann, Paris 1952).
- [8] E. THOMAS, *Fields of tangent k-planes on manifolds*, Inventiones math. 3 (1967), 334–347.
- [9] J. H. C. WHITEHEAD, *The immersion of an open 3-manifold in Euclidean 3-space*, Proc. Lond. Phil. Soc. (3) 11 (1961), 81–90.

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