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Foliations on Open Manifolds, II

by Anthony Phillips

1. Introduction

Consider an n-dimensional smooth Riemannian manifold M which is open (i.e. has no compact components) and on M a field σ of tangent k-planes. This note gives a sufficient condition for σ to be homotopic to an integrable field. The condition is stated in terms of the complementary q-plane field $\sigma^{\perp}$ $(q=n-k)$, which we may consider as a q-dimensional subbundle of TM, the tangent bundle of M.

THEOREM: *If the structural group of $\sigma^{\perp}$ can be reduced to a discrete group, then σ is homotopic to an integrable field.*

REMARKS: This theorem was suggested by the following result of Ehresmann ([1, p. 38], [2, p. 364]): Let $N\subset M$ be an embedded submanifold; then N is a leaf of a foliation of a neighborhood of N in M if and only if the structural group of the normal bundle of N in M can be reduced to a discrete group. It gives a partial answer to a question posed by Reeb [7], Haefliger [2] and Thomas [8]. Since the one-dimensional orthogonal group is discrete, this gives a simpler proof of [6], Theorem 1.2: every $(n-1)$-plane field on M is homotopic to an integrable field. The restriction to open manifolds allows the use of submersion theory.

The proof of this theorem is given in the next two sections; the last section contains an example.

Early drafts of this note contain a much more restricted theorem. I am very grateful to André Haefliger for pointing out this generalisation.

2. Proof of Theorem

Let $\tilde{M}$ be the universal cover of M, with $p:\tilde{M}\to M$ the projection, and denote by $\alpha:\tilde{M}\to\tilde{M}$ the covering transformation corresponding to the element α of the fundamental group π of M. We will use the notation $\alpha_*:T\tilde{M}\to T\tilde{M}$, etc., for the differential of $\alpha:\tilde{M}\to\tilde{M}$, etc. A field τ of tangent planes on $\tilde{M}$ satisfying

$$\alpha_*\tau=\tau \quad \text{for} \quad \alpha\in\pi \tag{*}$$

projects on M to a field $p_*\tau$ which is integrable if and only if τ is. The fields σ, $\sigma^{\perp}$ are lifted up by p to fields $p^*\sigma$, $p^*\sigma^{\perp}$ on $\tilde{M}$ satisfying (*). Let us give $\tilde{M}$ the Riemannian metric pulled back from M by p; then $p^*\sigma^{\perp}=(p^*\sigma)^{\perp}$.

The universal covering $p:\tilde{M}\to M$ is a principal bundle over M with group π. The bundle $\sigma^\perp$, having discrete structural group, is isomorphic to $\tilde{M}\times_r R^q$, the q-plane bundle associated to $p:\tilde{M}\to M$ by a representation $r:\pi\to O(q)$. (Compare [3], Lemma 1.) To simplify notation, let $r_\alpha = r(\alpha)$. As usual, we construct $\tilde{M}\times_r R^q$ by dividing $\tilde{M}\times R^q$ by the relation $(x, y)\sim(\alpha(x), r_\alpha(y))$, for $\alpha\in\pi$. Let $\varphi:\sigma^\perp\to\tilde{M}\times_r R^q$ be the isomorphism, and $\tilde{p}:\tilde{M}\times R^q\to\tilde{M}\times_r R^q$ the canonical projection.

There exists a unique trivialisation $\Phi:p^*\sigma^\perp\to\tilde{M}\times R^q$ making the following square of bundle maps commute, as can easily be verified.

$$\begin{array}{ccc} p^*\sigma^\perp & \xrightarrow{\Phi} & \tilde{M}\times R^q \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ \sigma^\perp & \xrightarrow{\varphi} & \tilde{M}\times_r R^q \end{array}$$

The reason for defining Φ is to obtain the tangent bundle map $H:T\tilde{M}\to TR^q$ by the composition

$$T\tilde{M}\longrightarrow p^*\sigma^\perp \xrightarrow{\Phi} \tilde{M}\times R^q \xrightarrow{\text{proj.}} R^q = TR_0^q \subset TR^q$$

(the first map is orthogonal projection, and TR_0^q denotes the tangent space at the origin.) This map has kernel $\ker H = p^*\sigma$; furthermore it is easy to check that H is a π-equivariant epimorphism in the sense that $H\circ\alpha_* = (r_\alpha)_*\circ H$, for $\alpha\in\pi$, and that $\operatorname{rank}(H)=q$.

LEMMA: *The map H is homotopic through π-equivariant epimorphisms to the differential f_* of a submersion $f:\tilde{M}\to R^q$.*

This lemma is proved in the next section. It yields the proof of the theorem, as follows. If H_t is the homotopy, $0\leqslant t\leqslant 1$, then $\ker H_t$ defines a homotopy between $p^*\sigma$ and the k-plane field $\ker(f_*)$, which is tangent to the foliation given by the manifolds $\{f=\text{constant}\}$. By equivariance each $\ker H_t$ satisfies $(*)$, so $p_*\ker H_t$, $0\leqslant t\leqslant 1$, gives a homotopy between σ and the integrable field $p_*\ker(f_*)$.

3. Proof of Lemma

First, following [5], Corollary 1.2, realize M as $M=\bigcup_{i=1}^\infty U_i$, an expanding union of compact manifolds with boundary, such that U_0 is an n-disc and either a) U_{i+1} retracts into U_i through embeddings which leave U_{i-2} fixed, or b) U_{i+1} is U_i with a handle of index $\leqslant n-1$ attached. We may assume that the $p^{-1}U_i$ give a similar decomposition for $\tilde{M}$; of course now in case b) $p^{-1}U_{i+1}$ will be $p^{-1}U_i$ with one handle attached for each element of π. This can be guaranteed, for instance, by taking neighborhoods of the simplexes of an $(n-1)$-dimensional spine of M (which exists by [9], Theorem 3.2) in a sufficiently fine triangulation.

For convenience in indexing, we will parametrize the homotopy by $[0, \infty]$. Let U_0^* be one component of $p^{-1}U_0$; then U_0^* is an n-disc and, by Lemma 2.1 of [5], $H \mid U_0^*$ is homotopic through bundle epimorphisms to the differential f'_* of a submersion $f': U_0^* \to R^q$. Let V_0 be an open set containing U_0^* and such that $V_0 \cap \alpha(V_0) = \emptyset$ for $\alpha \in \pi$, and let H'_t, $t \in [0, 1]$, be a homotopy, fixed outside V_0, between H and a bundle epimorphism H'_1 equal to f'_* on U_0^*. Such a homotopy, through bundle epimorphisms, exists by [5], Lemma 5.1. A π-equivariant homotopy $H_t: T\tilde{M} \to TR^q$, $0 \leqslant t \leqslant 1$, is now defined by $\tilde{H}_t = H$ outside $p^{-1}p(V_0)$, and $H_t|_{\alpha(x)} = (r_\alpha)_* \circ H'_t|_x \circ \alpha_*^{-1}$ for $x \in V_0$, and $\alpha \in \pi$. Also define $f_0: p^{-1}U_0 \to R^q$ by $f_0 \circ \alpha(x) = r_\alpha \circ f'(x)$, for $x \in U_0^*$, $\alpha \in \pi$.

Induction hypothesis: between $t=0$ and $t=k$ we have deformed H through π-equivariant epimorphisms to a map H_k which over $p^{-1}U_k$ is the differential of a submersion f_k. Observe that the deformation between k and $k+1$ will leave this map fixed on $p^{-1}U_{k-2}$. The induction step will thus imply the existence of a well-defined homotopy, since for any $x \in p^{-1}U_k$, $H_t|_x = H_{k+2}|_x$ for $t \geqslant k+2$.

Proof of induction step: cases a) and b) must be distinguished. In case a) there is a homotopy h_t, $t \in [0, 1]$, of embeddings of U_{k+1} in itself, joining the identity map of U_{k+1} to $h_1: U_{k+1} \to U_k$, and such that each h_t is the identity on U_{k-2}. Covering this homotopy defines a unique similar homotopy $\tilde{h}_t$ on $p^{-1}U_{k+1}$. Let $f_{k+1} = f_k \circ \tilde{h}_1$. Extend the homotopy $(h_t)_*: TU_{k+1} \to TU_{k+1}$ to a homotopy of bundle epimorphisms $L_t: TM \to TM$ (see [5], Lemma 5.1); cover this homotopy to define $\tilde{L}_t: T\tilde{M} \to T\tilde{M}$, and finally let $H_{k+t} = H_k \circ \tilde{L}_t$, for $t \in [0, 1]$.

In case b), let U_{k+1}^* be one of the components of $p^{-1}U_{k+1} - p^{-1}U_k$. It follows from the proof of [5], Lemma 6.2 that $H_k \mid U_{k+1}^*$ is homotopic through epimorphisms fixed near $p^{-1}U_k$ to f'_*, where $f': U_{k+1}^* \to R^q$ is a submersion extending f_k. Proceed as in the case $k=0$ to define f_{k+1} and a homotopy from H_k to H_{k+1} satisfying the induction hypothesis.

4. An Example

Consider the punctured projective space $P^6 - x$. This manifold is doubly covered by $S^5 \times R$. On $S^5 \times R$ the 2-plane field spanned by v (the pullback of a non-zero vectorfield on P^5 via the projections $S^5 \times R \to S^5 \to P^5$) and $w = \partial/\partial t$ is invariant under the antipodal map and so defines a 2-plane field on $P^6 - x$. This bundle clearly has discrete structural group; it now follows that $P^6 - x$ has a 4-dimensional foliation.

The existence of such a foliation cannot be directly deduced from submersion theory, for the following argument shows there can be no submersion from $P^6 - x$ to a 2-dimensional manifold M. Such a submersion f would split $T(P^6 - x)$ as $\eta^2 \oplus \xi^4$, where $\eta = f^* TM$. Note first that for $i \leqslant 5$ the i^{th} Stiefel-Whitney class [4] of $P^6 - x$ is $w_i(P^6 - x) = a^i$, where a generates $H^1(P^6 - x; Z_2) = Z_2$. The equation $w_5(P^6 - x) = w_1\eta\, w_4\xi + w_2\eta\, w_3\xi$ shows that $w_1\eta$ and $w_2\eta$ cannot both be 0. Suppose $w_1\eta \neq 0$, that is $w_1\eta = a$. This would imply $a^3 = w_1(f^*TM)^3 = f^*(w_1TM)^3 = 0$. Similarly $w_2\eta \neq 0$ is also impossible.

Finally, note that the 2-plane field spanned by v and w on $S^5 \times R$ is integrable, and thus projects to give an integrable 2-plane field on $P^6 - x$. This manifold therefore carries foliations of dimensions 1 (integrate a non-zero vectorfield), 2, 4 and 5 (see the Remarks in § 1). On the other hand, a straightforward argument with Stiefel-Whitney classes shows that there can exist no 3-plane field on $P^6 - x$.

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