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A Note on the Fundamental Theorem of Projective Geometry

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Introduction

The aim of this note is to prove a generalisation to commutative rings of the classical fundamental theorem of projective geometry. In § 1, we introduce the notions of projective spaces and projectivities. In § 2, we prove the main theorem. The method of proof is similar to the proof of the theorem in the classical case as found for example in ARTIN [1]. The proof, as in the classical case, is elementary, but is trickier. In § 3, we give an example to show that a bijection between projective spaces of the same dimension which preserves collinear points is not necessarily a projectivity. This is in contrast to what happens in the case of projective spaces over fields.

§ 1. Projective Spaces and Projectivities

Let A be a commutative ring with 1 and let M be a free A -module. Let $P(M)$ denote the set of all A -free direct summands of rank 1 of M . This set is called the *projective space associated to M* . Clearly, any element of $P(M)$ is of the form Ae where e is a unimodular element of M , i.e. there exists a linear form $g: M \rightarrow A$ with $g(e)=1$. If (e_1, \dots, e_n) is a basis for the A -module M and $e = \sum a_i e_i$, then we note that e is unimodular if and only if $\sum_{1 \leq i \leq n} A e_i = A$. If the ring A is such that every projective module of rank 1 is free, then $P(M)$ coincides with the usual projective space of algebraic geometry [2, p. 13].

DEFINITION. Let M and N be free modules over commutative rings A and B respectively. A map $\alpha: P(M) \rightarrow P(N)$ is called a *projectivity* if α is bijective and for $p_1, p_2, p_3 \in P(M)$, we have $\alpha p_1 \subset \alpha p_2 + \alpha p_3$ in N if and only if $p_1 \subset p_2 + p_3$ in M .

This definition generalises the classical notion of projectivity between projective spaces over fields.

We note that by the very definition, $\alpha^{-1}: P(N) \rightarrow P(M)$ is also a projectivity. For later purposes, we need the following

LEMMA 1. With the notation above, if e_1, \dots, e_n is a basis of M and $e \in M$ a unimodular element such that $Ae \subset \sum_{1 \leq i \leq k} A e_i$, then $\alpha Ae \subset \sum_{1 \leq i \leq k} \alpha A e_i$.

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Proof. We prove the lemma by induction on k . Let $e = \sum_{1 \leq i \leq k} a_i e_i$. Then $e' = \sum_{1 \leq i \leq k-1} a_i e_i + e_k$ is unimodular and $Ae \subset Ae' + Ae_k$. By definition this implies that $\alpha Ae \subset \alpha Ae' + \alpha Ae_k$. Let $e'' = \sum_{1 \leq i \leq k-2} a_i e_i + e_k$. Since $e' \in Ae'' + Ae_{k-1}$, we again have $\alpha Ae' \subset \alpha Ae'' + \alpha Ae_{k-1}$. We thus have $\alpha Ae \subset \alpha Ae'' + \alpha Ae_{k-1} + \alpha Ae_k$. By induction,

$$\alpha Ae'' \subset \sum_{1 \leq i \leq k-2} \alpha Ae_i + \alpha Ae_k \quad \text{and hence} \quad \alpha Ae \subset \sum_{1 \leq i \leq k} \alpha Ae_i.$$

Let A and B be rings and $\sigma: A \rightarrow B$ a homomorphism. If M and N are modules over A and B respectively, then a map $\Phi: M \rightarrow N$ is called σ -semilinear if Φ is additive and $\Phi(am) = \sigma(a) \Phi(m)$ for all $a \in A, m \in M$. If M and N are free modules over A and B of the same rank and $\Phi: M \rightarrow N$ a σ -semilinear map which takes a basis (e_1, \dots, e_n) of M into a basis of N , then if $e = \sum a_i e_i$ is a unimodular element of M , then $\Phi(e) = \sum \sigma(a_i) \Phi(e_i)$ is unimodular in N . For, if $\sum \lambda_i a_i = 1, \lambda_i \in A$, we have $\sum \sigma(\lambda_i) \sigma(a_i) = 1$ which implies $\Phi(e) = \sum \sigma(a_i) \Phi(e_i)$ is unimodular. It is clear that we have an induced map $P(\Phi): P(M) \rightarrow P(N)$ by setting for any unimodular element e of M , $P(\Phi)(Ae) = B\Phi(e)$. We then have the following rather obvious

PROPOSITION 1: *With the same notation as above, for any $p_1, p_2, p_3 \in P(M)$ with $p_1 \subset p_2 + p_3$, $P(\Phi)p_1 \subset P(\Phi)p_2 + P(\Phi)p_3$. If σ is an isomorphism, then $P(\Phi)$ is a projectivity.*

§ 2 The Theorem

Our object in this section is to prove the following theorem which generalises to commutative rings the classical "Fundamental theorem of projective geometry".

THEOREM. *Let M and N be free modules of finite rank ≥ 3 over commutative rings A and B respectively. If $\alpha: P(M) \rightarrow P(N)$ is a projectivity, then there exists an isomorphism $\sigma: A \rightarrow B$ and a σ -semilinear isomorphism $\Phi: M \rightarrow N$ such that $\alpha = P(\Phi)$. If $\sigma_i: A \rightarrow B, i=1, 2$, are isomorphisms and $\Phi_i: M \rightarrow N$ are σ_i -semilinear isomorphisms such that $P(\Phi_1) = P(\Phi_2)$, then there exists a $b \in B$ such that $\Phi_1 = b \cdot \Phi_2$ and $\sigma_1 = \sigma_2$.*

Proof. Let e_1, \dots, e_n be a basis for M and let $\alpha Ae_i = Bf_i, 1 \leq i \leq n$. We assert that f_1, \dots, f_n generate the B -module N . Since any element of N is a linear combination of elements of a basis for N , it is enough to check that any unimodular element $f \in N$ is a linear combination of f_1, \dots, f_n . If $e \in M$ is a unimodular element with $\alpha Ae = Bf$ and $e = \sum_{1 \leq i \leq n} a_i e_i$, we have $Ae \subset \sum_{1 \leq i \leq n} Ae_i$ and by lemma 1, we get $Bf \subset \sum_{1 \leq i \leq n} Bf_i$.

This proves that f_1, \dots, f_n generate N . Since B is a commutative ring, this implies that $\text{rank } N \leq n$. Since α^{-1} is also a projectivity, it follows that $\text{rank } M = \text{rank } N$ and f_1, \dots, f_n is a basis for N .

Let $\alpha A e_1 = B f_1$ and $\alpha A e_2 = B g_2$. Now $e_1 + e_2$ is unimodular and $A(e_1 + e_2) \subset A e_1 + A e_2$ which implies that $\alpha A(e_1 + e_2) \subset B f_1 + B g_2$. Hence $\alpha A(e_1 + e_2) = B(b_1 f_1 + b_2 g_2)$. Since $A e_2 \subset A e_1 + A(e_1 + e_2)$ we have $B g_2 \subset B f_1 + B(b_1 f_1 + b_2 g_2)$. Thus $g_2 = b f_1 + c(b_1 f_1 + b_2 g_2)$. Since f_1, g_2 are independent, it follows that $c b_2 = 1$, i.e. b_2 is a unit in B . Similarly b_1 is also a unit. Writing $f_2 = b_1^{-1} b_2 g_2$, we see that f_2 is unimodular, $B f_2 = B g_2$ and $\alpha A(e_1 + e_2) = B(f_1 + f_2)$. Doing this for any $i > 1$, we get a basis f_1, f_2, \dots, f_n of N such that

$$\begin{aligned} \alpha A e_i &= B f_i & 1 \leq i \leq n \\ \alpha A(e_1 + e_i) &= B(f_1 + f_i) & 2 \leq i \leq n. \end{aligned} \quad (1)$$

It is clear as before that for any $a \in A$ $\alpha A(e_1 + a e_2) = B(b_1 f_1 + b_2 f_2)$ with b_1 a unit of B . Thus we can write

$$\alpha A(e_1 + a e_2) = B(f_1 + \sigma(a) f_2), \quad (2)$$

where $\sigma: A \rightarrow B$ is a well defined map. Clearly

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(1) = 1. \quad (3)$$

For any fixed $i > 2$, we can similarly define $\tau: A \rightarrow B$ by

$$\alpha A(e_1 + a e_i) = B(f_1 + \tau(a) f_i) \quad (4)$$

and we have

$$\tau(0) = 0 \quad \text{and} \quad \tau(1) = 1. \quad (5)$$

Since $e_1 + a e_2 + a' e_i \in A(e_1 + a e_2) + A e_i$, we have $\alpha A(e_1 + a e_2 + a' e_i) \subset B(f_1 + \sigma(a) f_2) + B f_i$. Hence $\alpha A(e_1 + a e_2 + a' e_i) = B(b(f_1 + \sigma(a) f_2) + b' f_i)$. Similarly, $\alpha A(e_1 + a e_2 + a' e_i) = B(c(f_1 + \tau(a') f_i) + c' f_2)$.

Combining the above equations, we find that

$$\alpha A(e_1 + a e_2 + a' e_i) = B(f_1 + \sigma(a) f_2 + \tau(a') f_i). \quad (6)$$

Since $a e_2 + e_i \in A(e_1 + a e_2 + e_i) + A e_1$, using (6) and (5) we have $\alpha A(a e_2 + e_i) = B(b(f_1 + \sigma(a) f_2 + f_i) + c f_1)$. Since $\alpha A(a e_2 + e_i) \subset B f_2 + B f_i$, we get $b + c = 0$ and this proves

$$A(a e_2 + e_i) = B(\sigma(a) f_2 + f_i). \quad (7)$$

Now using (6) and (5), we have for $a, a' \in A$, $\alpha A(e_1 + (a + a') e_2 + e_i) = B(f_1 + \sigma(a + a') f_2 + f_i)$. But $\alpha A(e_1 + (a + a') e_2 + e_i) \subset \alpha A(e_1 + a e_2) + \alpha A(a' e_2 + e_i)$. Using (7), we therefore have $\alpha A(e_1 + (a + a') e_2 + e_i) \subset B(f_1 + \sigma(a) f_2) + B(\sigma(a') f_2 + f_i)$. Using the above, we see that for $a, a' \in A$, we have

$$\sigma(a + a') = \sigma(a) + \sigma(a'). \quad (8)$$

Now for $a, a' \in A$, we have, using (6), that $\alpha A(e_1 + a a' e_2 + a e_i) = B(f_1 + \sigma(a a') f_2 +$

$+\tau(a)f_i)$. On the other hand, $\alpha A(e_1 + aa'e_2 + ae_i) \subset \alpha A e_1 + \alpha A(a'e_2 + e_i)$ which implies that $\alpha A(e_1 + aa'e_2 + ae_i) = B(bf_1 + b'(\sigma(a')f_2 + f_i))$. Comparing coefficients, we find that $\sigma(aa') = \tau(a)\sigma(a')$. Setting $a' = 1$, we get

$$\sigma(a) = \tau(a) \quad \text{for all } a \in A \quad (9)$$

and

$$\sigma(aa') = \sigma(a)\sigma(a') \quad \text{for } a, a' \in A. \quad (10)$$

Thus, the map $\sigma: A \rightarrow B$ defined by (2) is a homomorphism. Replacing α by α^{-1} , we can define a homomorphism $\sigma': B \rightarrow A$ satisfying

$$\alpha^{-1} B(f_1 + bf_2) = A(e_1 + \sigma'(b)e_2)$$

and clearly σ and σ' are inverses of each other. Thus $\sigma: A \rightarrow B$ is an isomorphism.

We now show that, for $a_2, \dots, a_n \in A$, we have

$$\alpha A(e_1 + a_2 e_2 + \dots + a_n e_n) = B(f_1 + \sigma(a_2)f_2 + \dots + \sigma(a_n)f_n). \quad (11)$$

We can assume by induction that

$$\alpha A(e_1 + a_2 e_2 + \dots + a_{n-1} e_{n-1}) = B(f_1 + \sigma(a_2)f_2 + \dots + \sigma(a_{n-1})f_{n-1}).$$

Since

$$\alpha A(e_1 + a_2 e_2 + \dots + a_n e_n) \subset \alpha A(e_1 + a_2 e_2 + \dots + a_{n-1} e_{n-1}) + \alpha A e_n,$$

we have

$$\alpha A(e_1 + a_2 e_2 + \dots + a_n e_n) = B(b(f_1 + \sigma(a_2)f_2 + \dots + \sigma(a_{n-1})f_{n-1}) + b'f_n).$$

On the other hand, we also have

$$\alpha A(e_1 + a_2 e_2 + \dots + a_n e_n) \subset \alpha A(e_1 + a_n e_n) + \alpha A e_2 + \dots + \alpha A e_{n-1}.$$

Comparing coefficients we find that $b' = b\sigma(a_n)$ and this proves (11).

If $a_2, \dots, a_n \in A$ are such that $a_2 e_2 + \dots + a_n e_n \in M$ is unimodular, we have

$$\alpha A(a_2 e_2 + \dots + a_n e_n) \subset A(e_1 + a_2 e_2 + \dots + a_n e_n) + \alpha A e_1.$$

Using (11) we have

$$\alpha A(a_2 e_2 + \dots + a_n e_n) = B(b(f_1 + \sigma(a_2)f_2 + \dots + \sigma(a_n)f_n) + b'f_1).$$

We also have

$$\alpha A(a_2 e_2 + \dots + a_n e_n) \subset Bf_2 + \dots + Bf_n.$$

Combining these two facts, we get

$$\alpha A(a_2 e_2 + \dots + a_n e_n) = B(\sigma(a_2)f_2 + \dots + \sigma(a_n)f_n). \quad (12)$$

We now assert that for any $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$ and $i = 2, \dots, n$,

$$\begin{aligned} \alpha A(e_i + a_1 e_1 + \dots + a_{i-1} e_{i-1} + a_{i+1} e_{i+1} + \dots + a_n e_n) \\ = B(f_i + \sigma(a_1)f_1 + \dots + \sigma(a_n)f_n). \end{aligned} \quad (13)$$

To prove (13), we first observe, using (1) and (12) that $\alpha A(e_i + e_j) = B(f_i + f_j)$ for any $j \neq i$. Fixing an i and replacing e_1 by e_i , we can repeat the previous arguments to get an isomorphism $\varrho: A \rightarrow B$ such that for $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$, we have the following equation:

$$\alpha A(e_i + a_1 e_1 + \dots + a_{i-1} e_{i-1} + a_{i+1} e_{i+1} + \dots + a_n e_n) = B(f_i + \varrho(a_1) f_1 + \dots + \varrho(a_n) f_n). \quad (14)$$

instead of (11).

Taking in (14) $a_1 = 0$ and comparing this equation with (12), we find that $\sigma = \varrho$. Now (14) gives (13).

Let $e = \sum_{1 \leq i \leq n} a_i e_i \in M$ be a unimodular element. We now show that

$$\alpha A(a_1 e_1 + \dots + a_n e_n) = B(\sigma(a_1) f_1 + \dots + \sigma(a_n) f_n). \quad (15)$$

Since for $i = 1, 2, 3$, we have $\alpha A e \subset \alpha A e_i + \alpha A(e_i + \dots + \widehat{a_i e_i} + \dots)$ (where $\widehat{}$ indicates that the corresponding term is omitted), we can write $\alpha A e = B f$ where

$$\begin{aligned} f &= b_1 \sigma(a_1) f_1 + c_1 \sigma(a_2) f_2 + c_1 \sigma(a_3) f_3 + \dots \\ &= c_2 \sigma(a_1) f_1 + b_2 \sigma(a_2) f_2 + c_2 \sigma(a_3) f_3 + \dots \\ &= c_3 \sigma(a_1) f_1 + c_3 \sigma(a_2) f_2 + b_3 \sigma(a_3) f_3 + \dots \end{aligned}$$

Comparing coefficients, we find

$$\left. \begin{aligned} b_1 \sigma(a_1) \sigma(a_2) &= c_3 \sigma(a_1) \sigma(a_2) = c_1 \sigma(a_1) \sigma(a_2) \\ \text{and for every } i \geq 3, \text{ we have} \\ b_1 \sigma(a_1) \sigma(a_i) &= c_2 \sigma(a_1) \sigma(a_i) = c_1 \sigma(a_1) \sigma(a_i). \end{aligned} \right\} \quad (16)$$

Since $e = \sum a_i e_i$ is unimodular, it follows that $\sum \sigma(a_i) f_i$ is unimodular and hence there exist $k_1, \dots, k_n \in B$ such that $\sum \sigma(a_i) k_i = 1$. Set

$$d = b_1 \sigma(a_1) k_1 + c_1 \sigma(a_2) k_2 + \dots + c_1 \sigma(a_n) k_n.$$

Using the equations (16), we easily verify that $d \sigma(a_1) = b_1 \sigma(a_1)$ and $d \sigma(a_i) = c_1 \sigma(a_i)$ for $i \geq 2$. Then d is a unit and (15) is proved.

Let $\Phi: M \rightarrow N$ be the σ -semilinear isomorphism $M \rightarrow N$ defined by $\Phi(e_i) = f_i$. The equation (15) shows that $\alpha = P(\Phi)$. The proof of the second statement of the theorem is the same as in the classical case which can be found for instance in E. ARTIN [1, chap. II].

§ 3 A Counter-Example

If M, N are finite dimensional vector spaces of the same rank over fields A and B respectively and if $\alpha: P(M) \rightarrow P(N)$ is a bijection which is such that for any $p_1, p_2, p_3 \in P(M)$ with $p_1 \subset p_2 + p_3$, we have $\alpha p_1 \subset \alpha p_2 + \alpha p_3$, it can be proved (see for instance Artin [1, chap. II]) that α is a projectivity. We now give an example to show that this need not be the case if A and B are arbitrary rings.

Let K be a field; let $A = K\langle x \rangle$ be the ring of formal power series in x and B the

quotient field of A . The canonical inclusion $\sigma: A \rightarrow B$ induces a σ -semilinear map $A^3 \rightarrow B^3$ which in turn gives rise to a map $P(\sigma): P(A^3) \rightarrow P(B^3)$.

PROPOSITION 2.*) *The map $P(\sigma)$ is a bijection such that for any $p_1, p_2, p_3 \in P(A^3)$ with $p_1 \subset p_2 + p_3$, we have $P(\sigma) p_1 \subset P(\sigma) p_2 + P(\sigma) p_3$. However $P(\sigma)$ is not a projectivity.*

Proof. Let $(a_1, a_2, a_3), (a'_1, a'_2, a'_3)$ be unimodular elements of A^3 which represent the same element of $P(B^3)$. We then have $a, a' \in A, a \neq 0, a' \neq 0$ such that $a'(a'_1, a'_2, a'_3) = a(a_1, a_2, a_3)$, i.e. $a'a'_i = aa_i, 1 \leq i \leq 3$. If $\sum_{1 \leq i \leq 3} a_i k_i = 1$, we have $a'\lambda = a$ with $\lambda = \sum a_i k_i A$. Similarly, $a\mu = a'$ for some $\mu \in A$. This implies that a and a' differ by a unit of A and hence $A(a_1, a_2, a_3) = A(a'_1, a'_2, a'_3)$. This proves that $P(\sigma)$ is injective. Given any element of $P(B^3)$, we can write it in the form Be where $e \in A^3$. Dividing if necessary by a suitable power of x , we may assume that at least one coordinate of e has a nonzero constant term and hence is a unit in A . Therefore we may assume that e is a unimodular element of A^3 and this proves that $P(\sigma)$ is surjective. If $p_1, p_2, p_3 \in P(A^3)$ are such that $p_1 \subset p_2 + p_3$, it is trivial to check that $P(\sigma) p_1 \subset P(\sigma) p_2 + P(\sigma) p_3$. Now, $P(\sigma) A(1, 0, 0) = B(1, 0, 0) = B(x, 0, 0) \subset P(\sigma) A(x, 1, 0) + P(\sigma) A(0, 1, 0)$. However, $(1, 0, 0) \notin A(x, 1, 0) + A(0, 1, 0)$. This shows that $P(\sigma)$ is not a projectivity.

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*) (Added in proof.) This proposition and its proof are valid equally for any unique factorisation domain.