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# A Note on the Fundamental Theorem of Projective Geometry

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## Introduction

The aim of this note is to prove a generalisation to commutative rings of the classical fundamental theorem of projective geometry. In § 1, we introduce the notions of projective spaces and projectivities. In § 2, we prove the main theorem. The method of proof is similar to the proof of the theorem in the classical case as found for example in ARTIN [1]. The proof, as in the classical case, is elementary, but is trickier. In § 3, we give an example to show that a bijection between projective spaces of the same dimension which preserves collinear points is not necessarily a projectivity. This is in contrast to what happens in the case of projective spaces over fields.

## § 1. Projective Spaces and Projectivities

Let  $A$  be a commutative ring with 1 and let  $M$  be a free  $A$ -module. Let  $P(M)$  denote the set of all  $A$ -free direct summands of rank 1 of  $M$ . This set is called the *projective space associated to  $M$* . Clearly, any element of  $P(M)$  is of the form  $Ae$  where  $e$  is a unimodular element of  $M$ , i.e. there exists a linear form  $g: M \rightarrow A$  with  $g(e) = 1$ . If  $(e_1, \dots, e_n)$  is a basis for the  $A$ -module  $M$  and  $e = \sum a_i e_i$ , then we note that  $e$  is unimodular if and only if  $\sum_{1 \leq i \leq n} A e_i = A$ . If the ring  $A$  is such that every projective module of rank 1 is free, then  $P(M)$  coincides with the usual projective space of algebraic geometry [2, p. 13].

**DEFINITION.** Let  $M$  and  $N$  be free modules over commutative rings  $A$  and  $B$  respectively. A map  $\alpha: P(M) \rightarrow P(N)$  is called a *projectivity* if  $\alpha$  is bijective and for  $p_1, p_2, p_3 \in P(M)$ , we have  $\alpha p_1 \subset \alpha p_2 + \alpha p_3$  in  $N$  if and only if  $p_1 \subset p_2 + p_3$  in  $M$ .

This definition generalises the classical notion of projectivity between projective spaces over fields.

We note that by the very definition,  $\alpha^{-1}: P(N) \rightarrow P(M)$  is also a projectivity. For later purposes, we need the following

**LEMMA 1.** *With the notation above, if  $e_1, \dots, e_n$  is a basis of  $M$  and  $e \in M$  a unimodular element such that  $Ae \subset \sum_{1 \leq i \leq k} A e_i$ , then  $\alpha Ae \subset \sum_{1 \leq i \leq k} \alpha A e_i$ .*

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*Proof.* We prove the lemma by induction on  $k$ . Let  $e = \sum_{1 \leq i \leq k} a_i e_i$ . Then  $e' = \sum_{1 \leq i \leq k-1} a_i e_i + e_k$  is unimodular and  $Ae \subset Ae' + Ae_k$ . By definition this implies that  $\alpha Ae \subset \alpha Ae' + \alpha Ae_k$ . Let  $e'' = \sum_{1 \leq i \leq k-2} a_i e_i + e_k$ . Since  $e' \in Ae'' + Ae_{k-1}$ , we again have  $\alpha Ae' \subset \alpha Ae'' + \alpha Ae_{k-1}$ . We thus have  $\alpha Ae \subset \alpha Ae'' + \alpha Ae_{k-1} + \alpha Ae_k$ . By induction,

$$\alpha Ae'' \subset \sum_{1 \leq i \leq k-2} \alpha Ae_i + \alpha Ae_k \quad \text{and hence} \quad \alpha Ae \subset \sum_{1 \leq i \leq k} \alpha Ae_i.$$

Let  $A$  and  $B$  be rings and  $\sigma: A \rightarrow B$  a homomorphism. If  $M$  and  $N$  are modules over  $A$  and  $B$  respectively, then a map  $\Phi: M \rightarrow N$  is called  $\sigma$ -semilinear if  $\Phi$  is additive and  $\Phi(am) = \sigma(a)\Phi(m)$  for all  $a \in A, m \in M$ . If  $M$  and  $N$  are free modules over  $A$  and  $B$  of the same rank and  $\Phi: M \rightarrow N$  a  $\sigma$ -semilinear map which takes a basis  $(e_1, \dots, e_n)$  of  $M$  into a basis of  $N$ , then if  $e = \sum a_i e_i$  is a unimodular element of  $M$ , then  $\Phi(e) = \sum \sigma(a_i)\Phi(e_i)$  is unimodular in  $N$ . For, if  $\sum \lambda_i a_i = 1, \lambda_i \in A$ , we have  $\sum \sigma(\lambda_i)\sigma(a_i) = 1$  which implies  $\Phi(e) = \sum \sigma(a_i)\Phi(e_i)$  is unimodular. It is clear that we have an induced map  $P(\Phi): P(M) \rightarrow P(N)$  by setting for any unimodular element  $e$  of  $M, P(\Phi)(Ae) = B\Phi(e)$ . We then have the following rather obvious

**PROPOSITION 1:** *With the same notation as above, for any  $p_1, p_2, p_3 \in P(M)$  with  $p_1 \subset p_2 + p_3, P(\Phi)p_1 \subset P(\Phi)p_2 + P(\Phi)p_3$ . If  $\sigma$  is an isomorphism, then  $P(\Phi)$  is a projectivity.*

### § 2 The Theorem

Our object in this section is to prove the following theorem which generalises to commutative rings the classical ‘‘Fundamental theorem of projective geometry’’.

**THEOREM.** *Let  $M$  and  $N$  be free modules of finite rank  $\geq 3$  over commutative rings  $A$  and  $B$  respectively. If  $\alpha: P(M) \rightarrow P(N)$  is a projectivity, then there exists an isomorphism  $\sigma: A \rightarrow B$  and a  $\sigma$ -semilinear isomorphism  $\Phi: M \rightarrow N$  such that  $\alpha = P(\Phi)$ . If  $\sigma_i: A \rightarrow B, i = 1, 2,$  are isomorphisms and  $\Phi_i: M \rightarrow N$  are  $\sigma_i$ -semilinear isomorphisms such that  $P(\Phi_1) = P(\Phi_2)$ , then there exists a  $b \in B$  such that  $\Phi_1 = b \cdot \Phi_2$  and  $\sigma_1 = \sigma_2$ .*

*Proof.* Let  $e_1, \dots, e_n$  be a basis for  $M$  and let  $\alpha Ae_i = Bf_i, 1 \leq i \leq n$ . We assert that  $f_1, \dots, f_n$  generate the  $B$ -module  $N$ . Since any element of  $N$  is a linear combination of elements of a basis for  $N$ , it is enough to check that any unimodular element  $f \in N$  is a linear combination of  $f_1, \dots, f_n$ . If  $e \in M$  is a unimodular element with  $\alpha Ae = Bf$  and  $e = \sum_{1 \leq i \leq n} a_i e_i$ , we have  $Ae \subset \sum_{1 \leq i \leq n} Ae_i$  and by lemma 1, we get  $Bf \subset \sum_{1 \leq i \leq n} Bf_i$ .

This proves that  $f_1, \dots, f_n$  generate  $N$ . Since  $B$  is a commutative ring, this implies that  $\text{rank } N \leq n$ . Since  $\alpha^{-1}$  is also a projectivity, it follows that  $\text{rank } M = \text{rank } N$  and  $f_1, \dots, f_n$  is a basis for  $N$ .

Let  $\alpha A e_1 = B f_1$  and  $\alpha A e_2 = B g_2$ . Now  $e_1 + e_2$  is unimodular and  $A(e_1 + e_2) \subset A e_1 + A e_2$  which implies that  $\alpha A(e_1 + e_2) \subset B f_1 + B g_2$ . Hence  $\alpha A(e_1 + e_2) = B(b_1 f_1 + b_2 g_2)$ . Since  $A e_2 \subset A e_1 + A(e_1 + e_2)$  we have  $B g_2 \subset B f_1 + B(b_1 f_1 + b_2 g_2)$ . Thus  $g_2 = b f_1 + c(b_1 f_1 + b_2 g_2)$ . Since  $f_1, g_2$  are independent, it follows that  $c b_2 = 1$ , i.e.  $b_2$  is a unit in  $B$ . Similarly  $b_1$  is also a unit. Writing  $f_2 = b_1^{-1} b_2 g_2$ , we see that  $f_2$  is unimodular,  $B f_2 = B g_2$  and  $\alpha A(e_1 + e_2) = B(f_1 + f_2)$ . Doing this for any  $i > 1$ , we get a basis  $f_1, f_2, \dots, f_n$  of  $N$  such that

$$\begin{aligned} \alpha A e_i &= B f_i & 1 \leq i \leq n \\ \alpha A(e_1 + e_i) &= B(f_1 + f_i) & 2 \leq i \leq n. \end{aligned} \quad (1)$$

It is clear as before that for any  $a \in A$   $\alpha A(e_1 + a e_2) = B(b_1 f_1 + b_2 f_2)$  with  $b_1$  a unit of  $B$ . Thus we can write

$$\alpha A(e_1 + a e_2) = B(f_1 + \sigma(a) f_2), \quad (2)$$

where  $\sigma: A \rightarrow B$  is a well defined map. Clearly

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(1) = 1. \quad (3)$$

For any fixed  $i > 2$ , we can similarly define  $\tau: A \rightarrow B$  by

$$\alpha A(e_1 + a e_i) = B(f_1 + \tau(a) f_i) \quad (4)$$

and we have

$$\tau(0) = 0 \quad \text{and} \quad \tau(1) = 1. \quad (5)$$

Since  $e_1 + a e_2 + a' e_i \in A(e_1 + a e_2) + A e_i$ , we have  $\alpha A(e_1 + a e_2 + a' e_i) \subset B(f_1 + \sigma(a) f_2) + B f_i$ . Hence  $\alpha A(e_1 + a e_2 + a' e_i) = B(b(f_1 + \sigma(a) f_2) + b' f_i)$ . Similarly,  $\alpha A(e_1 + a e_2 + a' e_i) = B(c(f_1 + \tau(a') f_i) + c' f_2)$ .

Combining the above equations, we find that

$$\alpha A(e_1 + a e_2 + a' e_i) = B(f_1 + \sigma(a) f_2 + \tau(a') f_i). \quad (6)$$

Since  $a e_2 + e_i \in A(e_1 + a e_2 + e_i) + A e_1$ , using (6) and (5) we have  $\alpha A(a e_2 + e_i) = B(b(f_1 + \sigma(a) f_2 + f_i) + c f_1)$ . Since  $\alpha A(a e_2 + e_i) \subset B f_2 + B f_i$ , we get  $b + c = 0$  and this proves

$$A(a e_2 + e_i) = B(\sigma(a) f_2 + f_i). \quad (7)$$

Now using (6) and (5), we have for  $a, a' \in A$ ,  $\alpha A(e_1 + (a + a') e_2 + e_i) = B(f_1 + \sigma(a + a') f_2 + f_i)$ . But  $\alpha A(e_1 + (a + a') e_2 + e_i) \subset \alpha A(e_1 + a e_2) + \alpha A(a' e_2 + e_i)$ . Using (7), we therefore have  $\alpha A(e_1 + (a + a') e_2 + e_i) \subset B(f_1 + \sigma(a) f_2) + B(\sigma(a') f_2 + f_i)$ . Using the above, we see that for  $a, a' \in A$ , we have

$$\sigma(a + a') = \sigma(a) + \sigma(a'). \quad (8)$$

Now for  $a, a' \in A$ , we have, using (6), that  $\alpha A(e_1 + a a' e_2 + a e_i) = B(f_1 + \sigma(a a') f_2 +$

$+\tau(a)f_i$ ). On the other hand,  $\alpha A(e_1+aa'e_2+ae_i)\subset\alpha Ae_1+\alpha A(a'e_2+e_i)$  which implies that  $\alpha A(e_1+aa'e_2+ae_i)=B(bf_1+b'(\sigma(a')f_2+f_i))$ . Comparing coefficients, we find that  $\sigma(aa')=\tau(a)\sigma(a')$ . Setting  $a'=1$ , we get

$$\sigma(a)=\tau(a) \quad \text{for all } a\in A \quad (9)$$

and

$$\sigma(aa')=\sigma(a)\sigma(a') \quad \text{for } a, a'\in A. \quad (10)$$

Thus, the map  $\sigma:A\rightarrow B$  defined by (2) is a homomorphism. Replacing  $\alpha$  by  $\alpha^{-1}$ , we can define a homomorphism  $\sigma':B\rightarrow A$  satisfying

$$\alpha^{-1}B(f_1+bf_2)=A(e_1+\sigma'(b)e_2)$$

and clearly  $\sigma$  and  $\sigma'$  are inverses of each other. Thus  $\sigma:A\rightarrow B$  is an isomorphism.

We now show that, for  $a_2, \dots, a_n \in A$ , we have

$$\alpha A(e_1+a_2e_2+\dots+a_n e_n)=B(f_1+\sigma(a_2)f_2+\dots+\sigma(a_n)f_n). \quad (11)$$

We can assume by induction that

$$\alpha A(e_1+a_2e_2+\dots+a_{n-1}e_{n-1})=B(f_1+\sigma(a_2)f_2+\dots+\sigma(a_{n-1})f_{n-1}).$$

Since

$$\alpha A(e_1+a_2e_2+\dots+a_n e_n)\subset\alpha A(e_1+a_2e_2+\dots+a_{n-1}e_{n-1})+\alpha Ae_n,$$

we have

$$\alpha A(e_1+a_2e_2+\dots+a_n e_n)=B(b(f_1+\sigma(a_2)f_2+\dots+\sigma(a_{n-1})f_{n-1})+b'f_n).$$

On the other hand, we also have

$$\alpha A(e_1+a_2e_2+\dots+a_n e_n)\subset\alpha A(e_1+a_n e_n)+\alpha Ae_2+\dots+\alpha Ae_{n-1}.$$

Comparing coefficients we find that  $b'=b\sigma(a_n)$  and this proves (11).

If  $a_2, \dots, a_n \in A$  are such that  $a_2e_2+\dots+a_n e_n \in M$  is unimodular, we have

$$\alpha A(a_2e_2+\dots+a_n e_n)\subset A(e_1+a_2e_2+\dots+a_n e_n)+\alpha Ae_1.$$

Using (11) we have

$$\alpha A(a_2e_2+\dots+a_n e_n)=B(b(f_1+\sigma(a_2)f_2+\dots+\sigma(a_n)f_n)+b'f_1).$$

We also have

$$\alpha A(a_2e_2+\dots+a_n e_n)\subset Bf_2+\dots+Bf_n.$$

Combining these two facts, we get

$$\alpha A(a_2e_2+\dots+a_n e_n)=B(\sigma(a_2)f_2+\dots+\sigma(a_n)f_n). \quad (12)$$

We now assert that for any  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$  and  $i=2, \dots, n$ ,

$$\begin{aligned} \alpha A(e_i+a_1e_1+\dots+a_{i-1}e_{i-1}+a_{i+1}e_{i+1}+\dots+a_n e_n) \\ = B(f_i+\sigma(a_1)f_1+\dots+\sigma(a_n)f_n). \end{aligned} \quad (13)$$

To prove (13), we first observe, using (1) and (12) that  $\alpha A(e_i + e_j) = B(f_i + f_j)$  for any  $j \neq i$ . Fixing an  $i$  and replacing  $e_1$  by  $e_i$ , we can repeat the previous arguments to get an isomorphism  $\varrho: A \rightarrow B$  such that for  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$ , we have the following equation:

$$\alpha A(e_i + a_1 e_1 + \dots + a_{i-1} e_{i-1} + a_{i+1} e_{i+1} + \dots + a_n e_n) = B(f_i + \varrho(a_1) f_1 + \dots + \varrho(a_n) f_n). \tag{14}$$

instead of (11).

Taking in (14)  $a_1 = 0$  and comparing this equation with (12), we find that  $\sigma = \varrho$ . Now (14) gives (13).

Let  $e = \sum_{1 \leq i \leq n} a_i e_i \in M$  be a unimodular element. We now show that

$$\alpha A(a_1 e_1 + \dots + a_n e_n) = B(\sigma(a_1) f_1 + \dots + \sigma(a_n) f_n). \tag{15}$$

Since for  $i = 1, 2, 3$ , we have  $\alpha A e \subset \alpha A e_i + \alpha A(e_i + \dots + \widehat{a_i e_i} + \dots)$  (where  $\widehat{\phantom{x}}$  indicates that the corresponding term is omitted), we can write  $\alpha A e = B f$  where

$$\begin{aligned} f &= b_1 \sigma(a_1) f_1 + c_1 \sigma(a_2) f_2 + c_1 \sigma(a_3) f_3 + \dots \\ &= c_2 \sigma(a_1) f_1 + b_2 \sigma(a_2) f_2 + c_2 \sigma(a_3) f_3 + \dots \\ &= c_3 \sigma(a_1) f_1 + c_3 \sigma(a_2) f_2 + b_3 \sigma(a_3) f_3 + \dots \end{aligned}$$

Comparing coefficients, we find

$$\left. \begin{aligned} b_1 \sigma(a_1) \sigma(a_2) &= c_3 \sigma(a_1) \sigma(a_2) = c_1 \sigma(a_1) \sigma(a_2) \\ \text{and for every } i \geq 3, \text{ we have} \\ b_1 \sigma(a_1) \sigma(a_i) &= c_2 \sigma(a_1) \sigma(a_i) = c_1 \sigma(a_1) \sigma(a_i). \end{aligned} \right\} \tag{16}$$

Since  $e = \sum a_i e_i$  is unimodular, it follows that  $\sum \sigma(a_i) f_i$  is unimodular and hence there exist  $k_1, \dots, k_n \in B$  such that  $\sum \sigma(a_i) k_i = 1$ . Set

$$d = b_1 \sigma(a_1) k_1 + c_1 \sigma(a_2) k_2 + \dots + c_1 \sigma(a_n) k_n.$$

Using the equations (16), we easily verify that  $d \sigma(a_1) = b_1 \sigma(a_1)$  and  $d \sigma(a_i) = c_1 \sigma(a_i)$  for  $i \geq 2$ . Then  $d$  is a unit and (15) is proved.

Let  $\Phi: M \rightarrow N$  be the  $\sigma$ -semilinear isomorphism  $M \rightarrow N$  defined by  $\Phi(e_i) = f_i$ . The equation (15) shows that  $\alpha = P(\Phi)$ . The proof of the second statement of the theorem is the same as in the classical case which can be found for instance in E. ARTIN [1, chap. II].

### § 3 A Counter-Example

If  $M, N$  are finite dimensional vector spaces of the same rank over fields  $A$  and  $B$  respectively and if  $\alpha: P(M) \rightarrow P(N)$  is a bijection which is such that for any  $p_1, p_2, p_3 \in P(M)$  with  $p_1 \subset p_2 + p_3$ , we have  $\alpha p_1 \subset \alpha p_2 + \alpha p_3$ , it can be proved (see for instance Artin [1, chap. II]) that  $\alpha$  is a projectivity. We now give an example to show that this need not be the case if  $A$  and  $B$  are arbitrary rings.

Let  $K$  be a field; let  $A = K\langle x \rangle$  be the ring of formal power series in  $x$  and  $B$  the

quotient field of  $A$ . The canonical inclusion  $\sigma:A\rightarrow B$  induces a  $\sigma$ -semilinear map  $A^3\rightarrow B^3$  which in turn gives rise to a map  $P(\sigma):P(A^3)\rightarrow P(B^3)$ .

**PROPOSITION 2.\*** *The map  $P(\sigma)$  is a bijection such that for any  $p_1, p_2, p_3\in P(A^3)$  with  $p_1\subset p_2+p_3$ , we have  $P(\sigma)p_1\subset P(\sigma)p_2+P(\sigma)p_3$ . However  $P(\sigma)$  is not a projectivity.*

*Proof.* Let  $(a_1, a_2, a_3), (a'_1, a'_2, a'_3)$  be unimodular elements of  $A^3$  which represent the same element of  $P(B^3)$ . We then have  $a, a'\in A, a\neq 0, a'\neq 0$  such that  $a'(a'_1, a'_2, a'_3)=a(a_1, a_2, a_3)$ , i.e.  $a'a'_i=aa_i, 1\leq i\leq 3$ . If  $\sum_{1\leq i\leq 3} a_i k_i=1$ , we have  $a'\lambda=a$  with  $\lambda=\sum a_i k_i A$ . Similarly,  $a\mu=a'$  for some  $\mu\in A$ . This implies that  $a$  and  $a'$  differ by a unit of  $A$  and hence  $A(a_1, a_2, a_3)=A(a'_1, a'_2, a'_3)$ . This proves that  $P(\sigma)$  is injective. Given any element of  $P(B^3)$ , we can write it in the form  $Be$  where  $e\in A^3$ . Dividing if necessary by a suitable power of  $x$ , we may assume that at least one coordinate of  $e$  has a nonzero constant term and hence is a unit in  $A$ . Therefore we may assume that  $e$  is a unimodular element of  $A^3$  and this proves that  $P(\sigma)$  is surjective. If  $p_1, p_2, p_3\in P(A^3)$  are such that  $p_1\subset p_2+p_3$ , it is trivial to check that  $P(\sigma)p_1\subset P(\sigma)p_2+P(\sigma)p_3$ . Now,  $P(\sigma)A(1, 0, 0)=B(1, 0, 0)=B(x, 0, 0)\subset P(\sigma)A(x, 1, 0)+P(\sigma)A(0, 1, 0)$ . However,  $(1, 0, 0)\notin A(x, 1, 0)+A(0, 1, 0)$ . This shows that  $P(\sigma)$  is not a projectivity.

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\*) (Added in proof.) This proposition and its proof are valid equally for any unique factorisation domain.