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Homotopy-associative H-spaces which are Sphere Bundles over Spheres

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1. Introduction

Let $S^q \to E \to S^n$ be a q-sphere bundle over the n-sphere (q, n positive). The following theorem is proven, first in [2], and then more briefly in [3].

THEOREM 1. *If E is an H-space, then either*
(i) *both q and n belong to the set* $\{1, 3, 7\}$, *or*
(ii) *the pair* (q, n) *is* $(1, 2)$, $(3, 4)$ *or* $(3, 5)$.

In this note we study the further restrictions imposed on the pair (q, n), in case E supports a homotopy-associative H-space multiplication. Precisely, our main result is the following.

THEOREM 2. *If E supports a homotopy-associative H-space multiplication, then either*
(i) *both q and n belong to the set* $\{1, 3\}$, *or*
(ii) *the pair* (q, n) *is* $(1, 2)$, $(3, 5)$ *or* $(3, 7)$.

Remark: Product bundles in case (i), and $SO(3)$, $SU(3)$ and $Sp(2)$, respectively, in case (ii), provide examples of Lie groups E with $S^q \to E \to S^n$ a fibration. Thus, the restrictions in Theorem 2 are the best possible.

COROLLARY. $S^3 \times S^7$ *can not support homotopy-associative H-space multiplications.*

2. Proof of Main Result

LEMMA 1. *Let X be (the total space of) a* 3*-sphere bundle over* S^4. *If X is an H-space, then X has the homotopy type of* S^7.

Proof: The mod 2 cohomology of X can not be an exterior algebra on two generators of respective dimensions 3 and 4, by ADAMS' theorem in [1]. Thus X is a mod 2 cohomology 7-sphere. By a classical theorem of BOREL, X is a mod p cohomology 7-sphere, for all odd primes p. Therefore, X is an integral cohomology 7-sphere, and a generator of $\pi_7(X)$ is a homotopy equivalence.

LEMMA 2. *If X is (the total space of) a q-sphere bundle over S^n, with* $(q, n) = (1, 7)$ *or* $(7, 1)$, *then the universal covering space $\tilde{X}$ of X has the homotopy type of* S^7.

(The proof is an elementary exercise in covering space theory, together with an application of a theorem of WHITEHEAD [5].)

LEMMA 3. *If X is a homotopy-associative H-space and* $H^*(X; \mathbf{Z})$ *is ring isomorphic* *to* $H^*(S^3 \times S^7; \mathbf{Z})$, *then the* mod 3 *Steenrod operation*

$$\mathscr{P}_3^1 : H^3(X; \mathbf{Z}_3) \to H^7(X; \mathbf{Z}_3)$$

is an isomorphism.

Proof: Let $P_3 X$ be the projective 3-space of (some homotopy-associative multiplication on) X, as defined by STASHEFF [4]. $H^*(P_3 X; \mathbf{Z})$ is torsion-free, and contains a truncated polynomial algebra on two generators x and y (of degrees 4 and 8, respectively), truncated at height 4. The relation $y^3 = \mathscr{P}_3^4(y) = \mathscr{P}_3^1 \mathscr{P}_3^3(y)$ implies that $\mathscr{P}_3^3(y) = \pm x y^2$. Consequently, $\mathscr{P}_3^1(x) = \pm y + \alpha x^2$ (for some $\alpha \in \mathbf{Z}_3$), which implies that $\mathscr{P}_3^1 : H^4(\Sigma X;\ \mathbf{Z}_3) \to H^8(\Sigma X;\ \mathbf{Z}_3)$ is an isomorphism (naturality with respect to inclusion of suspension $\Sigma X \to P_3 X$). Desuspending, we obtain Lemma 3.

LEMMA 4. *If X is (the total space of) a 7-sphere bundle over the 7-sphere, then X can not support a homotopy-associative H-space multiplication.*

(We omit the proof of Lemma 4, as it is essentially the same as the proof that S^7 can not be a homotopy-associative H-space.)

Proof of Theorem 2: By Theorem 1, it suffices to exclude the following pairs (q, n): (3, 4), (1, 7), (7, 1), (7, 3) and (7,7).

(3, 4) is eliminated by Lemma 1; (7, 7) by Lemma 4; and both (7, 1) and (1, 7) are excluded by Lemma 2, if we observe that the universal covering space of a homotopy-associative H-space is again a homotopy-associative H-space.

Suppose now that $S^7 \xrightarrow{i} E \xrightarrow{p} S^3$ is a bundle with fibre S^7 and base S^3. The Serre exact sequence for cohomology implies that $i^* : H^7(E; \mathbf{Z}_3) \to H^7(S^7; \mathbf{Z}_3)$ is an isomorphism. But then the following

$$\begin{array}{ccc} H^7(E; \mathbf{Z}_3) & \xrightarrow{i^*} & H^7(S^7; \mathbf{Z}_3) \\ \uparrow \mathscr{P}_3^1 & & \uparrow \mathscr{P}_3^1 \\ H^3(E; \mathbf{Z}_3) & \xrightarrow{i^*} & H^3(S^7; \mathbf{Z}_3) \end{array}$$

commutative diagram, together with Lemma 3, implies that E can not be homotopy-associative. This completes the proof of Theorem 2.

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