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# Homotopy-associative H-spaces which are Sphere Bundles over Spheres

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## 1. Introduction

Let  $S^q \rightarrow E \rightarrow S^n$  be a  $q$ -sphere bundle over the  $n$ -sphere ( $q, n$  positive). The following theorem is proven, first in [2], and then more briefly in [3].

**THEOREM 1.** *If  $E$  is an H-space, then either*

- (i) *both  $q$  and  $n$  belong to the set  $\{1, 3, 7\}$ , or*
- (ii) *the pair  $(q, n)$  is  $(1, 2)$ ,  $(3, 4)$  or  $(3, 5)$ .*

In this note we study the further restrictions imposed on the pair  $(q, n)$ , in case  $E$  supports a homotopy-associative  $H$ -space multiplication. Precisely, our main result is the following.

**THEOREM 2.** *If  $E$  supports a homotopy-associative H-space multiplication, then either*

- (i) *both  $q$  and  $n$  belong to the set  $\{1, 3\}$ , or*
- (ii) *the pair  $(q, n)$  is  $(1, 2)$ ,  $(3, 5)$  or  $(3, 7)$ .*

*Remark:* Product bundles in case (i), and  $SO(3)$ ,  $SU(3)$  and  $Sp(2)$ , respectively, in case (ii), provide examples of Lie groups  $E$  with  $S^q \rightarrow E \rightarrow S^n$  a fibration. Thus, the restrictions in Theorem 2 are the best possible.

**COROLLARY.**  $S^3 \times S^7$  *can not support homotopy-associative H-space multiplications.*

## 2. Proof of Main Result

**LEMMA 1.** *Let  $X$  be (the total space of) a 3-sphere bundle over  $S^4$ . If  $X$  is an H-space, then  $X$  has the homotopy type of  $S^7$ .*

*Proof:* The mod 2 cohomology of  $X$  can not be an exterior algebra on two generators of respective dimensions 3 and 4, by ADAMS' theorem in [1]. Thus  $X$  is a mod 2 cohomology 7-sphere. By a classical theorem of BOREL,  $X$  is a mod  $p$  cohomology 7-sphere, for all odd primes  $p$ . Therefore,  $X$  is an integral cohomology 7-sphere, and a generator of  $\pi_7(X)$  is a homotopy equivalence.

**LEMMA 2.** *If  $X$  is (the total space of) a  $q$ -sphere bundle over  $S^n$ , with  $(q, n) = (1, 7)$  or  $(7, 1)$ , then the universal covering space  $\tilde{X}$  of  $X$  has the homotopy type of  $S^7$ .*

(The proof is an elementary exercise in covering space theory, together with an application of a theorem of WHITEHEAD [5].)

**LEMMA 3.** *If  $X$  is a homotopy-associative H-space and  $H^*(X; \mathbf{Z})$  is ring isomorphic*

to  $H^*(S^3 \times S^7; \mathbf{Z})$ , then the mod 3 Steenrod operation

$$\mathcal{P}_3^1: H^3(X; \mathbf{Z}_3) \rightarrow H^7(X; \mathbf{Z}_3)$$

is an isomorphism.

*Proof:* Let  $P_3 X$  be the projective 3-space of (some homotopy-associative multiplication on)  $X$ , as defined by STASHEFF [4].  $H^*(P_3 X; \mathbf{Z})$  is torsion-free, and contains a truncated polynomial algebra on two generators  $x$  and  $y$  (of degrees 4 and 8, respectively), truncated at height 4. The relation  $y^3 = \mathcal{P}_3^4(y) = \mathcal{P}_3^1 \mathcal{P}_3^3(y)$  implies that  $\mathcal{P}_3^3(y) = \pm x y^2$ . Consequently,  $\mathcal{P}_3^1(x) = \pm y + \alpha x^2$  (for some  $\alpha \in \mathbf{Z}_3$ ), which implies that  $\mathcal{P}_3^1: H^4(\Sigma X; \mathbf{Z}_3) \rightarrow H^8(\Sigma X; \mathbf{Z}_3)$  is an isomorphism (naturality with respect to inclusion of suspension  $\Sigma X \rightarrow P_3 X$ ). Desuspending, we obtain Lemma 3.

LEMMA 4. *If  $X$  is (the total space of) a 7-sphere bundle over the 7-sphere, then  $X$  can not support a homotopy-associative  $H$ -space multiplication.*

(We omit the proof of Lemma 4, as it is essentially the same as the proof that  $S^7$  can not be a homotopy-associative  $H$ -space.)

*Proof of Theorem 2:* By Theorem 1, it suffices to exclude the following pairs  $(q, n): (3, 4), (1, 7), (7, 1), (7, 3)$  and  $(7, 7)$ .

$(3, 4)$  is eliminated by Lemma 1;  $(7, 7)$  by Lemma 4; and both  $(7, 1)$  and  $(1, 7)$  are excluded by Lemma 2, if we observe that the universal covering space of a homotopy-associative  $H$ -space is again a homotopy-associative  $H$ -space.

Suppose now that  $S^7 \xrightarrow{i} E \xrightarrow{p} S^3$  is a bundle with fibre  $S^7$  and base  $S^3$ . The Serre exact sequence for cohomology implies that  $i^*: H^7(E; \mathbf{Z}_3) \rightarrow H^7(S^7; \mathbf{Z}_3)$  is an isomorphism. But then the following

$$\begin{array}{ccc} H^7(E; \mathbf{Z}_3) & \xrightarrow{i^*} & H^7(S^7; \mathbf{Z}_3) \\ \uparrow \mathcal{P}_3^1 & & \uparrow \mathcal{P}_3^1 \\ H^3(E; \mathbf{Z}_3) & \xrightarrow{i^*} & H^3(S^7; \mathbf{Z}_3) \end{array}$$

commutative diagram, together with Lemma 3, implies that  $E$  can not be homotopy-associative. This completes the proof of Theorem 2.

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