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## Knot Cobordism Groups in Codimension Two<sup>1)</sup>

J. LEVINE

In [3] and [7] the relationship of *cobordism* between knotted  $n$ -spheres in  $(n+2)$ -space ( $n$ -knots) is introduced and studied. Cobordism is weaker than isotopy but, on the other hand, is the more natural concept for studying singularities of submanifolds of codimension two (see [2], [3]). Moreover cobordism has the advantage that cobordism classes form an abelian group  $C_n$ , for each dimension, under connected sums.

KERVAIRE [7] has shown that  $C_n=0$  when  $n$  is even i.e. all  $n$ -knots are null-cobordant (slice knots). When  $n$  is odd, FOX and MILNOR [3] for  $n=1$ , and KERVAIRE [7] for  $n \geq 3$ , have shown that  $C_n$  is infinitely-generated.

In this work, the groups  $C_{2n-1}$ , for  $n \geq 2$ , will be given a purely algebraic description. A relation, which we shall call cobordism, will be introduced into certain collections of matrices. Under "block addition" the cobordism classes form an abelian group for each collection. Two of these,  $G_+$  and  $G_-$ , are of special interest. We construct a homomorphism:

$$\phi_n: C_{2n-1} \rightarrow G_{\varepsilon_n}, \quad \varepsilon_n = (-1)^n$$

and our main result is that  $\phi_n$  is an isomorphism for  $n \geq 3$ ,  $\phi_2$  is an isomorphism onto a certain specified subgroup  $G_+^0$  of index 2, and  $\phi_1$  is onto. Thus the graded group  $\{C_n: n \geq 4\}$  is periodic with period four of the form  $\{0, G_-, 0, G_+\}$ , but  $C_1$  and  $C_3$  seem to violate this periodicity.  $C_1$  is still undetermined.

We next study the group  $G_\varepsilon$  by introducing some invariants whose relation to the Alexander polynomial and quadratic form of a knot will be immediately recognized. In particular, this will provide a more general setting for the result of FOX and MILNOR [3] and KERVAIRE [7] on the form of the Alexander polynomial of a slice knot and the cobordism invariance of the Minkowski units and signature of the quadratic form of a knot [14], [15]. We determine the values these invariants may assume and show that they are *not* faithful. In fact, we find that  $G_\varepsilon$  contains a linearly independent set. This has also been proved by MILNOR [23].

In conclusion, we use some of our considerations to construct examples, in every odd dimension, of knots whose complements are homotopy equivalent, yet are not cobordant. This generalizes the example of the granny knot and square knot in dimension 1. It is interesting to compare this to the result [9] that a knot whose complement is a homotopy circle is necessarily unknotted (at least in dimensions  $\neq 2$  or 3). Moreover, in half the cases, the two knots are not even diffeomorphic; in the rest,

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it is known that the differential structure of a knot depends only on the homotopy type of its complement [10].

### Cobordism of Matrices

1. All matrices will have integer entries. A square matrix  $N$  is *null-cobordant* if it is congruent to a matrix of the form  $\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix}$  where  $N_i$  are square matrices of the same size. Note that  $N$  must have an even number of rows.

If  $A_1, A_2$  are matrices, we define the "block sum"  $A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ . We say square matrices  $A_1$  and  $A_2$  are *cobordant* if  $A_1 \oplus (-A_2)$  is null-cobordant. This is a reflexive and symmetric relation, but not necessarily transitive. For transitivity to hold, we will have to restrict the collection of matrices.

2. LEMMA 1: Suppose  $A$  and  $N$  are square matrices and  $N$  and  $A \oplus N$  are null-cobordant. If some linear combination  $\lambda N + \mu N'$  ( $\lambda, \mu$  integers and  $N'$  the transpose of  $N$ ) is non-singular i.e. determinant  $\neq 0$ , then  $A$  is null-cobordant.

Note that some restriction on  $N$  is necessary, since a large zero matrix would not work.

*Proof:* Consider  $B = A \oplus N$  as a bilinear form on  $Z^{2n} = Z^{2m} \oplus Z^k \oplus Z^l$ , with the properties that  $k=l$ ,  $Z^{2m}$  and  $Z^k \oplus Z^l$  are orthogonal with respect to  $B$ ,  $B|Z^{2m} = A$ ,  $B|Z^k \oplus Z^l = N$ ,  $B|Z^k = 0$ , and there exists  $\alpha_1, \dots, \alpha_n \in Z^{2n}$ , linearly independent elements, with  $B(\alpha_i, \alpha_j) = 0$ ,  $1 \leq i, j \leq n$ . To prove the lemma it suffices to find linearly independent elements  $\beta_1, \dots, \beta_m \in Z^{2m}$  satisfying  $B(\beta_i, \beta_j) = 0$ ,  $1 \leq i, j \leq m$ . Let us write  $\alpha_i = x_i + y_i + z_i$  where  $x_i \in Z^{2m}$ ,  $y_i \in Z^k$ ,  $z_i \in Z^l$ . We will need the well-known fact:

(1) If  $\gamma_1, \dots, \gamma_n \in Z^m$ , then there exists a non-singular matrix  $P = (p_{ij})$  such that if  $\gamma'_i = \sum p_{ij} \gamma_j$ , then  $\gamma'_1, \dots, \gamma'_r$  are linearly independent, while  $\gamma'_{r+1} = \dots = \gamma'_n = 0$ , for some  $r$ .

Applying this to  $\{z_i\} \subset Z^l$ , we may assume that  $z_1, \dots, z_r$  are independent, while  $z_{r+1} = \dots = z_n = 0$ . Note that  $r \leq k = l$ . Applying (1) now to  $\{x_{r+1}, \dots, x_n\}$ , we may assume  $\{x_{r+1}, \dots, x_{r+s}\}$  is linearly independent and  $x_{r+s+1} = \dots = x_n = 0$ . Then  $A(x_i, x_i) = B(\alpha_i, \alpha_i) = 0$  for  $r+1 \leq i, j \leq r+s$ , so it suffices to show  $s \geq m$ . But  $N(y_i, z_j) = B(\alpha_i, \alpha_j) = 0$  for  $i > r+s$ ; similarly  $N(z_j, y_i) = 0$  for  $i > r+s$ . If we define  $N_0 = \lambda N + \mu N'$ , as specified in the hypothesis, then  $y_{r+s+1}, \dots, y_n$  lie in the subgroup  $Y \subset Z^k$  orthogonal to  $z_1, \dots, z_r$  with respect to  $N_0$ . Since  $N_0$  is non-singular and  $\{z_1, \dots, z_r\}$  are linearly independent,  $Y$  has rank  $\leq k - r$ . Thus  $n - r - s \leq k - r$  (note  $\{y_i\} = \{\alpha_i\}$ , for  $i > r+s$ , and so are linearly independent) and this implies  $s \geq n - k = m$ .

3. It now follows from Lemma 1 that, in the collection of square matrices  $A$  satisfying:

(2)  $\lambda A + \mu A'$  is non-singular for some integers  $\lambda, \mu$ ,

cobordism is an equivalence relation. In fact, if  $A_1, A_2$  and  $A_2, A_3$  are cobordant pairs, then  $A_1 \oplus (-A_2)$  and  $A_2 \oplus (-A_3)$  are null-cobordant. Therefore,  $A \oplus (-A_2) \oplus A_2 \oplus (-A_3)$  is null-cobordant, and, since  $A_2 \oplus (-A_2)$  is null-cobordant, it follows from Lemma 1 that  $A_1$  and  $A_3$  are cobordant.

For a fixed pair of integers  $\lambda, \mu$ , consider the collection of matrices  $A$  satisfying (2). This collection is closed under block sum; moreover block sum preserves cobordism. It follows that the set  $G_{\lambda, \mu}$  of cobordism classes has the structure of an abelian semi-group. But if  $A$  satisfies (2), so does  $-A$ , which is obviously a cobordism inverse. Thus  $G_{\lambda, \mu}$  is an abelian group. Notice that the matrices satisfying (2), except that  $\lambda A + \mu A'$  is required to be *unimodular*, if there are any, define a subgroup of  $G_{\lambda, \mu}$ .

4. We will say that  $A$  has *property*  $\varepsilon$  ( $\varepsilon = \pm 1$ ) if  $A + \varepsilon A'$  is unimodular. It follows from § 3 that the cobordism classes of matrices with property  $\varepsilon$  form an abelian group  $G_\varepsilon$  under block sum. It follows immediately from Lemma 1 that the zero class in  $G_\varepsilon$  consists precisely of the null-cobordant matrices of type  $\varepsilon$ .

A matrix with property  $\varepsilon$  must have an even number of rows. If  $\varepsilon = -1$ , this is a familiar property of non-singular skew-symmetric matrices. If  $\varepsilon = +1$ , it is a property of unimodular even quadratic forms (see e.g. [11, Theorem 1]).

If  $A$  has property  $+$ , then  $A + A'$  has signature a multiple of 8 (see [11, Theorem 1]). Let  $G_+^0$  be the subgroup of  $G_+$  of index 2, defined by matrices  $A$  with property  $+$  and signature  $(A + A')$  a multiple of 16. Note that signature  $(A + A')$  depends only on the cobordism class of  $A$ .

### Cobordism of Knots

5. If  $n \geq 1$  is an integer, an  $n$ -knot is a smooth oriented submanifold of  $S^{n+2}$ , homeomorphic to  $S^n$ . Two  $n$ -knots  $K_0, K_1$  are *cobordant* if there exists a smooth oriented submanifold  $V$  of  $I \times S^{n+2}$ , with  $\partial V = (1 \times K_1) \cup (0 \times (-K_0))$ . An  $n$ -knot is *null-cobordant* (slice knot) if it is cobordant to the standard imbedded  $S^n \subset S^{n+2}$  – equivalently, if it bounds an imbedded  $(n+1)$ -disk in  $D^{n+3}$ .

Cobordism is weaker than isotopy e.g. the square knot is null-cobordant. The cobordism classes of  $n$ -knots form an abelian group  $C_n$ , under *connected sums*. See [7, Ch. III] for more detail and proofs – our  $C_n$  is larger than Kervaire's, since he only allows knots *diffeomorphic* to  $S^n$ . The negative of the cobordism class of a knot  $K$  is represented by the image of  $K$ , with reversed orientation, under a reflection of  $S^{n+2}$ ; we denote this knot by  $-K$ . As mentioned in the introduction, Kervaire [7] has shown that  $C_n = 0$  for  $n$  even; we therefore concentrate on odd  $n$ .

6. A  $(2n-1)$ -knot  $K$  is *simple* if the homotopy groups of its complement  $S^{2n+1} - K$  coincide with those of the circle in dimensions  $< n$ . Note that this is the



most one can ask if  $K$  is knotted (see [9]). If  $K$  is simple, it is proved in [9], that  $K$  bounds an  $(n-1)$ -connected submanifold of  $S^{2n+1}$  – and conversely.

### Construction of $\phi_n$

7. Let  $K$  be a  $(2n-1)$ -knot; then  $K$  bounds an oriented  $2n$ -dimensional submanifold  $V$  of  $S^{2n+1}$  (see e.g. [9; Lemma 2]). We can define a pairing:

$$\theta: H_n(V) \otimes H_n(V) \rightarrow \mathbb{Z}$$

by  $\theta(\alpha \otimes \beta) = L(\alpha \otimes i_*(\beta))$ , where  $L$  denotes linking number and  $i: V \rightarrow S^{2n+1} - V$  is defined by translation in the positive normal direction. We have the formula:

$$\theta(\alpha \otimes \beta) + (-1)^n \theta(\beta \otimes \alpha) = -\alpha \cdot \beta \quad (3)$$

where  $\alpha \cdot \beta$  is the intersection number in  $V$  (see [10, § 2.5]).

A basis for the torsion-free part of  $H_n(V)$  determines a matrix representing  $\theta$  – such a matrix  $A$  will be called a *Seifert matrix for  $K$* . It follows directly from (3) that  $A$  satisfies property  $(-1)^n$ , since the intersection pairing of  $H_n(V)$  is unimodular.

If  $A_1, A_2$  are Seifert matrices for  $K_1, K_2$ , it is easily seen that  $A_1 \oplus A_2$  is a Seifert matrix for the connected sum  $K_1 \# K_2$  and  $-A_1$  is a Seifert matrix for  $-K_1$ .

Let  $K$  be a simple  $(2n-1)$ -knot,  $n \geq 2$ ; then  $\pi_n(S^{2n+1} - K)$  is a module over the group ring  $A$  of  $\pi_1(S^{2n+1} - K) \approx \mathbb{Z}$ . If  $t$  is a generator of  $\pi_1(S^{2n+1} - K)$  and  $A$  is a Seifert matrix for  $K$ , then  $tA + (-1)^n A'$ , viewed as a matrix with entries in  $A$ , is a relation matrix for  $\pi_n(S^{2n+1} - K)$  (see [7, p. 255]). More generally, for any  $(2n-1)$ -knot  $K$ ,  $tA + (-1)^n A'$  is a relation matrix for  $H_n(\tilde{X}; Q)$  ( $Q$ =rationals) where  $\tilde{X}$  is the infinite cyclic covering of the complement of  $K$  (see [10, § 2]).

8. LEMMA 2: *If  $K$  is a null-cobordant  $(2n-1)$ -knot and  $A$  is a Seifert matrix for  $K$ , then  $A$  is null-cobordant.*

*Proof:* Let  $V$  be the oriented submanifold of  $S^{2n+1}$  bounded by  $K$  from which  $A$  is defined. It is required to find a linearly independent subset  $\alpha_1, \dots, \alpha_r \in H_n(V)$  such that  $\text{rank } H_n(V) = 2r$  and  $\theta(\alpha_i, \alpha_j) = 0$  for  $1 \leq i, j \leq r$ .

Let  $\Delta$  be a smooth  $(2n+1)$ -disk in  $D^{2n+2}$  bounded by  $K$ . Consider the closed manifold  $\bar{V} = V \cup \Delta$  (corner at  $K$ ) oriented consistent with  $V$ .  $\bar{V}$  bounds a submanifold  $W$  of  $D^{2n+2}$ . To see this, we apply the Thom-Pontriagin construction. Let  $v$  be a unit normal vector field to  $\bar{V}$  in  $D^{2n+2}$ . Extending this to a framed submanifold of  $D^{2n+2}$  is equivalent to a more standard problem. Let  $U$  be an open tubular neighborhood of  $\Delta$  in  $D^{2n+2}$ . Then  $(\bar{V}, v)$  is trivially isotopic to a framed submanifold of  $\partial M$ , where  $M = D^{2n+2} - U$ . According to the Thom-Pontriagin construction, we get an obstruction to extending  $(\bar{V}, v)$  in  $H^2(M, \partial M)$ , and this is the only obstruction. But  $H^2(M, \partial M) \approx H^2(D^{2n+2}, S^{2n+1} \cup \Delta) \approx H^1(S^{2n+1} \cup \Delta) = 0$ . Thus  $W$  exists.

Now consider the inclusion  $j: V \rightarrow W$ . If  $\alpha, \beta \in \text{Ker } \{j_*: H_n(V) \rightarrow H_n(W)\}$ , then  $\theta(\alpha \otimes \beta) = 0$ , because  $\alpha$  bounds a chain in  $W$  and  $i_*(\beta)$  bounds a chain in  $D^{2n+2} - W$  obtained by translating off  $W$  a chain bounded by  $\beta$  in  $W$  (recall  $i: V \rightarrow S^{2n+1} - V$ ). Thus, to prove Lemma 2 it suffices to show that  $\text{Ker } j_*$  has rank  $r$ .

Now consider the exact sequence:

$$H_{n+1}(W) \xrightarrow{\lambda} H_{n+1}(W, V) \xrightarrow{\delta} H_n(V) \xrightarrow{j_*} H_n(W) \xrightarrow{\lambda'} H_n(W, V). \quad (4)$$

By duality (and  $V = \partial W$ -disk), we have  $H_n(W, V) \approx H^{n+1}(W)$  and  $H_{n+1}(W, V) \approx H^n(W)$ . Moreover, modulo torsion, the homomorphisms  $\lambda$  and  $\lambda'$  correspond to the homomorphisms  $H_{n+1}(W) \rightarrow \text{Hom}\{H_n(W), \mathbb{Z}\}$ ,  $H_n(W) \rightarrow \text{Hom}\{H_{n+1}(W), \mathbb{Z}\}$  determined by the intersection pairing:  $H_n(W) \otimes H_{n+1}(W) \rightarrow \mathbb{Z}$ .

In particular,  $\text{rank}(\text{image } \lambda) = \text{rank}(\text{image } \lambda')$ . From this, and the following well-known fact:

If  $A \xrightarrow{g} B \xrightarrow{h} C$  is an exact sequence of abelian-groups, then  $\text{rank } B = \text{rank}(\text{image } g) + \text{rank}(\text{image } h)$ .

We deduce from (4) that:  $\text{rank}(\text{Ker } j_*) = \text{rank}(\text{image } \delta) = 1/2 \text{rank } H_n(V) = r$  as desired.

9. Let  $\varepsilon_n = (-1)^n$ . We can now construct a homomorphism  $\phi_n: C_{2n-1} \rightarrow G_{\varepsilon_n}$  by assigning to the cobordism class of a knot, the cobordism class of any Seifert matrix. By § 7 and Lemma 2,  $\phi_n$  is well-defined and a homomorphism.

### The Main Theorem

THEOREM:  $\phi_n$  is: (a) an isomorphism onto  $G_{\varepsilon_n}$  for  $n \geq 3$   
 (b) an isomorphism onto  $G_+^0$  for  $n = 2$   
 (c) an epimorphism for  $n = 1$ .

10. To prove the onto parts of the theorem we use the following result:

LEMMA 3: If  $A$  is a matrix satisfying property  $\varepsilon_n$ , then there exists a simple  $(2n-1)$ -knot for which  $A$  is a Seifert matrix, for any  $n \geq 1$ . If  $n = 2$ , it is necessary to assume that  $A + A'$  is a matrix representation of the intersection pairing  $H_2(V) \otimes H_2(V) \rightarrow \mathbb{Z}$  for some simply-connected closed 4-manifold  $V$ .

For  $n = 1$ , this is a classical result [17]. For  $n \geq 2$ , a proof is given in [7, Ch. II, § 6].

Using Lemma 3, the onto statements for  $n \neq 2$  follow immediately. We now consider  $n = 2$ .

Suppose  $K$  is a 3-knot bounding the 4-manifold  $V$  in  $S^5$ . Then  $K$  is diffeomorphic to  $S^3$  and we can put a disk on  $\partial V$  to form a closed manifold  $\bar{V}$ . If  $A$  is a Seifert matrix for  $K$ , then  $A + A'$  represents the intersection pairing of  $\bar{V}$ . It follows from

Rohlin's Theorem that signature  $(A + A')$  is a multiple of 16, since  $V$  is parallelizable (it has a trivial normal bundle in Euclidean space) (see e.g. [8]). Thus  $\phi_2(C_3) \subset G_+^0$ .

Suppose  $\alpha \in G_+^0$ . We can represent  $\alpha$  by an arbitrarily large matrix  $A$ , for which  $A + A'$  is indefinite – for example, by adding to a given representative a number of copies of  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Now, in [11], Milnor constructs a closed 4-manifold  $M$  with index 16 and second Betti number 22. By forming the connected sum of copies of  $M$  and  $S^2 \times S^2$ , it follows that we can construct a closed 4-manifold  $V$  with any index  $S$  a multiple of 16, and second Betti-number any even integer  $\geq 11/8 \cdot S$ . Thus we construct  $V$  whose intersection pairing  $H_2(V) \otimes H_2(V) \rightarrow \mathbb{Z}$  has the same signature and rank as  $A + A'$  (for suitable  $A$ ). But, by the classification of indefinite unimodular integral quadratic forms [11; Th. 1, 2], it follows that  $A + A'$  is actually a matrix representative of the intersection pairing of  $V$ . By Lemma 3,  $A$  is a Seifert matrix for a 3-knot and therefore  $\alpha \in \phi_2(C_3)$ . Thus  $\phi_2(C_3) = G_+^0$ .

11. The theorem will now follow from the following two lemmas.

LEMMA 4: *Every  $(2n-1)$ -knot is cobordant to a simple knot.*

LEMMA 5: *If  $n \geq 2$  and  $K$  is a simple  $(2n-1)$ -knot with a null-cobordant Seifert matrix, then  $K$  is null-cobordant.*

*Proof of Lemma 4:* We only need consider  $n \geq 2$ . The idea is to extend a knot  $K$  to an  $(n-1)$ -connected submanifold  $V_0$  of  $D^{2n+2}$  which can then be “engulfed” in the boundary of a smaller disk  $D_0^{2n+2}$  in the interior of  $D^{2n+2}$ . Then  $V_0$  bounds a simple knot  $K_0$  in  $\partial D_0^{2n+2}$  and the annular region  $\overline{D^{2n+2}} - D_0^{2n+2}$  gives a cobordism between  $K$  and  $K_0$ .

Let  $V$  be a submanifold of  $S^{2n+1}$  bounded by  $K$ . In [7, Ch. III, § 3] Kervaire shows how to add handles to  $V$  in  $D^{2n+2}$  in such a way that the corresponding “surgeries” simplify  $V$  in a prescribed manner. His arguments, although presented only in the case of even-dimensional knots, work equally well in the odd-dimensional case, but only up to one dimension below the middle. The result is a  $(2n+1)$ -dimensional submanifold  $W$  of  $D^{2n+2}$ , with an imbedding  $i: V \times I \rightarrow W$  satisfying:

(a)  $W \cap S^{2n+1} = V$  and  $i(x, t) = 1/2(t+1)x$ , considering  $V \subset D^{2n+2}$ , and using scalar multiplication in  $R^{2n+2}$ .

(b)  $\partial W = V \cup i(\partial V \times I) \cup V_0$ , where  $V_0 \cap V = \emptyset$  and  $V_0 \cap i(\partial V \times I) = \partial V_0 = i(\partial V \times 0)$

(c)  $V_0$  is  $(n-1)$ -connected

(d)  $W$  is obtained from  $i(V \times I)$  by attaching handles of index  $\leq n$  to  $i(V \times 0)$ .

Note that  $W$  has a corner at  $i(\partial V \times 1) = K$ .

We now wish to apply the engulfing theorem of HIRSCH and ZEEMAN to imbed a  $(2n+2)$ -disk  $D_0^{2n+2}$  in the interior of  $D^{2n+2}$  so that  $D_0^{2n+2} \cap W = \partial D_0^{2n+2} \cap W = V_0$ . We may formally apply [4, Th. 2]; in the notation of [4], we let  $X = V_0$  and  $V$  (of [4])

$= D^{2n+2}$  with a “cut” along  $W$  (alternatively, the complement in  $D^{2n+2}$  of a tubular neighborhood of  $W$ ). The hypothesis of [4, Th. 2] are satisfied as follows:

- (i)  $V_0$  is  $n$ -collapsible by (c) and e.g. [16, Lemma 2.7],
- (ii)  $D^{2n+2} - W$  is 1-connected by (d), and
- (iii)  $2n+1 \geq n+3$ , since  $n \geq 2$ .

Now  $D^{2n+2} - D_0^{2n+2}$  is an  $h$ -cobordism between  $S^{2n+1}$  and  $\partial D^{2n+1}$ ; according to [18] ( $n \geq 2$ ) it is diffeomorphic to  $S^{2n+1} \times I$ . If  $h$  is such a diffeomorphism, then  $h_i(\partial V \times I)$  is a cobordism from  $K$  to  $h(\partial V_0)$ , the latter being simple since it bounds  $h(V_0)$  in  $S^{2n+1}$ .

12. *Proof of Lemma 5:* Suppose  $V$  is an  $(n-1)$ -connected submanifold of  $S^{2n+1}$ , bounded by  $K$ . By the assumption of Lemma 5 and the fact that  $\phi_n$  is well-defined, any matrix associated with  $K$  is null-cobordant. Thus,  $H_n(V)$  has a basis  $\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r$  such that  $\theta(\alpha_i, \alpha_j) = 0$  for  $1 \leq i, j \leq r$ .

Suppose  $n \geq 3$ ; then  $\{\alpha_i\}$  are represented by disjoint imbedded  $n$ -spheres  $S_i \subset V$ . This follows from Whitney's procedure [22] applied, first, to the self-intersections of  $S_i$ , and, then, to the intersections of  $S_i$  and  $S_j$ , using  $\alpha_i \cdot \alpha_j = \theta(\alpha_i, \alpha_j) + \theta(\alpha_j, \alpha_i) = 0$ . Next we extend each  $S_i$  to an imbedded  $(n+1)$ -disk  $d_i$  in  $D^{2n+2}$ . Since the intersection number of  $d_i$  and  $d_j$  is  $\theta(\alpha_i, \alpha_j)$ , we can again apply Whitney's procedure ( $n+1 \geq 3$ ) to insure that the  $\{d_i\}$  are disjoint. Now  $d_i$  can be taken as the core of a handle  $h_i$  attached to  $V$ . This is done by extending a normal field to  $V$  in  $S^{2n+1}$  to a normal field  $v_i$  to  $d_i$  in  $D^{2n+2}$  – the obstruction can be identified with  $\theta(\alpha_i, \alpha_i) = 0$  – and taking  $h_i$  to be the orthogonal complement to  $v_i$  in a tubular neighborhood of  $d_i$ .

The handles  $\{h_i\}$  induce surgeries on  $V$  resulting in a submanifold  $\Delta$  of  $D^{2n+2}$ , bounded by  $K$ . The computation in [12, § 6] shows that  $\Delta$  is contractible. If  $n \geq 3$ , it follows from [18] that  $\Delta$  is a  $2n$ -disk and, therefore,  $K$  is null-cobordant.

13. If  $n=2$ , the above argument fails in two places:

- (a) representing  $\{\alpha_i\}$  by disjoint imbedded spheres, and
- (b) the assertion that  $\Delta$  is a disk.

We use results of WALL [20], [21] on 4-manifolds to repair the argument. Since  $K$  is diffeomorphic to  $S^3$ , we form the smooth closed manifold  $\bar{V}$  by putting a disk on  $\partial V$ . Since  $V$  is parallelizable and has index zero, it follows from [21, Th. 1 and Lemma 2] that  $\bar{V}$  is  $h$ -cobordant to the boundary of a handlebody with handles of index 2. But then it follows easily from [21, Th. 3] that the connected sum of  $\bar{V}$  with enough copies of  $S^2 \times S^2$  is diffeomorphic to the boundary of such a handlebody  $W$ . Since we may perform these connected sums with copies of  $S^2 \times S^2$  in  $S^5$ , we may assume that  $\bar{V} = \partial W$ .

The handlebody decomposition of  $W$  provides us with a family  $\{S'_i\}$  of disjoint imbedded  $n$ -spheres in  $\bar{V}$  – the boundaries of the transverse disks of the handles. If

$\alpha'_i \in H_2(\bar{V})$  is the homology class of  $S'_i$ , then an easy homology argument shows that  $\{\alpha'_i\}$  is half of a basis of  $H_2(\bar{V})$ . Also  $\alpha'_i \cdot \alpha = 0'_j$ , all  $i, j$ . Since the intersection pairing on  $H_2(\bar{V})$  is unimodular, both  $\{\alpha_i\}$  and  $\{\alpha'_i\}$  extend to bases  $\{\alpha_i; \beta_i\}$ ,  $\{\alpha'_i; \beta'_i\}$  satisfying  $\alpha_i \cdot \beta_j = \alpha'_i \cdot \beta'_j = \delta_{ij}$ ,  $\beta_i \cdot \beta_j = \beta'_i \cdot \beta'_j = 0$ . It then follows from [20, Th. 2], that  $\bar{V}$  admits a diffeomorphism  $h$  onto itself such that  $h_*(\alpha'_i) = \alpha_i$ . We can therefore take  $S_i = h(S'_i)$ .

Observe that the result of surgery on  $\bar{V}$ , using the  $\{S'_i\}$ , is diffeomorphic to  $S^4$ , because this is equivalent to removing the handles of  $W$  (since  $\pi_3(SO_2) = 0$ , the framing of the  $\{S'_i\}$  is irrelevant). Thus, applying  $h^{-1}$ ,  $\Delta$  must be a 4-disk.

This completes the proof of Lemma 5, and the Theorem.

### Alexander Polynomial

14. We now begin a purely algebraic study of the groups  $G_\varepsilon$ . For an integral matrix  $A$  of type  $\varepsilon$ , we define an integral, i.e. integer coefficients, polynomial:  $\Delta_A(t) = \text{determinant}(tA + \varepsilon A')$ .

**PROPOSITION 1:** *If  $A$  has  $2\mu$  rows, then*

$$(1) \Delta_A(t) = t^{2\mu} \Delta_A(t^{-1}),$$

$$(2) \Delta_A(-\varepsilon) \text{ is square,}$$

$$(3) \Delta_A(1) = (-\varepsilon)^\mu.$$

*Proof:* (1) follows directly from the definition. (2) follows from the fact that skew-symmetric matrices have square determinants. This fact together with property -1 implies (3), for  $\varepsilon = -1$ . For  $\varepsilon = +1$ ,  $\Delta_A(1)$  is the determinant of an even unimodular quadratic form of rank  $2\mu$ . If this form has signature  $2S$ , then the determinant is  $(-1)^{\mu-S}$ . But  $S \equiv 0 \pmod{8}$  for such forms, which implies (3).

As a converse to Proposition 1, we have:

**PROPOSITION 2:** *Let  $\varepsilon = \pm 1$ ,  $\mu$  be a positive integer and  $\Delta(t)$  an integral polynomial satisfying (1), (2) and (3). Then there exists a square integral matrix  $A$  satisfying property  $\varepsilon$ , such that  $\Delta(t) = \Delta_A(t)$ .*

*Proof:* We adapt the construction in [17]. For  $\mu = 1$ , the most general form for such a  $\Delta(t)$  is  $at^2 + (1-2a)t + a$  for  $\varepsilon = -1$ , and  $a(a+1)t^2 - (2a(a+1)+1)t + a(a+1)$  for  $\varepsilon = +1$ , where  $a$  is an integer. We may set  $A = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$  or  $\begin{pmatrix} a(a+1) & 2a+1 \\ 0 & 1 \end{pmatrix}$ , for  $\varepsilon = -1$  or  $+1$ , respectively.

Assume the proposition for all  $\mu < \mu_0$ . In addition, assume that  $A$  may be chosen so that the matrix obtained from  $tA + \varepsilon A'$  by deleting the first row and column is  $(t+\varepsilon)(t-\varepsilon)^{2\mu-2}$ . This is true for  $\mu = 1$  of the above choices. We may write a given  $\Delta(t)$ , satisfying (1), (2) and (3), for  $\mu = \mu_0$ , in the form:

$$\Delta(t) = a(t+\varepsilon)^2(t-\varepsilon)^{2\mu_0-2} - \varepsilon t \Delta_0(t)$$

for some polynomial  $\Delta_0(t)$ . It may be checked directly that  $\Delta_0(t)$  satisfies (1), (2) and

(3) for  $\mu = \mu_0 - 1$ . Thus there exists a square matrix  $A_0$  with  $2\mu_0 - 2$  rows, satisfying property  $\varepsilon$ , such that  $\Delta_{A_0}(t) = \Delta_0(t)$  and satisfying the additional hypothesis for  $\mu = \mu_0 - 1$ . We now define  $A$  by:

$$A = \begin{pmatrix} 0 & 1 & -a & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ a & -1 & \hline 0 & 0 & & A_0 & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{pmatrix}$$

It is easily checked that  $A$  satisfies property  $\varepsilon$  and  $\Delta_A(t) = \Delta(t)$ .

15. The polynomial  $\Delta_A(t)$ , clearly an invariant of the congruence class of  $A$ , is also important in studying the cobordism class of  $A$  because of:

**PROPOSITION 3:** *If  $A$  is a null-cobordant matrix of type  $\varepsilon$ , with  $2\mu$  rows, then  $\Delta_A(t) = (-\varepsilon t)^\mu \theta(t) \theta(t^{-1})$  for some integral polynomial  $\theta(t)$ .*

*Proof:* Since  $\Delta_A(t)$  is an invariant of the congruence class of  $A$ , we may assume  $A = \begin{pmatrix} 0 & A_1 \\ A_2 & A_3 \end{pmatrix}$ , where  $A_i$  are  $(\mu \times \mu)$ -matrices. Then  $tA + \varepsilon A' = \begin{pmatrix} 0 & tA_1 + \varepsilon A'_1 \\ tA_2 + \varepsilon A'_2 & tA_3 + \varepsilon A'_3 \end{pmatrix}$  and  $\Delta_A(t) = (-1)^\mu \det(tA_1 + \varepsilon A'_1) \det(tA_2 + \varepsilon A'_2)$ . But  $tA_2 + \varepsilon A'_2 = \varepsilon t(t^{-1}A'_1 + \varepsilon A_2)$ ; and we may set  $\theta(t) = \det(tA_1 + \varepsilon A'_1)$ .

Consider the family of integral polynomials satisfying (1)–(3) of Proposition 1 for a given  $\varepsilon = \pm 1$ , and some  $\mu$  (which is uniquely determined by (1)). Define an equivalence relation among these polynomials by:  $\Delta_1(t) \sim \Delta_2(t)$  if and only if  $\Delta_1(t) \Delta_2(t) = (-\varepsilon t)^\mu \theta(t) \theta(t^{-1})$  for some  $\mu$  and  $\theta(t)$ . If  $P_\varepsilon$  denote the set of equivalence classes, polynomial multiplication induces an abelian group structure on  $P_\varepsilon$  in which every element has order two.

By Proposition 3, a homomorphism  $G_\varepsilon \rightarrow P_\varepsilon$  is induced by  $A \mapsto \Delta_A(t)$ . Proposition 2 implies it is an epimorphism.

### The Quadratic Form

16. If  $A$  satisfies property  $\varepsilon$ , then  $A + A'$  represents an even quadratic form with odd determinant (when  $\varepsilon = -1$ , it differs from the unimodular  $A - A'$  by even entries). From Section 3 we see that the cobordism classes of quadratic forms with non-zero determinant form an abelian group. Let  $K_+, K_-$  be the subgroups defined by restricting the quadratic forms to be even and their determinants to be  $\pm 1$ , and odd, respectively. Then it follows immediately that  $A \mapsto A + A'$  defines homomorphisms  $G_\varepsilon \rightarrow K_\varepsilon$ . Surjectivity when  $\varepsilon = +1$  is obvious; when  $\varepsilon = -1$  we need:

**LEMMA 6:** *If  $B$  is a symmetric integral matrix with even diagonal entries and odd*

determinant, then there exists an integral matrix  $A$ , satisfying property  $-1$ , such that  $A + A' = B$ .

*Proof:* Let  $B_0$  be a skew-symmetric matrix such that  $B \equiv B_0 \pmod{2}$ , i.e. corresponding entries of  $B$  and  $B_0$  have the same parity. This can be accomplished, for example, by changing the diagonal entries of  $B$  to zero, and changing the sign of all entries below the diagonal. Clearly  $B_0$  has odd determinant. Now  $B_0$  is congruent to a matrix of the form  $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ , where  $A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$ ; since  $\det B_0$  is odd, the  $a_i$  are odd. Thus  $B$  is congruent to a matrix  $C = (c_{ij})$ , where  $c_{ij}$  is even unless either (i)  $i$  is odd and  $j = i + 1$ , or (ii)  $i$  is even and  $j = i - 1$  – in these cases  $c_{ij}$  is odd. We define  $\bar{A} = (\bar{a}_{ij})$ , by

- (i)  $\bar{a}_{ij} = 1/2 c_{ij}$ , if  $c_{ij}$  is even,
- (ii)  $\bar{a}_{ij} = 1/2 (c_{ij} + 1)$ , if  $i$  is odd and  $j = i + 1$ , and
- (iii)  $\bar{a}_{ij} = 1/2 (c_{ij} - 1)$ , if  $i$  is even,  $j = i - 1$ .

Then  $C = \bar{A} + \bar{A}'$  and  $\bar{A}$  is congruent to the desired  $A$  by means of the same congruence which transforms  $C$  to  $B$ .

17. If  $A$  is a commutative ring with 1, we define the Witt group of *even* unimodular quadratic forms over  $A$  to be the group (under block sum) of equivalence classes of such forms under the relation defined as follows:  $A \sim B$  if and only if  $A \oplus kU$  and  $B \oplus lU$  are congruent, for some  $k, l$ , where  $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . An inverse for  $A$  is given by  $-A^{-1}$ , since, if  $P = \begin{pmatrix} I - CA^{-1} & -C \\ A^{-1} & I \end{pmatrix}$ , where  $A = C + C'$ , then

$$P(A \oplus (-A^{-1}))P' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

This generalizes the classical Witt group for  $A$  a field of characteristic  $\neq 2$  (see [1] and [5]).  $W(A)$  is easily seen to coincide with the Grothendieck group of even unimodular quadratic forms over  $A$ , divided by the subgroup generated by  $U$ .

The following lemma says that  $W(A)$  is the same as the group of cobordism classes of even unimodular quadratic forms over  $A$ .

**LEMMA 7:** *Let  $B$  be an even unimodular quadratic form over  $A$ . Then  $B$  is null-cobordant if and only if  $B$  is congruent, over  $A$ , to  $U \oplus U \oplus \cdots \oplus U$ .*

*Proof:* We may assume  $B = \begin{pmatrix} 0 & B_1 \\ B_2 & B_3 \end{pmatrix}$ , where  $B_1, B_2$  are invertible over  $A$ , and  $B_3$  has even diagonal entries.

Define  $T = \begin{pmatrix} B_1^{-1} & 0 \\ AB_1^{-1} & I \end{pmatrix}$ , where  $B_3 = A + A'$ , and  $I$  is the identity matrix. Then  $TBT'$  has the form  $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ .



18. Recall that the rational 2-adic integers  $Q(2)$  is the ring of all rational numbers with odd denominator. It follows from Lemma 7 that there are homomorphisms  $K_+ \rightarrow W(Z)$ ,  $K_- \rightarrow W(Q(2))$ , defined by regarding a quadratic form representing an element of  $K_-$  (or  $K_+$ ) as an even unimodular form over  $Q(2)$  (or  $Z$ ).

Define  $A_\varepsilon$  to be  $Z$  if  $\varepsilon = +1$ , and  $Q(2)$  if  $\varepsilon = -1$ .

PROPOSITION 4: *The above homomorphisms are isomorphisms  $K_\varepsilon \approx W(A_\varepsilon)$ .*

This follows immediately from Lemma 7 if  $\varepsilon = +1$ . For  $\varepsilon = -1$ , we first prove another lemma.

19. Suppose  $A \subset \Gamma$  are principal ideal domains such that every element of  $\Gamma/A$  has finite  $A$ -order i.e. if  $\alpha \in \Gamma$ , then there exists  $\lambda \in A$  such that  $\lambda \alpha \in A$ . Let  $C_A$  and  $C_\Gamma$  be the group of cobordism classes of even non-degenerate (i.e. non-zero determinant) quadratic forms over  $A$  and  $\Gamma$  respectively.

LEMMA 8: *The natural homomorphism  $C_A \rightarrow C_\Gamma$  is injective.*

*Proof:* Suppose  $A$  is a non-degenerate even quadratic form over  $A$ , null-cobordant over  $\Gamma$ . If  $A$  is considered as a bilinear form on a free  $A$ -module  $V$ , then there exists a direct summand  $S$  of  $V \otimes_A \Gamma$ , whose  $\Gamma$ -dimension is half the  $A$ -dimension of  $V$ , such that  $A|_S = 0$ . Let  $S_0 = S \cap V$ ; it is easily seen from the hypotheses on  $A$  and  $\Gamma$  that  $S_0$  is a submodule of  $V$  of half its dimension. But  $S_0$  is also a direct summand, since  $V/S_0$  is torsion free and  $A$  is a principal ideal domain.

COROLLARY: *Under the conditions given above on  $A$  and  $\Gamma$ ,  $W(A) \rightarrow W(\Gamma)$  is injective.*

This follows from Lemmas 7 and 8.

It follows from Lemmas 7 and 8 that  $K_- \rightarrow W(Q(2))$  is injective. Surjectivity follows from the observation that if  $A$  is any even unimodular quadratic form over  $Q(2)$ , then, for some odd integer  $a$ ,  $aA$  is an even integral quadratic form, with odd determinant, and, if  $a$  is square,  $aA$  is congruent to  $A$  over  $Q(2)$ .

20. Summarizing, we have defined epimorphisms:

$$G_\varepsilon \rightarrow W(A_\varepsilon)$$

induced by  $A \mapsto A + A'$ .  $A$  represents an element of the kernel if and only if  $A + A'$  is null-cobordant (over the integers).

21. We now make some general remarks about the Witt groups  $W(A_\varepsilon)$ . If  $\varepsilon = +1$ ,  $A_\varepsilon = Z$  and there is an isomorphism  $W(Z) \approx Z$ , defined by  $A \mapsto 1/8$  (signature  $A$ ) (see [11] and [5, Appendix]).

If  $\varepsilon = -1$ ,  $A_\varepsilon = Q(2)$ . Letting  $Q$  be the rational numbers, we have a *monomorphism*



$W(Q(2)) \rightarrow W(Q)$  – see Corollary to Lemma 8. There are several well-known invariants of rational quadratic forms:

- (a)  $W(Q) \rightarrow Q^+/(Q^+)^2$ , defined by determinant, multiplied by a sign,
- (b)  $W(Q) \rightarrow \mathbb{Z}$ , defined by signature.

These are homomorphisms, but the following is not.

- (c)  $W(Q) \rightarrow \{-1, +1\}$ , defined by the Minkowski unit  $C_p$  for every prime  $p$ . (see [15]).

These form a complete set of invariants for  $W(Q)$ , i.e. two elements of  $W(Q)$  are equal if and only if their determinants, signatures and Minkowski units coincide (see [6]). Such is, therefore, also true for  $W(Q(2))$ . Moreover the range of values taken by these invariants on elements of  $W(Q(2))$  is known ([6, Theorems 29 and 45]).

It follows readily from these facts, and the additivity formulae for  $C_p$  (see [15 (2.5)]) that every element of  $W(Q(2))$  has order 1, 2, 4 or  $\infty$ . For example, any element of  $W(Q(2))$  with determinant 3 and signature 0 must, by considering  $C_3$ , have order 4. Also see [5, Appendix A].

### The Range of the Invariants

22. We have so far defined epimorphisms:  $G_\varepsilon \rightarrow P_\varepsilon$  and  $G_\varepsilon \rightarrow W(\Lambda_\varepsilon)$ . We now consider their direct sum:

$$\phi_\varepsilon: G_\varepsilon \rightarrow P_\varepsilon \oplus W(\Lambda_\varepsilon), \quad \varepsilon = \pm 1.$$

We would like to calculate the image of  $\phi_\varepsilon$ , and investigate injectivity.

We shall see that the image of  $\phi_\varepsilon$  is defined by the relation  $\Delta_A(\varepsilon) = \text{determinant}(A + A')$ . Let  $U_\varepsilon$  be the multiplicative groups of units in  $\Lambda_\varepsilon$  and  $U_\varepsilon^2$  the subgroup of square units. We define homomorphisms:

$$d_\varepsilon: P_\varepsilon \oplus W(\Lambda_\varepsilon) \rightarrow U_\varepsilon/U_\varepsilon^2$$

by  $d_\varepsilon(A, B) = (-1)^{\mu+e} \Delta(\varepsilon)$  determinant  $B$ , where  $\mu$  is determined from  $A$  as in Proposition 1, and  $\text{rank } B = 2q$ .

We also define epimorphisms  $U_\varepsilon/U_\varepsilon^2 \rightarrow \mathbb{Z}_2$  by  $a \mapsto (a-1)/2 \pmod{2}$ , where we use the fact that  $\Lambda_\varepsilon/2\Lambda_\varepsilon \approx \mathbb{Z}_2$ . If  $\varepsilon = +1$ , this is an isomorphism.

23. PROPOSITION 5: *The following sequence is exact:*

$$G_\varepsilon \xrightarrow{\phi_\varepsilon} P_\varepsilon \oplus W(\Lambda_\varepsilon) \xrightarrow{d_\varepsilon} U_\varepsilon/U_\varepsilon^2 \rightarrow \mathbb{Z}_2 \rightarrow 0$$

Thus  $\phi_+$  is onto.

*Proof: Exactness at  $U_\varepsilon/U_\varepsilon^2$ :* If  $\varepsilon = +1$ , it is shown in the proof of Proposition 1, that an even unimodular quadratic form of rank  $2q$  has determinant  $(-1)^e$ . This fact, and (3) of Proposition 1 imply  $d_+ = 0$ .

Suppose  $\varepsilon = -1$ . If  $\Delta(t)$  satisfies (1) and (3) of Proposition 1, an easy computation shows that  $\Delta(1) - \Delta(-1)$  is a multiple of 4. Thus, when  $\varepsilon = -1$ ,  $\Delta(-1) \equiv (-1)^\mu \pmod{4}$ . Furthermore, if  $B$  is an even unimodular quadratic form over  $R(2)$  of rank  $2\varrho$ , then  $B$  is congruent to a block sum of  $\varrho$   $2 \times 2$ -matrices (see [5] or [6]). But an even unimodular quadratic form of rank 2 is easily checked to have a determinant  $\equiv -1 \pmod{4}$ . It follows that determinant  $B \equiv (-1)^\varrho \pmod{4}$ . From these considerations, if  $(\Delta, B) \in P_- \oplus W(Q(2))$ , then  $d_-(\Delta, B) \equiv 1 \pmod{4}$ .

To complete the proof of exactness at  $U_-/U_-^2$ , we notice that any integer of the form  $4a-1$  can be realized as  $\Delta(-1)$  for some representative  $\Delta(t)$  of an element of  $P_-$  with  $\mu=1$ ; e.g. let  $\Delta(t) = at^2 + (1-2a)t + a$ .

24. *Exactness at  $P_\varepsilon \oplus W(\Lambda_\varepsilon)$* : The relation  $\Delta_A(\varepsilon) = \det(A + A')$  implies that  $\text{Image } \phi_\varepsilon \subset \text{Kernel } d_\varepsilon$ , since  $d_\varepsilon(\Delta_A, A + A') = (-1)^{\mu+\varrho} \Delta_A(\varepsilon)^2$ , and  $\varrho = \mu$  by Proposition 1.

We now show that  $\text{Kernel } d_\varepsilon \subset \text{Image } \phi_\varepsilon$ . Suppose  $\Delta(t)$ , satisfying (1), (2) and (3), also satisfies:  $(-1)^\mu \Delta(\varepsilon)$  is square i.e.  $(\Delta, 0) \in \text{Kernel } d_\varepsilon$ . Then, if  $A$  is the matrix constructed in the proof of Proposition 2, we will show that  $A + A'$  is null-cobordant i.e.  $(\Delta, 0) = \phi_\varepsilon(A)$ .

$$\text{If } \mu=1 \text{ and } \varepsilon=+1, \text{ then } A + A' = \begin{pmatrix} 2a(a+1) & 2a+1 \\ 2a+1 & 2 \end{pmatrix}.$$

If we subtract  $a$  times the second row from the first, and then perform the corresponding column operation, we obtain a null-cobordant matrix. If  $\mu=1$  and  $\varepsilon=-1$ , then  $A + A' = \begin{pmatrix} 2a & 1 \\ 1 & 2 \end{pmatrix}$ ; but  $-\Delta(-1)$  being square implies  $a = b(1-b)$  for some integer  $b$ . If we subtract  $b$  times the second row from the first, and then perform the corresponding column operation, we obtain a null-cobordant matrix. We now proceed to the inductive step. If  $\Delta(t)$  satisfies the given conditions, so does  $\Delta_0(t)$ ; we then assume  $A_0 + A'_0$  is null-cobordant. Now  $A + A' = U \oplus (A_0 + A'_0)$ , where  $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , which is clearly null-cobordant, if  $A_0 + A'_0$  is.

We have shown that  $P_\varepsilon \cap \text{Kernel } d_\varepsilon \subset \text{Image } \phi_\varepsilon$ . This, together with the facts that (i) the composition of  $\phi_\varepsilon$  with projection to  $W(\Lambda_\varepsilon)$  is onto, and (ii)  $\text{Image } \phi_\varepsilon \subset \text{Kernel } d_\varepsilon$ , implies that  $\text{Kernel } d_\varepsilon = \text{Image } \phi_\varepsilon$  by the following argument. Suppose  $(\Delta, B) \in \text{Kernel } d_\varepsilon$ . By (i),  $(\Delta', B) \in \text{Image } \phi_\varepsilon$ , for some  $\Delta'$ . Then  $(\Delta, B) - (\Delta', B) = (\Delta \cdot \Delta', 0) \in \text{Kernel } d_\varepsilon$ , by (ii). But we have shown that  $(\Delta \cdot \Delta', 0) \in \text{Image } \phi_\varepsilon$ ; therefore,  $(\Delta, B) = (\Delta', B) + (\Delta \cdot \Delta', 0) \in \text{Image } \phi_\varepsilon$ .

### More Invariants

25. We now see that  $G_\varepsilon$  is very large.

**PROPOSITION 6:**  $G_\varepsilon$  contains an infinite linearly independent set.

*Proof:* We define a new invariant. Given a matrix  $A$  of type  $\varepsilon$  and  $\xi$  a complex

number of unit norm, we consider the Hermitian form:

$$B_\xi = \begin{cases} \frac{\xi A + A'}{\xi + 1} & \xi \neq -1 \\ i(A' - A) & \xi = -1 \end{cases}$$

The signature of  $B_\xi$  is well-defined. Let  $S_A$  be the unit circle in the complex plane with the zeros of determinant  $(\xi A + A')$  (as a function of  $\xi$ ) removed. If  $\sigma_A(\xi)$  = signature  $B_\xi$  for  $\xi \in S_A$ , then we will show that  $\sigma_A$  is continuous.

Recall the characterization of signature given in [6, § 3]; the arguments there apply with only slight modification to Hermitian forms. A non-singular Hermitian matrix  $M$  is regular if the sequence of principal minors  $1, D_1, \dots, D_n$  has no two consecutive zeros;  $D_i$  is the determinant of the submatrix of  $M$  formed by the first  $i$  rows and columns. The following two facts are of importance;

(a). If  $M$  is regular, then the signature of  $M$  is the number of permanences of sign reduced by the number of changes of sign in the sequence  $1, D_1, \dots, D_n$ , where, if  $D_i = 0$  we may assign it either sign.

(b). Any non-singular Hermitian matrix  $M$  is congruent to a regular matrix i.e. there exists a non-singular matrix  $P$  such that  $P M \bar{P}'$  is regular ( $\bar{\phantom{x}}$  is complex conjugate).

Now suppose  $\xi \in S_A$ ,  $\xi \neq -1$ . Then  $B_\xi$  is congruent to a regular matrix  $P B_\xi \bar{P}'$ . Clearly if  $\eta \in S_A$  is near enough to  $\xi$ , then  $P B_\eta \bar{P}'$  is also regular and the non-zero minors  $D_i$  have the same sign in  $\xi$  and  $\eta$ . It then follows easily from (a) that  $B_\xi$  and  $B_\eta$  have the same signature.

To establish continuity at  $\xi = -1$ , notice that  $|\xi + 1| B_\xi$  has the same signature as  $B_\xi$ , for  $\xi \neq -1$ . Since  $|\xi + 1|/(\xi + 1) \rightarrow i$  as  $\xi \rightarrow -1$ ,  $B_{-1} = \lim_{\xi \rightarrow -1} |\xi + 1| B_\xi$  and we can apply the argument of the previous paragraph to  $|\xi + 1| B_\xi$ .

Notice that  $\sigma_A(-1) = 0$  for all  $A$ .

Clearly, if  $A$  is null-cobordant then  $\sigma_A = 0$ . Also note that, if  $A = A_1 \oplus A_2$ , then  $\sigma_A = \sigma_{A_1} + \sigma_{A_2}$  on  $S_A = S_{A_1} \cap S_{A_2}$ .

26. We first construct the linearly independent set for  $\varepsilon = -1$ . For any integer  $k \geq 1$ , define  $A_k = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$ . Then  $S_{A_k} = S_k$  is obtained by removing the points  $(1/2k^2) - 1 \pm i(4k^2 - 1)^{1/2}/2k^2$  from the unit complex circle. It is easily checked that  $\sigma_k(-1) = 0$ ,  $\sigma_k(1) = 2$ . If  $N_k$  is the component of  $S_k$  in which  $\sigma_k(\xi) = 2$ , then we see that  $N_1 \subset N_2 \subset \dots \subset N_k \subset N_{k+1} \subset \dots$ , where the inclusions are proper. To see that the elements of  $G_-$  determined by  $\{A_k\}$  are linearly independent, suppose  $\sum \oplus \lambda_k A_k = A$  is null-cobordant. If  $k$  is the largest integer for which  $\lambda_k \neq 0$  and  $\xi \in N_k - N_{k-1}$ , then  $\sigma_A(\xi) = 2\lambda_k$ . But  $\sigma_A = 0$ , and so  $\lambda_k = 0$ .

For  $\varepsilon=1$ , we take  $A_k = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ .

We leave it to the reader to check that  $\sigma(1)=\sigma(-1)=0$ ,  $\sigma(i)=2$  and, if  $N_k$  is the component of  $S_k$  containing  $i$ , then  $N_1 \subset N_2 \subset \dots$  are proper inclusions. We may now use the above argument to conclude that  $\{A_k\}$  represent linearly independent elements of  $G_+$ .

### Non-cobordant Knots With Homotopy Equivalent Complements

27. LEMMA 9: *If  $n \geq 2$ , then two simple  $(2n-1)$ -knots have homotopy equivalent complements if and only if their  $n$ -th homotopy groups are isomorphic as  $\Lambda$ -modules (see § 7).*

*Proof:* Note that the universal coverings of the complements have non-zero homology groups only in dimension  $n$  [13], which are there isomorphic to the  $n$ -th homotopy groups of the complements – the action of  $\Lambda$  corresponding to the action of the covering transformations. The result now follows by obstruction theory considerations.

Let  $K$  be a  $(2n-1)$ -knot whose quadratic form has non-zero signature. This is possible, for every  $n \geq 1$ , according to the theorem and § 20. Now consider  $K_1 = K \# K$  and  $K_2 = K \# (-K)$ .  $K_1$  has non-zero signature, while  $K_2$  is null-cobordant. If  $A$  is a Seifert matrix for  $K$  and  $A_1 = A \oplus A$ ,  $A_2 = A \oplus (-A)$ , then  $A_i$  is a Seifert matrix for  $K_i$ . Recall (see § 7) that  $tA_i + (-1)^n A'_i = R_i$  is a relation matrix for  $\pi_n(X_i)$  as a  $\Lambda$ -module;  $X_i$  is the complement of  $K_i$ . But, if  $P = I \oplus (-I)$ , then  $PA_1 = A_2$ ,  $PA'_1 = A'_2$ , and so  $PR_1 = R_2$ . Therefore, by Lemma 9,  $X_1$  and  $X_2$  are homotopy equivalent.

We have proved.

PROPOSITION 7: For every  $n \geq 2$ , there exist non-cobordant  $(2n-1)$ -knots with homotopy equivalent complements.

28. Note that, if  $n$  is even and  $\Sigma$  is a  $(2n-1)$ -knot, the signature of the quadratic form of  $\Sigma$  is also the signature of a parallelizable manifold bounded by  $\Sigma$  (see § 7(3)). In the construction above,  $K_1$  can be chosen to have signature 16, for  $n$  even  $\geq 4$ . It follows from the Index Theorem that  $K_1$  is not diffeomorphic to  $S^{2n-1}$ .  $K_2$  is diffeomorphic to  $S^{2n-1}$ , of course.

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