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Note on Poisson's Treatment of the Euler-Maclaurin Formula¹⁾

(To Hugo Hadwiger for his 60-th birthday)

By ALEXANDER OSTROWSKI (Basel)

1. In the Euler-Maclaurin Formula

$$\sum_{\mu=p}^q f(\mu) = \int_p^q f(t) dt + \sum_{v=1}^n \frac{B_{2v}}{(2v)!} (f^{(2v-1)}(q) - f^{(2v-1)}(p)) + R_n \quad (n \geq 1), \quad (1)$$

an expression for the remainder term was first given by POISSON [6] 1827. This is

$$R_n = \frac{1}{(2n+1)!} \int_p^q \overline{B_{2n+1}}(t) f^{(2n+1)}(t) dt, \quad (2)$$

where $\overline{B_{2n+1}}(t)$ is given by

$$B_{2n+1}(t) = \frac{2}{(2\pi)^{2n+1}} U_{2n+1}(t), \quad U_s(t) = \sum_{v=1}^{\infty} \frac{\sin 2v\pi t}{v^s} \quad (s > 1). \quad (3)$$

To obtain from here an estimate for $U_{2n+1}(t)$, POISSON used the obvious bound

$$|U_{2n+1}(t)| \leq \sum_{v=1}^{\infty} \frac{1}{v^{2n+1}}. \quad (4)$$

2. It has apparently not as yet been noticed, that the inequality (4) can be replaced with

$$|U_{2n+1}(t)| < 1 \quad (0 \leq t \leq 1, \quad n = 1, 2, \dots)^2) \quad (5)$$

¹⁾ The work on this paper was partly done under the Contract DA J A 37-67-C-0628 of the Institute of Mathematics, University of Basel with the US Department of the Army, European Research Office.

²⁾ This follows immediately from the relation

$$\left| \frac{B_{2n+1}(t)}{(2n+1)!} \right| < \frac{2}{(2\pi)^{2n+1}} \quad (0 \leq t \leq 1, \quad n \geq 1).$$

As to this formula, it is for $n > 1$ equivalent with a formula deduced by LEHMER on p. 538 of his paper, LEHMER [2] (see the formula without number, following immediately after LEHMER's formula (18) in his paper). However, this relation is also true for $n = 1$, as is immediately verified, using for instance

the value of $\text{Max}_{\langle 0, 1 \rangle} \left| \frac{B_3(t)}{6} \right| = .00801875$ (which is obtained from LEHMER's value of his M_3), since this

this is $< \frac{1}{4\pi^3} = .0806$.

However, the inequality (5) can be further improved. It is easy to see that we have in any case

$$\max_{\langle 0, \frac{1}{2} \rangle} |U_s(t)| > 1 - \frac{1}{3^s} \quad (s > 1) \tag{6}$$

(see section 4).

As to the upper limit of $|U_s(t)|$, we will prove that we have

$$|U_s(t)| < 1 - \frac{1}{3^s} \quad (s \geq 7). \tag{7}$$

On the other hand we have

$$\max_{\langle 0, \frac{1}{2} \rangle} |U_5(t)| = 1 - \frac{.496}{3^{5-}}, \tag{8}$$

$$\max_{\langle 0, \frac{1}{2} \rangle} |U_3(t)| = 1 - \frac{.147}{3^3}. \tag{9}$$

3. The inequality (7) will be proved for all real s by a rather elementary method, making no use of the results obtained in the last decades on the zeros of the Bernoullian Polynomials of even order. (NÖRLUND [4], LENSE [3], LEHMER [2], INKERI [1], OSTROWSKI [5].) As to the values (8) and (9), they are obtained using the connection between the Bernoullian Polynomials $B_5(t)$ and $B_3(t)$ with $U_5(t)$ and $U_3(t)$.

4. We obtain at once from (3)

$$U_s\left(\frac{1}{4}\right) = \sum_{v=1}^{\infty} \frac{\sin v \pi/2}{v^s} = \frac{1}{1^s} - \frac{1}{3^s} + \left(\frac{1}{5^s} - \frac{1}{7^s}\right) + \dots > 1 - \frac{1}{3^s} \quad (s > 1),$$

and this proves (6).

5. In order to prove (7) we introduce

$$\varphi(t) = \sin \pi t + \frac{\sin 2 \pi t}{2^s} + \frac{\sin 3 \pi t}{3^s} \tag{10}$$

and put

$$U_s(t/2) = \varphi(t) + R(t), \quad R(t) = \sum_{v=4}^{\infty} \frac{\sin v \pi t}{v^s}. \tag{11}$$

Then we have

$$\begin{aligned} |R(t)| &\leq \frac{1}{4^s} + \sum_{v=5}^{\infty} \frac{1}{v^s} < \frac{1}{4^s} + \int_4^{\infty} \frac{dx}{x^s} \\ &= \frac{1}{4^s} \left(1 + \frac{4}{s-1}\right) \leq \frac{5}{4^s} = \frac{5}{3} \cdot \frac{1}{3^s} \cdot \left(\frac{3}{4}\right)^s \leq \frac{5}{3} \cdot \left(\frac{3}{4}\right)^7 \frac{1}{3^s}, \\ |R(t)| &\leq \frac{.2225}{3^s} \quad (s \geq 7). \end{aligned} \tag{12}$$

6. We are now going to discuss $\text{Max}_{\langle -1, 1 \rangle} |\varphi(t)| = \text{Max}_{\langle 0, 1 \rangle} |\varphi(t)|$ and to prove that

$$|\varphi(t)| \leq 1 - \frac{.7232}{3^s} \quad (|t| \leq 1, s \geq 7). \quad (13)$$

Let

$$u = \frac{\pi}{2} - \pi t, \quad 0 \leq |u| \leq \frac{\pi}{2}, \quad \sigma = \sin u, \quad \gamma = \cos u. \quad (14)$$

Then we have

$$\varphi(t) = \gamma K, \quad K = 1 - \frac{1}{3^s} + \frac{2\sigma}{2^s} + \frac{4\sigma^2}{3^s}, \quad (15)$$

$$f(u) := -\frac{d}{du} \varphi(t) = \sin u - \frac{3 \sin 3u}{3^s} - \frac{2 \cos 2u}{2^s}, \quad (16)$$

$$F_2(\sigma) := 3^{s-1} f(u) = 4\sigma^3 + 2\left(\frac{3}{2}\right)^{s-1} \sigma^2 + (3^{s-1} - 3)\sigma - \left(\frac{3}{2}\right)^{s-1}. \quad (17)$$

As $F_2(1) > \left(\frac{3}{2}\right)^{s-1} > 0$ and by Descartes' rule, $F_2(\sigma)$ has exactly one positive zero, σ_0 , which is < 1 . On the other hand, as $F_2'(\sigma) = 12\sigma^2 + 4\left(\frac{3}{2}\right)^{s-1}\sigma + 3^{s-1} - 3$, we have for $|\sigma| \leq 1$ and $s \geq 7$:

$$|F_2'(\sigma)| \geq 3^{s-1} - 3 - 12 - 4\left(\frac{3}{2}\right)^{s-1} = 3^{s-1} \left(1 - \frac{1}{2^{s-3}}\right) - 15 \geq 729 - \frac{15}{16} - 15 > 0,$$

and we see that $F_2(\sigma)$ cannot have a zero in $\langle -1, 0 \rangle$.

7. Putting $\sigma = \frac{x}{2^{s-1}}$ into $F_2(\sigma)$ we obtain

$$F_1(x) := 6^{s-1} f(u) = \frac{x^3}{4^{s-2}} + 2\left(\frac{3}{4}\right)^{s-1} x^2 + (3^{s-1} - 3)x - 3^{s-1},$$

$$F_1(1) = \frac{1}{4^{s-2}} + 2\left(\frac{3}{4}\right)^{s-1} - 3 < 0.$$

We see that $F_1(x)$ has exactly one positive zero, ξ , which is > 1 . Writing $x = 1 + y$, $F_1(x) = 3F(y)$ we obtain

$$F(y) = \frac{16y^3}{3 \cdot 4^s} + \left[\frac{16}{4^s} + \frac{8}{9} \left(\frac{3}{4}\right)^s \right] y^2 + \left[\frac{16}{4^s} + \frac{16}{9} \left(\frac{3}{4}\right)^s + \frac{1}{9} 3^s - 1 \right] y - \left[1 - \frac{16}{3 \cdot 4^s} - \frac{8}{9} \left(\frac{3}{4}\right)^s \right].$$

For positive y it follows

$$F(y) > \left(\frac{1}{9} 3^s - 1\right) y - \left(1 - \frac{8}{9} \left(\frac{3}{4}\right)^s\right)$$

and therefore

$$F\left(\frac{2}{2^s}\right) > \frac{2}{9}\left(\frac{3}{2}\right)^s - 1 + \frac{8}{9}\left(\frac{3}{4}\right)^s - \frac{2}{2^s} > 0 \quad (s \geq 7).$$

8. We see that $\xi = 1 + \frac{2\theta}{2^s}$,

$$\sigma_0 = \frac{2}{2^s} + \theta \frac{4}{4^s}, \quad 0 < \theta < 1, \quad (s \geq 7), \tag{18}$$

$$\sigma_0 = \frac{2}{2^s}\left(1 + \frac{2\theta}{2^s}\right) < \frac{2}{2^s}\left(1 + \frac{1}{2^6}\right) = \frac{2}{2^s}\left(1 + \frac{1}{64}\right) \quad (s \geq 7). \tag{19}$$

Since $\varphi(t)$ vanishes for $t=0$ and $t=1$, the maximum of $\varphi(t)$ is attained at a point where $f(u)$ vanishes, that is to which corresponds $\sin u = \sigma_0$. The corresponding values of $\gamma = \cos u$ is then

$$\sqrt{1 - \sigma_0^2} < 1 - \sigma_0^2/2 < 1 - \frac{2}{4^s}.$$

9. We now have from (15)

$$|\varphi(t)| < \left(1 - \frac{2}{4^s}\right)K,$$

$$\begin{aligned} K &= 1 - \frac{1}{3^s} + K_1, \quad K_1 = \frac{2\sigma_0}{2^s} + \frac{4\sigma_0^2}{3^s} = \frac{2\sigma_0}{2^s}\left(1 + \frac{2\sigma_0}{\left(\frac{3}{2}\right)^s}\right) \\ &< \frac{4}{4^s}\left(1 + \frac{1}{64}\right)\left(1 + \frac{4 + \frac{1}{8}}{3^7}\right) < \left(1 + \frac{1}{56}\right)\frac{4}{4^s} < \frac{57}{14 \cdot 4^s}, \\ K &< 1 - \frac{1}{3^s} + \frac{57}{14 \cdot 4^s}, \end{aligned}$$

$$\begin{aligned} |\varphi(t)| &< \left(1 - \frac{2}{4^s}\right)\left(1 - \frac{1}{3^s} + \frac{57}{14 \cdot 4^s}\right) = 1 - \frac{1}{3^s} + \frac{29}{14 \cdot 4^s} + \frac{2}{12^s} - \frac{81}{16^s} \\ &< 1 - \frac{1}{3^s} + \frac{1}{3^s}\left(\frac{29}{14}\left(\frac{3}{4}\right)^s + \frac{2}{4^s}\right) \leq 1 - \frac{1}{3^s} + \frac{1}{3^s}\left(\frac{29}{14}\left(\frac{3}{4}\right)^7 + \frac{2}{4^7}\right), \\ |\varphi(t)| &< 1 - \frac{1}{3^s} + \frac{.2766}{3^s} = 1 - \frac{.7234}{3^s}, \end{aligned}$$

and by (12) and (13)

$$|U_s(t)| < 1 - \frac{.7234}{3^s} + \frac{.2225}{3^s} = 1 - \frac{.5009}{3^s},$$

which proves (7).

It is of some interest to observe that (7) and our proof of (7) remain valid if

$$U_s(t) = \sum_{v=1}^{\infty} \frac{\sin 2v\pi t}{v^s} \text{ is replaced with } \sum_{v=1}^N \frac{\sin 2v\pi t}{v^s} \text{ for any } N \geq 3.$$

9. To verify (8) and (9) we use the fact that (3) is in $\langle 0, 1 \rangle$ a polynomial, the so-called Bernoullian Polynomial $B_{2n+1}(t)$. We have in particular $B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$, $B_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t$. The maximum of $|B_3(t)|$ in $\langle 0, 1 \rangle$ is attained at $t = .211325$ and gives $|B_3(0.211325)| = .048113$. Since $\frac{2\pi^3}{3} = 20.67085$, we obtain $\text{Max}_{\langle 0, 1 \rangle} |U_3(t)| = 0.99454 = 1 - \frac{.147}{27}$, which gives (9).

The maximum of $|B_5(t)|$ in $\langle 0, 1 \rangle$ is attained at $t = .240335$ and the corresponding value of $|B_5(t)|$ is .0244582. Since $\frac{2\pi^5}{15} = 40,8026$, we obtain

$$\text{Max}_{\langle 0, 1 \rangle} |U_5(t)| = .997958 = 1 - \frac{.496}{243},$$

which proves (8).

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