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Note on Poissson's Treatment of the Euler–Maclaurin Formula [1)]

(To Hugo Hadwiger for his 60-th birthday)

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1. In the Euler-Maclaurin Formula

$$\sum_{\mu=p}^{q}{}' f(\mu) = \int_p^q f(t)\,dt + \sum_{\nu=1}^{n} \frac{B_{2\nu}}{(2\nu)!}\left(f^{(2\nu-1)}(q) - f^{(2\nu-1)}(p)\right) + R_n \qquad (n \geqq 1), \tag{1}$$

an expression for the remainder term was first given by POISSON [6] 1827. This is

$$R_n = \frac{1}{(2n+1)!}\int_p^q \overline{B_{2n+1}}(t)\, f^{(2n+1)}(t)\,dt, \tag{2}$$

where $\overline{B_{2n+1}}(t)$ is given by

$$B_{2n+1}(t) = \frac{2}{(2\pi)^{2n+1}}\, U_{2n+1}(t), \qquad U_s(t) = \sum_{\nu=1}^{\infty} \frac{\sin 2\nu\pi t}{\nu^s} \qquad (s > 1). \tag{3}$$

To obtain from here an estimate for $U_{2n+1}(t)$, POISSON used the obvious bound

$$|U_{2n+1}(t)| \leqq \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2n+1}}. \tag{4}$$

2. It has apparently not as yet been noticed, that the inequality (4) can be replaced with

$$|U_{2n+1}(t)| < 1 \qquad (0 \leqq t \leqq 1, \quad n = 1, 2, \ldots).^{2)} \tag{5}$$

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2) This follows immediately from the relation

$$\left|\frac{B_{2n+1}(t)}{(2n+1)!}\right| < \frac{2}{(2\pi)^{2n+1}} \qquad (0 \leqq t \leqq 1,\ n \geqq 1).$$

As to this formula, it is for $n > 1$ equivalent with a formula deduced by LEHMER on p. 538 of his paper, LEHMER [2] (see the formula without number, following immediately after LEHMER's formula (18) in his paper). However, this relation is also true for $n = 1$, as is immediately verified, using for instance the value of $\operatorname{Max}_{\langle 0,1\rangle}\left|\frac{B_3(t)}{6}\right| = .00801875$ (which is obtained from LEHMER's value of his M_3), since this this is $< \frac{1}{4\pi^3} = .0806$.

However, the inequality (5) can be further improved. It is easy to see that we have in any case

$$\max_{\langle 0,\frac{1}{2}\rangle} |U_s(t)| > 1 - \frac{1}{3^s} \quad (s > 1) \tag{6}$$

(see section 4).

As to the upper limit of $|U_s(t)|$, we will prove that we have

$$|U_s(t)| < 1 - \frac{\frac{1}{2}}{3^s} \quad (s \geqq 7). \tag{7}$$

On the other hand we have

$$\max_{\langle 0,\frac{1}{2}\rangle} |U_5(t)| = 1 - \frac{.496}{3^5}, \tag{8}$$

$$\max_{\langle 0,\frac{1}{2}\rangle} |U_3(t)| = 1 - \frac{.147}{3^3}. \tag{9}$$

3. The inequality (7) will be proved for all real s by a rather elementary method, making no use of the results obtained in the last decades on the zeros of the Bernoullian Polynomials of even order. (Nörlund [4], Lense [3], Lehmer [2], Inkeri [1], Ostrowski [5].) As to the values (8) and (9), they are obtained using the connection between the Bernoullian Polynomials $B_5(t)$ and $B_3(t)$ with $U_5(t)$ and $U_3(t)$.

4. We obtain at once from (3)

$$U_s(\tfrac{1}{4}) = \sum_{\nu=1}^{\infty} \frac{\sin \nu\pi/2}{\nu^s} = \frac{1}{1^s} - \frac{1}{3^s} + \left(\frac{1}{5^s} - \frac{1}{7^s}\right) + \cdots > 1 - \frac{1}{3^s} \quad (s > 1),$$

and this proves (6).

5. In order to prove (7) we introduce

$$\varphi(t) = \sin \pi t + \frac{\sin 2\pi t}{2^s} + \frac{\sin 3\pi t}{3^s} \tag{10}$$

and put

$$U_s(t/2) = \varphi(t) + R(t), \quad R(t) = \sum_{\nu=4}^{\infty} \frac{\sin \nu\pi t}{\nu^s}. \tag{11}$$

Then we have

$$|R(t)| \leqq \frac{1}{4^s} + \sum_{\nu=5}^{\infty} \frac{1}{\nu^s} < \frac{1}{4^s} + \int_4^{\infty} \frac{dx}{x^s}$$

$$= \frac{1}{4^s}\left(1 + \frac{4}{s-1}\right) \leqq \frac{\frac{5}{3}}{4^s} = \frac{5}{3} \cdot \frac{1}{3^s} \cdot \left(\frac{3}{4}\right)^s \leqq \frac{5}{3} \cdot \left(\frac{3}{4}\right)^7 \frac{1}{3^s},$$

$$|R(t)| \leqq \frac{.2225}{3^s} \quad (s \geqq 7). \tag{12}$$

6. We are now going to discuss $\underset{\langle -1,1\rangle}{\operatorname{Max}} |\varphi(t)| = \underset{\langle 0,1\rangle}{\operatorname{Max}} |\varphi(t)|$ and to prove that

$$|\varphi(t)| \leqq 1 - \frac{.7232}{3^s} \quad (|t| \leqq 1,\ s \geqq 7). \tag{13}$$

Let

$$u = \frac{\pi}{2} - \pi t, \quad 0 \leqq |u| \leqq \frac{\pi}{2}, \quad \sigma = \sin u, \quad \gamma = \cos u. \tag{14}$$

Then we have

$$\varphi(t) = \gamma K, \quad K = 1 - \frac{1}{3^s} + \frac{2\sigma}{2^s} + \frac{4\sigma^2}{3^s}, \tag{15}$$

$$f(u) := -\frac{d}{du}\varphi(t) = \sin u - \frac{3\sin 3u}{3^s} - \frac{2\cos 2u}{2^s}, \tag{16}$$

$$F_2(\sigma) := 3^{s-1} f(u) = 4\sigma^3 + 2\left(\frac{3}{2}\right)^{s-1}\sigma^2 + (3^{s-1} - 3)\,\sigma - \left(\frac{3}{2}\right)^{s-1}. \tag{17}$$

As $F_2(1) > (\frac{3}{2})^{s-1} > 0$ and by Descartes' rule, $F_2(\sigma)$ has exactly one positive zero, σ_0, which is < 1. On the other hand, as $F_2'(\sigma) = 12\sigma^2 + 4(\frac{3}{2})^{s-1}\sigma + 3^{s-1} - 3$, we have for $|\sigma| \leqq 1$ and $s \geqq 7$:

$$|F_2'(\sigma)| \geqq 3^{s-1} - 3 - 12 - 4\left(\frac{3}{2}\right)^{s-1} = 3^{s-1}\left(1 - \frac{1}{2^{s-3}}\right) - 15 \geqq 729 - \frac{15}{16} - 15 > 0,$$

and we see that $F_2(\sigma)$ cannot have a zero in $\langle -1, 0\rangle$.

7. Putting $\sigma = \frac{x}{2^{s-1}}$ into $F_2(\sigma)$ we obtain

$$F_1(x) := 6^{s-1} f(u) = \frac{x^3}{4^{s-2}} + 2\left(\frac{3}{4}\right)^{s-1} x^2 + (3^{s-1} - 3)\,x - 3^{s-1},$$

$$F_1(1) = \frac{1}{4^{s-2}} + 2\left(\frac{3}{4}\right)^{s-1} - 3 < 0.$$

We see that $F_1(x)$ has exactly one positive zero, ξ, which is > 1. Writing $x = 1 + y$, $F_1(x) = 3F(y)$ we obtain

$$F(y) = \frac{16}{3}\frac{y^3}{4^s} + \left[\frac{16}{4^s} + \frac{8}{9}\left(\frac{3}{4}\right)^s\right] y^2$$
$$+ \left[\frac{16}{4^s} + \frac{16}{9}\left(\frac{3}{4}\right)^s + \frac{1}{9}3^s - 1\right] y - \left[1 - \frac{16}{3}\frac{1}{4^s} - \frac{8}{9}\left(\frac{3}{4}\right)^s\right].$$

For positive y it follows

$$F(y) > (\tfrac{1}{9}3^s - 1)\,y - (1 - \tfrac{8}{9}(\tfrac{3}{4})^s)$$

and therefore

$$F\left(\frac{2}{2^s}\right) > \frac{2}{9}\left(\frac{3}{2}\right)^s - 1 + \frac{8}{9}\left(\frac{3}{4}\right)^s - \frac{2}{2^s} > 0 \quad (s \geqq 7).$$

8. We see that $\xi = 1 + \frac{2\theta}{2^s}$,

$$\sigma_0 = \frac{2}{2^s} + \theta\frac{4}{4^s}, \quad 0 < \theta < 1, \quad (s \geqq 7), \tag{18}$$

$$\sigma_0 = \frac{2}{2^s}\left(1 + \frac{2\theta}{2^s}\right) < \frac{2}{2^s}\left(1 + \frac{1}{2^6}\right) = \frac{2}{2^s}\left(1 + \frac{1}{64}\right) \quad (s \geqq 7). \tag{19}$$

Since $\varphi(t)$ vanishes for $t = 0$ and $t = 1$, the maximum of $\varphi(t)$ is attained at a point where $f(u)$ vanishes, that is to which corresponds $\sin u = \sigma_0$. The corresponding values of $y = \cos u$ is then

$$\sqrt{1 - \sigma_0^2} < 1 - \sigma_0^2/2 < 1 - \frac{2}{4^s}.$$

9. We now have from (15)

$$|\varphi(t)| < \left(1 - \frac{2}{4^s}\right)K,$$

$$K = 1 - \frac{1}{3^s} + K_1, \quad K_1 = \frac{2\sigma_0}{2^s} + \frac{4\sigma_0^2}{3^s} = \frac{2\sigma_0}{2^s}\left(1 + \frac{2\sigma_0}{(\frac{3}{2})^s}\right)$$

$$< \frac{4}{4^s}\left(1 + \frac{1}{64}\right)\left(1 + \frac{4 + \frac{1}{8}}{3^7}\right) < \left(1 + \frac{1}{56}\right)\frac{4}{4^s} < \frac{\frac{57}{14}}{4^s},$$

$$K < 1 - \frac{1}{3^s} + \frac{\frac{57}{14}}{4^s},$$

$$|\varphi(t)| < \left(1 - \frac{2}{4^s}\right)\left(1 - \frac{1}{3^s} + \frac{\frac{57}{14}}{4^s}\right) = 1 - \frac{1}{3^s} + \frac{\frac{29}{14}}{4^s} + \frac{2}{12^s} - \frac{8\frac{1}{7}}{16^s}$$

$$< 1 - \frac{1}{3^s} + \frac{1}{3^s}\left(\frac{29}{14}\left(\frac{3}{4}\right)^s + \frac{2}{4^s}\right) \leqq 1 - \frac{1}{3^s} + \frac{1}{3^s}\left(\frac{29}{14}\left(\frac{3}{4}\right)^7 + \frac{2}{4^7}\right),$$

$$|\varphi(t)| < 1 - \frac{1}{3^s} + \frac{.2766}{3^s} = 1 - \frac{.7234}{3^s},$$

and by (12) and (13)

$$|U_s(t)| < 1 - \frac{.7234}{3^s} + \frac{.2225}{3^s} = 1 - \frac{.5009}{3^s},$$

which proves (7).

It is of some interest to observe that (7) and our proof of (7) remain valid if $U_s(t) = \sum_{\nu=1}^{\infty} \frac{\sin 2\nu\pi t}{\nu^s}$ is replaced with $\sum_{\nu=1}^{N} \frac{\sin 2\nu\pi t}{\nu^s}$ for any $N \geqq 3$.

9. To verify (8) and (9) we use the fact that (3) is in $\langle 0, 1\rangle$ a polynomial, the so-called Bernoullian Polynomial $B_{2n+1}(t)$. We have in particular $B_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}t$, $B_5(t)=t^5-\frac{5}{2}t^4+\frac{5}{3}t^3-\frac{1}{6}t$. The maximum of $|B_3(t)|$ in $\langle 0, 1\rangle$ is attained at $t=.211325$ and gives $|B_3(0.211325)|=.048113$. Since $\frac{2\pi^3}{3}=20.67085$, we obtain $\underset{\langle 0,1\rangle}{\operatorname{Max}}|U_s(t)|=0.99454=1-\frac{.147}{27}$, which gives (9).

The maximum of $|B_5(t)|$ in $\langle 0, 1\rangle$ is attained at $t=.240335$ and the corresponding value of $|B_5(t)|$ is .0244582. Since $\frac{2\pi^5}{15}=40,8026$, we obtain

$$\underset{\langle 0,1\rangle}{\operatorname{Max}}|U_5(t)| = .997958 = 1-\frac{.496}{243},$$

which proves (8).

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