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Welding Riemann Surfaces and Transmission Problems with Shifts

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I. A real, 2-dimensional C^∞ manifold X is called an H-manifold whenever there is a coordinate system on X for which $\varphi_1 \circ \varphi_2^{-1}$ is either biholomorphic or bianti-holomorphic on $\varphi_2(U_1 \cap U_2)$ for every pair φ_1, φ_2. A closed subset S of X is called a C^{1+a} 1-complex on X if S is locally a star of C^{1+a} curves and is equipped with its minimal simplicial structure ([6], [7]); S is not necessarily oriented. From the bordered H-manifold $\bar{X}$ relative to X and S ([6], [7]). Intuitively this is done by cutting X along S and attaching to each resulting piece those 1-simplices of S which lie along it. Let pr be the projection of $\bar{X}$ onto X, $\bar{X}_\beta = \mathrm{pr}^{-1}(S)$, and $\{\sigma_\lambda^1 : \lambda \in \Lambda\}$ the 1-simplices of a simplicial structure on $\bar{X}_\beta$ for which $\mathrm{pr}(\sigma_\lambda^1)$ is a C^{1+a} curve on X for every $\lambda \in \Lambda$ (or equivalently, the projection of this simplicial structure onto S refines the minimal structure on S). Let $\varphi : \Lambda \to \Lambda$ be a bijective map such that

(i) $$\varphi(\lambda) \neq \lambda;$$
(ii) $$\varphi \circ \varphi(\lambda) = \lambda$$

for every $\lambda \in \Lambda$. Let $\alpha_\lambda : \sigma_\lambda^1 \to \sigma_{\varphi(\lambda)}^1$ be a C^{1+a} homeomorphism such that

$$\alpha_{\varphi(\lambda)} \circ \alpha_\lambda = \text{identity},$$

again for every $\lambda \in \Lambda$. Form the quotient space X^α from $\bar{X}$ by identifying two points $\bar{x}, \bar{y} \in \bar{X}_\beta$ whenever there are finitely many maps $\alpha_{\lambda_1}, \ldots, \alpha_{\lambda_n}$ for which $\alpha_{\lambda_n} \circ \cdots \circ \alpha_{\lambda_1}(\bar{x}) = \bar{y}$. Let $\chi : \bar{X} \to X^\alpha$ be the quotient map. It will be assumed throughout that $\chi^{-1}(x)$ is a finite set for every $x \in X^\alpha$. X^α is then a smooth manifold. A condition on the maps $\{\alpha_\lambda : \lambda \in \Lambda\}$ will be given and shown to be both necessary and sufficient in order that

(a) X^α can be given a unique H-structure for which

$$\chi \circ \mathrm{pr}^{-1} : X \backslash S \to X^\alpha \backslash \chi(\bar{X}_\beta)$$

is an H-homeomorphism (X^α is then said to be obtained uniquely from X by welding);

(b) $S^\alpha = \chi(\bar{X}_\beta)$ is a $C^{1+a'}$ 1-complex on X^α for some $a' \leqslant a$.

This condition is described as follows. For simplicity, suppose that every point on S and S^α has order >1 ([6]); the case when S or S^α has points of order $=1$ will be discussed later. Let $\bar{X}_j, j \in J$ denote the connected components of $\bar{X}$ and pr_j the map $\mathrm{pr} \mid \bar{X}_j$. The H-coordinate system on $\bar{X}$ can always be chosen so that for any

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coordinate neighborhood U of a point in $\overline{X}_j \cap \overline{X}_\beta$ and its corresponding coordinate $\bar{\bar{\varphi}}$, $\bar{\bar{\varphi}}(U)$ is a wedge in $\mathbf{C}$ with tip at the origin and $\psi = \varphi \circ \mathrm{pr}_j \circ \bar{\bar{\varphi}}^{-1}$ is angle preserving on $\bar{\bar{\varphi}}(U)$, where φ is a coordinate in an open set on X containing $\mathrm{pr}_j(U)$. For any vertex x of $\overline{X}_\beta$ and σ_λ^1 containing $x = \bar{\bar{\varphi}}_2 \circ \alpha_\lambda^1 \alpha_\lambda \circ \bar{\bar{\varphi}}_1^{-1}$ ($\bar{\bar{\varphi}}_1$ and $\bar{\bar{\varphi}}_2$ are coordinates at x and $\alpha_\lambda(x)$ respectively) is a map from the ray $\{te^{i\zeta}: 0 \leqslant t < M\}$ onto the ray $\{te^{i\eta}: 0 \leqslant \leqslant t < N\}$. Define $\alpha_\lambda''(t)$ by the rule $\alpha_\lambda'(te^{i\zeta}) = \alpha_\lambda''(t)\, e^{i\eta}$; $(d\alpha_\lambda''/dt)(0)$ exists and is nonzero. Denote $(d\alpha_\lambda''/dt)(0)$ by $\dot{\alpha}_\lambda(x)$; $\dot{\alpha}_\lambda(x)$ depends on the choice of coordinates. Choose an oriented coordinate neighborhood of every point in S^α which is the image under χ of a vertex of $\overline{X}_\beta$; this orientation induces one on a coordinate neighborhood of every vertex x of $\overline{X}_\beta$ and thereby an orientation on a piece of each of the two 1-simplices containing x. x is the initial point of one piece and the terminal point of the other. Denote the 1-simplex whose piece has x as initial point by σ_x^1 and by α_x the map α_λ whose domain is σ_x^1. For each vertex x of $\overline{X}_\beta$ the product $\prod\{\dot{\alpha}_y(y): \chi(y) = \chi(x)\}$ is independent of the choice of coordinates. The condition referred to above is that for every vertex $x \in \overline{X}_\beta$

$$\prod\{\dot{\alpha}_y(y): \chi(y) = \chi(x)\} = 1. \tag{1}$$

The condition is the same as (1) at a vertex x if we choose σ_x^1 to be the simplex whose piece has y as its terminal point for every y for which $\chi(y) = \chi(y)$, and so is also independent of the above choice of oriented coordinate neighborhoods.

Using the welding procedure and the results of [6], it can be shown that there is a large class of transmission problems with shifts generalizing those in the sense of Haseman and Carleman such that for each member of this class there is an associated holomorphic fibre bundle over a corresponding welded Riemann surface for which the solution space of the problem is functorally isomorphic to the space of global holomorphic sections in the associated bundle. Known results about holomorphic fibre bundles on Riemann surfaces can then be used to describe the solution spaces.

Finally, the welding and transmission problems on holomorphic families of Riemann surfaces are discussed.

Notation. A C^a map is one which is Hölder continuous with index a. It is C^{1+a} if it is continuously differentiable with C^a first partials. It is C_0^{1+a} if it is C^{1+a} with nowhere zero first partials. All curves are smooth, so a C^{1+a} curve is one described by a C_0^{1+a} map. $G(K)$ denotes the disc in $\mathbf{C}$ with center the origin and radius K, while $\ell(\zeta, K)$ is the ray $\{t\exp(i\zeta): 0 \leqslant t < K\}$. $D(K)$ and $\overline{D}(K)$ denote wedges of the form $\{w: |w| < K, \zeta < \arg w < \eta\}$ and $\{w: |w| < K, \zeta \leqslant \arg w \leqslant \eta\}$ respectively. $\mathbf{C}^+ = \{w: \mathrm{Im}\, w \geqslant \geqslant 0\}$, $\mathbf{C}^- = \mathbf{C} \setminus \mathrm{Int}\, \mathbf{C}^+$.

§ 1. The Local Problem

Let $\overline{X}$ be the disjoint union $\cup\{\overline{D}_\mu(K_\mu): \mu = 1, \dots, n\}$; clearly $\overline{X}_\beta = \cup\{bdy_{G(K_\mu)} \overline{D}_\mu(K_\mu): \mu = 1, \dots, n\}$. Let α_μ be a C_0^{1+a} homeomorphism of $\ell(\eta_\mu, K_\mu)$

onto $\ell(\zeta_{\mu+1}, K_{\mu+1})$. Form X^{α} relative to $\overline{X}_{\beta}$ and $\{\alpha_\mu, \alpha_\mu^{-1}:\mu=1,\dots,n\}$. X^{α} and S^{α} have properties (a) and (b) of the introduction if and only if

1°) there is a region G in $\mathbf{C}$ and a star of simple, $C^{1+\alpha'}$ curves ([6]) $\{\mathscr{C}_\mu:\mu=1,\dots,n\}$ at some point $w_0 \in G$ which divides G into n simply-connected domains G_μ, where $bdy_G G_\mu = \mathscr{C}_{\mu-1} \cup \mathscr{C}_\mu$;

2°) there exist $C^{1+\alpha'}$ homeomorphisms $A_\mu : \overline{D}_\mu(K_\mu) \to Cl_G G_\mu$ which are biholomorphic on $D_\mu(K_\mu)$ and satisfy

(i) $A_\mu(0) = w_0$

(ii) $A_{\mu+1}^{-1} \circ A_\mu = \alpha_\mu$ on $\ell(\eta_\mu, K_\mu)$; [3])

3°) (uniqueness) if G_μ^*, $\mathscr{C}^*$, A_μ^* also satisfy 1° and 2°, then there is a homeomorphism $\mathscr{L}: G \to G^*$ for which both $\mathscr{L}$ and $\mathscr{L}^{-1}$ are H-maps and $A_\mu = \mathscr{L} \circ A_\mu^*$ for every μ.

X^{α} and S^{α} are said to have properties (a) and (b) *locally* if there is an open neighborhood V of $\chi(0)$ in X^{α} for which $X^{\alpha} \cap V$ and $S^{\alpha} \cap V$ have properties (a) and (b).

In order to prove the necessity of condition (1) for the local problem, let α'_μ be the real-valued function defined by

$$\alpha_\mu(t \exp(i\eta_\mu)) = \alpha'_\mu(t) \exp(i\zeta_{\mu+1}).$$

Define $\zeta^{-1} = \sum_{\mu=1}^{n} (\eta_\mu - \zeta_\mu)$; a single-valued branch g_μ of $w^{2\pi\zeta}$ can be defined on each wedge $\overline{D}_\mu(K_\mu)$. Now

$$A_\mu^* = A_\mu \circ g_\mu^{-1} : g_\mu(\overline{D}_\mu(K_\mu)) \to G_\mu$$

is biholomorphic on $g_\mu(D_\mu(K_\mu))$ and therefore $C^{1+\alpha'}$ on $g_\mu(\overline{D}_\mu(K_\mu))$; A_μ^* is also angle preserving at the origin and so $(d/dw)\, A_\mu^*$ exists on $g_\mu(\overline{D}_\mu(K_\mu))$ and is nowhere zero. These facts are Kellogg's theorem ([3], [9]). It follows that $(d/dw)\,(A_\mu^*)^{-1}$ exists and is nonzero on $Cl_G G_\mu$. Now

$$(A_{\mu+1}^*)^{-1} \circ A_\mu^* = g_{\mu+1} \circ A_{\mu+1}^{-1} \circ A_\mu \circ g_\mu^{-1} = g_{\mu+1} \circ \alpha_\mu \circ g_\mu^{-1} \quad {}^{3)}$$

and so

$$\prod_{\mu=1}^{n} \left| \frac{d}{dt} \alpha'_\mu(0) \right| = \prod \left| \frac{d}{dt} g_{\mu+1} \circ \alpha_\mu \circ g_\mu^{-1} \right|^{2\pi\zeta}$$
$$= \prod \left| \frac{d}{dt} A_{\mu+1}^{*-1} \circ A_\mu^*(0) \right|^{2\pi\zeta} = \prod \left| \frac{d}{dw} A_{\mu+1}^{*-1}(0) \right|^{2\pi\zeta} \cdot \left| \frac{d}{dw} A_\mu^*(0) \right|^{2\pi\zeta} = 1;$$

(d/dt) means the following. The above maps, e.g. $(A_{\mu+1}^*)^{-1} \circ A_\mu^*$, are maps from one ray through the origin to another and so define real-valued functions of the real variable t just as α_μ defines α'_μ. It is these functions which are differentiated with respect to t. This proves the necessity of the condition.

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[3]) Indices are taken modulo n.

Now assume that

$$\prod_{\mu=1} \left| \frac{d}{dt} \alpha'_\mu(0) \right| = 1; \tag{1'}$$

since for $\alpha^*_\mu = g_{\mu+1} \circ \alpha_\mu \circ g_\mu^{-1}$ we have

$$\prod_{\mu=1}^{n} \left| \frac{d}{dt} \alpha^{*\prime}_\mu(0) \right|^{2\pi\zeta} = \prod_{\mu=1}^{n} \left| \frac{d}{dt} \alpha'_\mu(0) \right| = 1 = \prod_{\mu=1}^{n} \left| \frac{d}{dt} \alpha^{*\prime}_\mu(0) \right|,$$

there is no loss in generality in assuming that $\zeta^{-1} = 2\pi$. Of course the wedges may be rotated so that $\eta_\mu = \zeta_{\mu+1}$.

(1.1) LEMMA. *If a_μ are positive numbers for which $\prod_{\mu=1}^{n} a_\mu = 1$ $(n>1)$, then there are positive numbers $c_1, \ldots, c_n$ satisfying $c_{\mu+1}^{-1} \cdot a_\mu \cdot c_\mu = 1$ for all μ.*

Proof. Obvious.

Let c_μ be the constants determined in the lemma for $a_\mu = |(d/dt)\, \alpha'_\mu(0)|$, and denote also by c_μ the constant map on $\mathbf{C}$ with value c_μ. The maps $\alpha^*_\mu = c_{\mu+1}^{-1} \circ \alpha_\mu \circ c_\mu$ satisfy $|(d/dt)\, \alpha^{*\prime}_\mu(0)| = 1$, and so there is no loss in generality in assuming that $|(d/dt)\, \alpha'_\mu(0)| = 1$. In fact, since the functions α'_μ are increasing, we may assume that

$$\frac{d}{dt} \alpha'_\mu(0) = 1. \tag{2}$$

Finally, it may be assumed that there is μ and ν with $\mu \neq \nu$ and $\zeta_\nu = \zeta^\mu + \pi$. For if not, choose a ζ_μ and let $\zeta_0 = \zeta_\mu + \pi$; $\ell(\zeta_0, K_\nu)$ is contained in some $D_\nu(K_\nu)$. Form $\overline{\overline{D_\nu(K_\nu)}}$ relative to $\ell(\zeta_0, K_\nu)$ and let $\bar{D}_1(K_\nu) \cup \bar{D}_2(K_\nu) = \overline{\overline{D_\nu(K_\nu)}}$, while α is the unique welding correspondence between $\bar{D}_1(K_\nu)$ and $\bar{D}_2(K_\nu)$ which induces in the obvious way the identity map on $\ell(\zeta_0, K_\nu)$. Take all other welding correspondences as before. Condition (2) is then satisfied by the new system of welding correspondences $\{\alpha_\mu : \mu = 1, \ldots, n\} \cup \{\alpha\}$, so if (2) is sufficient for the existence of a local weld then X^α formed relative to $\{\alpha_\mu : \mu = 1, \ldots, n\}$ and $\tilde{X}^\alpha$ formed relative to $\{\alpha_\mu : \mu = 1, \ldots, n\} \cup \{\alpha\}$ are H-isomorphic in an open neighborhood of $\chi(0)$. This is a direct result of the choice of α.

Consequently, the sufficiency of (1) for the existence of a local weld is proved once we have proved the following proposition.

Let the wedges $D_\mu(K_\mu)$, $\mu = 1, \ldots, n$ be such that $\eta_\mu = \zeta_{\mu+1}$ and denote $\ell(\zeta_\mu, K_\mu)$ by $\ell_\mu(K_\mu)$. Let $\alpha_\mu : \ell_\mu(K_\mu) \to \ell_\mu(K_{\mu+1})$ be a homeomorphism such that $\alpha_\mu(0) = 0$.

(1.2) PROPOSITION. *Let $\bar{X} = \cup \bar{D}_\mu(K_\mu)$ with welding correspondences α_μ as above which are C_0^{1+a} on $\ell_\mu(K_\mu)$. Then $\bar{X}$ can be welded locally if*

1° *there exists p and q, $1 \leqslant p < q \leqslant n$ such that $\zeta_p = \zeta_q + \pi$;*

2° *$(d/dt)\, \alpha'_\mu(0) = 1$ for every $\mu = 1, \ldots, n$.*

Note that assumption 1° requires that $n \geqslant 2$.

The procedure for proving this proposition will be to weld each of the chains

$\{\bar{D}_p(K_p),\ldots,\bar{D}_{q-1}(K_{q-1})\}$ and $\{\bar{D}_q(K_q),\ldots,\bar{D}_n(K_n),\ldots,\bar{D}_{p-1}(K_{p-1})\}$ so that the results are half discs, and then to weld these two half discs. The chains are welded by welding two wedges at a time, which in turn can be reduced to welding two half discs. Thus the proof of proposition (1.2) for $n>2$ reduces essentially to the proof for $n=2$. This special case will follow from known results about quasiconformal mapping.

For $n=2$ we may take $\bar{D}_1(K_1)$ (resp. $\bar{D}_2(K_2)$) to be the upper (resp. lower) half disc of $G(K_1)$ (resp. $G(K_2)$). The maps α_1 and α_2^{-1} together define a C_0^{1+a} map α from $\mathrm{bdy}_{C^+}\bar{D}_1(K_1)$ onto $\mathrm{bdy}_{C^-}D_2(K_2)$ for which $(d/dt)\,\alpha'(0)=1$. For ease of notation we will suppress the constants K_ν in the notation $\bar{D}_\nu(K_\nu)$.

The procedure for finding A_1 and A_2 satisfying 2° above is due to C. Blanc ([2]; also see [4]). The map

$$P(x, y) = \alpha(x) + i(\alpha(x+y) - \alpha(x)), \qquad w = x + iy$$

is defined and is a C_0^{1+a} homeomorphism on $V \cap \bar{D}_1$ for some open neighborhood V of the origin. Since α is a C_0^{1+a} map, $P(x, y)$ is a quasiconformal map ([8]). $P(x, y)$ satisfies $P(x, 0)=\alpha(x)$ and $P^{-1}(x, y)=\alpha^{-1}(x)+i(\alpha^{-1}(x+y)-\alpha^{-1}(x))$. $P^{-1}(x, y)$ is also quasiconformal and satisfies the usual Beltrami equation for

$$\mu(x, y) = (-1+i)\left[\frac{d\alpha^{-1}}{dt}(x+y) - \frac{d\alpha^{-1}}{dt}(x)\right] \cdot\left[\frac{d\alpha^{-1}}{dt}(x+y) + \frac{d\alpha^{-1}}{dt}(x) + i\left\{\frac{d\alpha^{-1}}{dt}(x+y) - \frac{d\alpha^{-1}}{dt}(x)\right\}\right]^{-1}.$$

Define

$$\mu^*(x, y) := \begin{cases} \mu(x, y), & \in V^* \cap D_1 \\ 0\ , & \notin V^* \cap D_1; \end{cases} \tag{3}$$

clearly $\mu^*(x, y)$ is well-defined for some open neighborhood V^* of 0, and is a C^a function on V^*. It is known ([1]) that there is then a C^{1+a} homeomorphism A on an open neighborhood V' of 0 satisfying the Beltrami equation

$$A_{\bar{w}} = \mu^* A_w;$$

now $A \circ P$ and A are 1-quasiconformal homeomorphisms on $U \cap D_1$ and $U \cap D_2$ respectively and therefore biholomorphic maps, where $U = V \cap V' \cap V^*$. Moreover, $\mathscr{C} = A \mid bdy_{C^-}\bar{D}_2$ is a C^{1+a} simple curve and $A^{-1} \circ (A \circ P) \mid bdy_{C^+}\bar{D}_1 = \alpha$. The uniqueness of the weld follows from [4], theorem 1 and so proposition (1.2) is proved for $n=2$.

It should be noted that $(d/dt)\,\alpha(0)=1$ has not been used, but rather only the fact that $(d/dt)\,\alpha_1(0)=(d/dt)\,\alpha_2^{-1}(0)\neq 0$ so that α_1 and α_2^{-1} together define a C_0^{1+a} map α. Therefore proposition (1.2) for $n=2$ will provide for a general H-manifold X the local weld at points on $\bar{X}_\beta$ which are not vertices of the given simplicial structure on $\bar{X}_\beta$. What $(d/dt)\,\alpha(0)=1$ does mean is that for any two biholomorphic A_1 and A_2 defining the local weld in (1.2) for $n=2$,

$$\frac{d}{dw}A_1(0)=\frac{d}{dw}A_2(0).$$

This is the basic fact which allows us to weld piece by piece when $n>2$.

In order to prove (1.2) for $n>2$, we first weld the wedges $\bar{D}_p(K_p)$ and $\bar{D}_{p+1}(K_{p+1})$ and then prepare the result for welding to $\bar{D}_{p+2}(K_{p+2})$. Extend $\bar{D}_p(K_p)$ and $\bar{D}_{p+1}(K_{p+1})$ to half discs with bounding line segments $\ell'_p(K_p)$ $(=\{t\cdot\exp(i\zeta_p):-K'_p\leqslant\leqslant t\leqslant{}'_pK\})$ and $\ell'_p(K_{p+1})$ respectively, and extend α_p to a C_0^{1+a} homeomorphism α_p of $\ell'_p(K_p)$ onto $\ell'_p(K_{p+1})$. There is an open neighborhood U_p of $\chi(0)$ in X^α for which the two half discs have properties (a) and (b) on $\chi^{-1}(U_p)$ for the welding correspondence α_p. Let A'_{p+1} and A'_p denote the biholomorphic welding maps, the domains of A'_p and A'_{p+1} being subsets of the half discs containing $\bar{D}_p(K_p)$ and $\bar{D}_{p+1}(K_{p+1})$ respectively. The curve $A_{p+1}\mid l_{p+1}(K_{p+1})\cap\chi^{-1}(U_p)$ is C^{1+a}; this follows from Kellogg's theorem. Extend this curve from $A'_{p+1}(0)$ through the image set $A_p(\chi^{-1}(U_p))\cup A'_{p+1}(\chi^{-1}(U_p))$ to its boundary in a simple, C^{1+a} way so that it does not intersect $A'_p(\bar{D}_p(K_p)\cap\chi^{-1}(U_p))\cup A'_{p+1}(\bar{D}_{p+1}(K_{p+1})\cap\chi^{-1}(U_p))$ except at $A'_{p+1}(0)$ $=A'_p(0)$. Let R_p be the region in $\mathbf{C}$ bounded by this extended curve and the boundary of the image set and containing $A'_p(\chi^{-1}(U_p))\cup A'_{p+1}(\chi^{-1}(U_p))$. Map this region by biholomorphic f_p onto a bounded half disc whose bounding line segment is a finite segment of $\ell'_{p+1}(\infty)$, which does not intersect $D_{p+2}(K_{p+2})$, and such that $f_p\circ A'_p(0)=$ $=f_p\circ A'_{p+1}(0)=0$ while the image of the above extended curve is on $\ell'_{p+1}(\infty)$ (a slightly smaller U_p may have to be chosen for this last part). Let $B_p=f_p\circ A'_p$, $B_{p+1}=$ $=f_p\circ A'_{p+1}$; since $(d/dw)\,A'_p(0)=(d/dw)\,A'_{p+1}(0)$, f_p may also be normalized by the condition

$$\left|\frac{d}{dw}B_p(0)\right|=\left|\frac{d}{dw}B_{p+1}(0)\right|=1\,.$$

Now weld $B_p(\bar{D}_p(K_p)\cap\chi^{-1}(U_p))\cup B_{p+1}(\bar{D}_{p+1}(K_{p+1})\cap\chi^{-1}(U_p))$ to $\bar{D}_{p+2}(K_{p+2})$ in exactly the same way for the welding correspondence $\alpha_{p+1}\circ B_{p+1}^{-1}$. Denote the resulting welding maps by B'_{p+1} and B_{p+2}, the domains of B'_{p+1} and B_{p+2} containing $B_{p+1}(\bar{D}_{p+1}(K_{p+1})\cap\chi^{-1}(U_{p+1}))$ and $D_{p+2}(K_{p+2})\cap\chi^{-1}(U_{p+1})$ respectively, where $U_{p+1}\subseteq U_p$ is also an open neighborhood of $\chi(0)$ in X^α. Continue this process until in an open neighborhood U_{q-2} of $\chi(0)$ in X^α the wedges $\bar{D}_p(K_p)\cap\chi^{-1}(U_{q-2})$ through $\bar{D}_{q-2}(K_{q-2})\cap\chi^{-1}(U_{q-2})$ have been welded together, and then determine the neighborhood $U_{q-1}\subseteq U_{q-2}$ and the welding maps A''_{q-2} and A'_{q-1} of the next weld. Because of condition 1° of the proposition and the fact that all maps so far determined are angle-preserving at the origin, the C^{1+a} extension of the curve $A'_{q-1}\mid\ell_{q-1}(K_{q-1})\cap$

$\cap \chi^{-1}(U_{q-1})$ will be taken to be $A''_{q-2} \circ B'_{q-3} \circ \cdots \circ B'_{p+1} \circ B_p \mid \ell_{p-1}(K_{p-1}) \cap \chi^{-1}(U_{q-1})$. Determine the map f_{q-1} as before with its proper normalization. The final result of welding the chain $\{\bar{D}_p(K_p), \ldots, \bar{D}_{q-1}(K_{q-1})\}$ is then a half-disc $\bar{\bar{\mathscr{D}}}_1(K'_{q-1})$ with bounding line segment $\ell'_{q-1}(K'_{q-1}) = \ell'_{q-1}(K'_{q-1})$ for some positive number K'_{q-1}.

Now weld the chain $\{\bar{D}_q(K_q), \ldots, \bar{D}_n(K_n), \ldots, \bar{D}_{p-1}(K_{p-1})\}$ in exactly the same way, starting with $\bar{D}_q(K_q)$ and $\bar{D}_{q+1}(K_{q+1})$. Use analogous notation throughout, i.e. the notation for the mappings, constants, etc. is determined by substituting q for p in the previous ones. Again, the result is a half disc $\bar{\bar{\mathscr{D}}}_2(K'_{p-1})$ with bounding line segment $\ell'_{p-1}(K'_{p-1}) = \ell'_{q-1}(K'_{p-1})$ for some positive number K'_{p-1}.

The final step is to weld these two half discs with the welding correspondence $\alpha: \ell'_{p-1}(K'_{q-1}) \to \ell'_{p-1}(K'_{p-1})$ given by

$$\beta = \alpha \mid \ell_{p-1}(K'_{q-1}) = B_{p-1} \circ \alpha_{p-1}^{-1} \circ B_p^{-1} \circ B'^{-1}_{p+1} \circ \cdots \circ B'^{-1}_{q-2},$$

while

$$\gamma = \alpha \mid \ell_{q-1}(K'_{q-1}) = B'_{p-2} \circ \cdots \circ B'_q \circ \alpha_{q-1} \circ B_{q-1}^{-1}.$$

Clearly both β and γ are C_0^{1+a} maps on their respective domains. Moreover, by condition 2° of the proposition and the chosen normalization of the maps f, one has $(d/dt)\,\beta'(0) = (d/dt)\,\gamma'(0) = 1$. Therefore α is a C_0^{1+a} map. Weld the half discs, V being the open neighborhood of $\chi(0)$ in X^α for this local weld and Γ_1, Γ_2 being the welding maps, where the domains of Γ_1 and Γ_2 are $\bar{\bar{\mathscr{D}}}_1(K'_{q-1}) \cap \chi^{-1}(V)$ and $\bar{\bar{\mathscr{D}}}_2(K'_{p-1}) \cap$ $\cap \chi^{-1}(V)$ respectively.

The welding maps A_μ, $\mu = 1, \ldots, n$ of 2°) are then given by

$$\begin{aligned}
A_p &= \Gamma_1 \circ B'_{q-2} \circ \cdots \circ B'_{p+1} \circ B_p \mid \bar{D}_p(K_p) \cap \chi^{-1}(V)\\
A_{p+1} &= \Gamma_1 \circ B'_{q-2} \circ \cdots \circ B'_{p+1} \circ B_{p+1} \mid \bar{D}_{p+1}(K_{p+1}) \cap \chi^{-1}(V)\\
&\vdots\\
A_{q-2} &= \Gamma_1 \circ B'_{q-2} \circ B_{q-2} \mid \bar{D}_{q-2}(K_{q-2}) \cap \chi^{-1}(V)\\
A_{q-1} &= \Gamma_1 \circ B_{q-1} \mid \bar{D}_{q-1}(K_{q-1}) \cap \chi^{-1}(V);
\end{aligned}$$

The A_q through A_{p-1} are given by replacing Γ_1 with Γ_2 and interchanging p and q in the above formulas. Each A_μ is biholomorphic on the interior of its domain; a straightforward computation shows that the equations 2°) (ii) are satisfied. That each curve $\mathscr{C}_\mu = A_\mu \mid \ell_\mu(K_\mu) \cap \chi^{-1}(V)$ is C^{1+a} follows from the constructions in the proof and repeated applications of Kellogg's theorem. The uniqueness of the weld follows in a straightforward way from the uniqueness in the case $n=2$, and so the proof of proposition (1.2) is complete.

Remark. It has been shown that the general local welding problem is solved by reducing it to the situation of proposition (1.2). If the original welding correspondences are C_0^{1+a} maps, the reduced ones are $C_0^{1+a'}$ for some $a' \leqslant a$, and as seen from the proof of (1.2), S^α will be a star of $C^{1+a'}$ curves in a neighborhood of $\chi(0)$. It is easy to show that a' may be given by

$$a' = a \cdot \min\left(1, (2\pi\zeta)^{-1}\right).$$

In fact, in terms of a given a, this is the maximum value of a' which will work.

§ 2. The General Case

The global welding problem is essentially done, since the problems of existence and uniqueness of the weld and whether or not S^α is a $C^{1+a'}$ 1-complex on X^α are local ones.[4] The only question remaining is the form of the condition (1) on the welding correspondences at vertices x of the simplicial structure on $\overline{X}_\beta$ for which either $\mathrm{pr}\,x$ is of order 1 in S or $\chi(x)$ has order 1 in S^α. The easiest way to give the condition in these cases is to allow "generalized" wedges as coordinate neighborhoods at vertices of $\overline{X}_\beta$ and in the local welding problem of § 1. A generalized wedge is a space $\overline{G}$ formed from a disc G in $\mathbf{C}$ with center 0 and a ray T originating at 0 and passing through ∂G; $\mathrm{Int}\,\overline{G}$ is holomorphically isomorphic to $G \backslash T$. Everything can be done as before as long as one is careful on the boundary of the generalized discs. The welding condition is essentially the same as (1) whenever x is such that $\mathrm{pr}\,x$ has order 1 and $\chi(x)$ has order >1. If $\chi(x)$ has order 1, or equivalently if $\alpha_x(x)=x$, let $\alpha'_x = \bar{\bar{\varphi}} \circ \alpha_x \circ \bar{\bar{\varphi}}^{-1}$, $\bar{\bar{\varphi}}$ a coordinate at x in $\overline{X}$. Then $(d/dt)\,\alpha''_x(x)$ is independent of the choice of $\bar{\bar{\varphi}}$, and in this case the welding condition is

$$\frac{d}{dt}\alpha''_x(x) = 1. \tag{1}$$

Summing up, we have

(2.1) THEOREM. *Let X be an H-manifold and $S \subseteq X$ a C^{1+a} 1-complex on X. Let $\{\sigma^1, \alpha_\lambda : \lambda \in \Lambda\}$ be defined as above and form X^α. Then X^α can be given a unique H-structure so that*

(i) $\chi \circ \mathrm{pr}^{-1} : X \backslash S \to X^\alpha \backslash S^\alpha$ *is an H-isomorphism;*
(ii) S^α *is a $C^{1+a'}$ 1-complex on X^α for some $a' \leqslant a$,*
if and only if for every vertex x of the simplicial structure on $\overline{X}_\beta$,

$$\prod \{\dot{\alpha}_y(y) : \chi(y) = \chi(x)\} = 1.$$

Remark. If in the welding problem S is a C^{n+a} 1-complex $(n>1)$, one may ask for conditions under which S^α would be a $C^{n+a'}$ 1-complex for some $a' \leqslant a$. The technique used here does not provide such conditions because the function $\mu^*(x, y)$

[4]) The Hölder index a' is locally determined. It can happen that no global Hölder constant on S^α exists.

defined in formula (3) is only of class C^a no matter how smooth the welding correspondence α is.

The application of theorem (2.1) to transmission problems with shifts requires that both X and X^α be Riemann surfaces, i.e. orientable H-manifolds. If X is a Riemann surface and X^α an H-manifold, it is useful to describe the orientability of X^α in terms of so-called welding signatures on $\bar{X}_\beta$.

Let $x \in \bar{X}_j$, $\mathrm{pr}_j := \mathrm{pr} \mid \bar{X}_j$, and φ a coordinate at $\mathrm{pr}\,\bar{x}$; $\varphi \circ \mathrm{pr}_j$ is then a coordinate at x. If $x \in \bar{X}_\beta$, let f be a biholomorphic map of $\varphi \circ \mathrm{pr}_j(U)$ onto a wedge with tip at the origin, f being angle-preserving on $\bar{X} \cap U$ und U a sufficiently small coordinate neighborhood of x. This describes the H-structure on $\bar{X}$ as in the introduction, and in fact gives $\bar{X}$ the structure of a bordered Riemann surface. Denote the wedge at $x \in \bar{X}_\beta$ by D_x, and suppose that x is a vertex of the simplicial structure on $\bar{X}_\beta$. Consider the set $\{D_y : \chi(y) = \chi(x)\}$. In order to put this set of wedges in the correct position for the local weld in § 1, some of them must be "turned over", i.e. mapped onto themselves by what is essentially the complex conjugation map $\gamma: \mathbf{C} \to \mathbf{C}$, $\gamma(w) = \bar{w}$. There is of course no unique way of determining which wedges will or will not be mapped by γ, but for each such choice one has welding signatures $\tilde{\sigma}(x)$, where $\tilde{\sigma}(x) = 1$ if D_x is mapped by γ and $\tilde{\sigma}(x) = 0$ otherwise. If $x \in \bar{X}_\beta$ is not a vertex, say $x \in \mathrm{Int}\,\sigma_\lambda^1$, an orientation on X induces by way of pr orientations on σ_λ^1 and $\sigma_{\varphi(\lambda)}^1$. If α_λ reverses these orientations, define $\tilde{\sigma}(x)$ and $\tilde{\sigma}(\alpha_\lambda(x))$ to both have value 0 or both have value 1, while if α_λ preserves orientation, define one to have value 0 and the other 1.

(2.2) THEOREM. *If X is a Riemann surface and X^α an H-manifold, obtained from X by welding, then $\bar{X}^\alpha$ can be given the structure of a Riemann surface if and only if welding signatures on $\bar{X}_\beta$ may be chosen so as to have constant value $\tilde{\sigma}(j)$ on each $\bar{X}_\beta \cap \bar{X}_j$, in which case the complex structure on X^α can be chosen so that $\chi_j := \chi \mid \mathrm{Int}\,\bar{X}_j$ is biholomorphic if $\tilde{\sigma}(j) = 0$ and biantiholomorphic if $\tilde{\sigma}(j) = 1$.*

Proof. Obvious.

Another topological property of the welded H-manifold X^α is the following

(2.3) THEOREM. *X^α is compact if and only if X is compact.*

§ 3. Welding Holomorphic Families of Riemann Surfaces

Let $\omega: \mathfrak{B} \to M$ be a holomorphic mapping of the complex manifold $\mathfrak{B}$ onto the complex manifold M. ω is called a *holomorphic family of Riemann surfaces* if for every $v \in \mathfrak{B}$ there is an open neighborhood U of v in $\mathfrak{B}$ and a biholomorphic map ψ_U of U onto $G \times \omega(U)$ such that if pr_2 denotes the projection $(w, x) \to x$, then $\omega = \mathrm{pr}_2 \circ \psi_U$; G is the unit disc $|w| < 1$ in $\mathbf{C}$. Let S be a closed subset of $\mathfrak{B}$ which is *locally a star of $C^{1+a(x)}$ curves depending holomorphically on $x \in M$* ([6]). Define a simplicial structure on S so that

(i) the restriction to each surface in the family is a simplicial structure as before;

(ii) the set of simplices of codimension 2 is a complex submanifold of $\mathfrak{V}$.

S together with such a structure is called a $C^{1+a(x)}$ *complex on* $\mathfrak{V}$ *of codimension* 1 *depending holomorphically on* $x\in M$. Let $\{\sigma_\lambda^1:\lambda\in\Lambda\}$ be the simplices of codimension 1 and $\varphi:\Lambda\to\Lambda$ as before. The welding correspondences $\alpha_\lambda:\sigma_\lambda^1\to\sigma_{\varphi(\lambda)}^1$ are $C^{1+a(x)}$ maps depending holomorphically on $x\in M$ ([6]) satisfying

$$\omega\circ\mathrm{pr}\circ\alpha_\lambda(v)=\omega\circ\mathrm{pr}(v)$$

for all $v\in\sigma_\lambda^1$, where pr is the projection map $\bar{\bar{\mathfrak{V}}}\to\mathfrak{V}$.

The local welding situation for holomorphic families of Riemann surfaces is essentially as follows. Let P be a polycylinder in $\mathbf{C}^n$, $I=\{t:0\leqslant t<1\}$, and $I'=\{t:-1<t<1\}$. In $D\times P$ we are given the family of curves $S(\mathscr{C}_\nu(t,z))=(a(z)+b_\nu(z)\,t,z)$, where $(t,z)\in I\times P$, a, $b_\nu:P\to\mathbf{C}$ are holomorphic and the b_ν are nowhere zero, and for every pair $1\leqslant\nu$, $\mu\leqslant n$ with $\mu\neq\nu$, $b_\nu(z)\neq b_\mu(z)$ for all $z\in P$. The welding correspondences α_ν are homeomorphisms of $S(\mathscr{C}_\nu)$ onto itself such that

1° $\alpha_\nu\mid\mathscr{C}_\nu(I\times\{z\})\times\{z\}$ is a $C_0^{1+a(z)}$ map of $\mathscr{C}_\nu(I\times\{z\})\times\{z\}$ onto itself for every $z\in P$;

2° $\alpha_\nu\mid S(\mathscr{C}_\nu(\{t\}\times P))$ is holomorphic for every $t\in I$.

The results of § 1 carry over to this situation with only minor adjustments in the proofs. For example, consider the case $n=2$ and $\arg b_1(z)=\arg b_2(z)+\pi$ for all $z\in P$. For each $z\in P$ define the affine transformation $\mathscr{L}_z(w)=a(z)+b_1(z)\,w$, and let $\mathscr{L}:\mathbf{C}\times P\to\mathbf{C}$ be the holomorphic map $\mathscr{L}(w,z)=\mathscr{L}_z(w)$. Define the map β of $I'\times P$ onto itself by

$$\beta(t,z)=\begin{cases}(\mathscr{L}_z^{-1}\,\mathrm{pr}_1\circ\alpha_1(\mathscr{L}(t,z),z),z), & t\geqslant 0\\ (\mathscr{L}_z^{-1}\,\mathrm{pr}_1\circ\alpha_2^{-1}(\mathscr{L}(t,z),z),z), & t<0.\end{cases}$$

Clearly there is β' such that

$$\beta(t,z)=(\beta'(t,z),z);$$

since β' is real-valued and holomorphic in z it is constant in z, i.e. there is a C_0^{1+a} map $\gamma:I'\to I'$ so that $\gamma(t)=\beta'(t,z)$. Then if A_1' and A_2' are the local welding maps for γ determined in § 1, clearly $A_\nu(w,z)=(\mathscr{L}(A_\nu'(w),z),z)$ $(\nu=1,2)$ are local welding maps for this special parameterized case.

The global welding problem for a holomorphic family of Riemann surfaces $\omega:\mathfrak{V}\to M$ and a $C^{1+a(z)}$ complex S of codimension 1 depending holomorphically on $x\in M$ requires more than simple adjustments in the proofs, however. In order to use the results of the local welding problem as it is presented above, one must as in § 2 be able to map coordinate neighborhoods of points $v\in\bar{\bar{\mathfrak{V}}}_\beta$ biholomorphically onto "holomorphic families of wedges". If $\dim M=0$, which is the situation of § 2, this can always be done. But when $\dim M>0$ there are restrictions.

Let $\mathscr{C}:\partial G\times P\to\mathbf{C}$ be a family of simple closed $C^{1+a(z)}$ curves depending holomorphically on $z\in P$ and denote by Ω_z the bounded region in $\mathbf{C}$ for which $\partial\Omega_z = =\mathscr{C}(\partial G\times\{z\})$. There is no loss in generality in assuming that for some $z^\circ\in P$, $\mathscr{C}\mid\partial G\times\{z^\circ\}$ is the boundary value of a biholomorphic map of G onto Ω_{z°.

(3.1) PROPOSITION. *Let $\mathscr{C}:\partial G\times P\to\mathbf{C}$ be as above. In order that there exist a map $f:G\times P\to\mathbf{C}$ satisfying*

(i) *f is holomorphic;*

(ii) *$f\mid G\times\{z\}$ is a biholomorphic map of G onto Ω_z for every $z\in P$,*

it is both necessary and sufficient that $\mathscr{C}$ be the boundary value of a map $\mathscr{C}:G\times P\to\mathbf{C}$ satisfying (i) *and* (ii).

Proof. ([5]).

By an *analytic complex on $\mathfrak{V}$ of codimension* 1 *depending holomorphically on $x\in M$* we mean a $C^{1+a(x)}$ complex S as above for which every $v\in\bar{\bar{\mathfrak{V}}}_\beta$ has a coordinate neighborhood in $\bar{\bar{\mathfrak{V}}}$ biholomorphically equivalent to a holomorphic family of wedges. In view of proposition (3.1) this means that S must be locally a star of analytic curves depending holomorphically on $x\in M$ and satisfy further conditions at the simplices of codimension 2 in the minimal simplicial structure on S.

If $\mathfrak{V}^\alpha$ is a complex manifold obtained by welding and S^α is a $C^{1+a'(x)}$ complex of codimension 1 depending holomorphically on $x\in M$, then $\omega^\alpha:\mathfrak{V}^\alpha\backslash S^\alpha\to M$ is a holomorphic family of Riemann surfaces, where

$$\omega^\alpha(v^\alpha)=\omega(\mathrm{pr}\circ\chi^{-1}(v^\alpha)),$$

and since $\omega\circ\mathrm{pr}\circ\alpha_\lambda(v)=\omega\circ\mathrm{pr}(v)$ for all $v\in\sigma^1_\lambda$ and all $\lambda\in\Lambda$, ω^α has a unique holomorphic extension to $\omega^\alpha:\mathfrak{V}^\alpha\to M$ which is also a holomorphic family of Riemann surfaces.

One remark about notation. Let $\alpha''(t,z)$ be defined analogously to $\alpha''(t)$ in the introduction for a point $v\in\bar{\bar{\mathfrak{V}}}_\beta$. Then $\dot{\alpha}_v(v)$ will now denote $(\partial/\partial t)\,\alpha''(0,z)$ where $(0,z)$ is the coordinate of v.

(3.2) THEOREM. *Let $\omega:\mathfrak{V}\to M$ be a holomorphic family of Riemann surfaces, S an analytic complex on $\mathfrak{V}\to M$ of codimension* 1 *depending holomorphically on $x\in M$, and $\{\alpha_\lambda:\lambda\in\Lambda\}$ a set of $C_0^{1+a(x)}$ welding correspondences depending holomorphically on $x\in M$. Suppose that there are welding signatures $\{\bar{\sigma}(\bar{v}):\bar{v}\in\bar{\bar{\mathfrak{V}}}_\beta\}$ satisfying $\bar{\sigma}(\bar{v})=0$ for all $\bar{v}\in\bar{\bar{\mathfrak{V}}}_\beta$. Then*

1° *$\omega^\alpha:\mathfrak{V}^\alpha\to M$ is a holomorphic family of Riemann surfaces obtained uniquely from $\omega:\mathfrak{V}\to M$, hence*

$$\chi\circ\mathrm{pr}^{-1}:\mathfrak{V}\backslash S\to\mathfrak{V}^\alpha\backslash S^\alpha$$

is biholomorphic; and

2° *S^α is a $C^{1+a'(x)}$ complex of codimension* 1 *depending holomorphically on $x\in M$ for some $a'(x)\leqslant a(x)$*

if and only if at every point v in some simplex of codimension 2 of the given simplicial structure on $\overline{\overline{\mathfrak{V}}}_\beta$,

$$\prod \{\dot{\alpha}_y(y): \chi(y) = \chi(v)\} = 1 .$$

Remark. Suppose that S is a family of simple, closed $C^{1+a(x)}$ curves depending holomorphically on $x \in M$ and denote the welding correspondences by α_1 and α_2. $\alpha'_\nu = \mathrm{pr} \circ \alpha_\nu \circ (\mathrm{pr} \mid \sigma^1_\nu)^{-1}$ is a map of S onto itself ($\nu = 1, 2$) such that $\alpha'_1 = \alpha'^{-1}_2$; if α'_1 is the boundary value of a biholomorphic map $\alpha'_1 : U \to \mathfrak{V}$ for some open neighborhood U of σ^1_1 in $\overline{\overline{\mathfrak{V}}}$, then the conclusions of theorem (3.2) hold. In general, the analyticity condition on the complex S in theorem (3.2) can be replaced by analyticity conditions on the welding correspondences. Moreover, the analyticity condition on S implies that each $\mathrm{pr}\,\sigma^1_\lambda$ is a pseudoconvex hypersurface in $\mathfrak{V}$, but nowhere strongly pseudoconvex. On the other hand, one has

(3.3) THEOREM. *If* $\mathrm{pr}\,\sigma^1_\lambda$ *is a strongly pseudoconvex hypersurface on* $\mathfrak{V}$ *and the conclusions of theorem* (3.2) *are true, then* $\mathrm{pr} \circ \alpha_\lambda \circ (\mathrm{pr} \mid \sigma^1_\lambda)^{-1}$ *is the boundary value of a biholomorphic map* $\mathrm{pr}\, U_\lambda \to \mathfrak{V}$ *for some open neighborhood* U_λ *of either* $\mathrm{Int}\,\sigma^1_\lambda$ *or* $\mathrm{Int}\,\sigma^1_{\lambda'}$ *on* $\overline{\overline{\mathfrak{V}}}$, *where* λ' *is that unique index satisfying* $\lambda' \neq \lambda$ *and* $\mathrm{pr}\,\sigma^1_\lambda = \mathrm{pr}\,\sigma^1_{\lambda'}$.

II. Let X be a Riemann surface, S a C^{1+a} 1-complex on X and $\{\sigma^1_\lambda, \alpha_\lambda : \lambda \in \Lambda\}$ welding data as in chapter I. satisfying the condition (1). Suppose that there are welding signatures $\{\tilde{\sigma}(\bar{x}) : \bar{x} \in \overline{X}_\beta\}$ which have constant value $\tilde{\sigma}(j)$ on each connected component $\overline{X}_j$ of $\overline{X}$. X^α is then a Riemann surface. Form $\overline{\overline{(X^\alpha)}}$ relative to X^α and S^α with the projection $\mathrm{pr}^\alpha : \overline{\overline{(X^\alpha)}} \to X^\alpha$ and denote it by $\overline{X}^\alpha$. Both $\overline{X}$ and $\overline{X}^\alpha$ are bordered Riemann surfaces for which $\mathrm{pr} : \mathrm{Int}\,\overline{X} \to X$ and $\mathrm{pr}^\alpha : \mathrm{Int}\,\overline{X}^\alpha \to X^\alpha$ are holomorphic maps. Clearly there is a homeomorphism $A : \overline{X} \to \overline{X}^\alpha$ such that $\mathrm{pr}^\alpha \circ A = \chi$; both A and A^{-1} are C^{1+a} maps. Accordingto (2.2), the complex structure on X^α, and thereby on $\overline{X}^\alpha$, can be chose so that $A \mid \mathrm{Int}\,\overline{X}_j$ is either biholomorphic or biantiholomorphic according to whether $\tilde{\sigma}(j) = 0$ or 1.

Let $\beta : \overline{X}_\beta \to \overline{X}_\beta$ and $\beta^\alpha : \overline{X}^\alpha_\beta \to \overline{X}^\alpha_\beta$ be the successor maps defined in [6] and define $\beta_\lambda = A^{-1} \circ \beta^\alpha \circ A$, $\beta_\lambda : \overline{X}_\beta \to \overline{X}_\beta$. For every $\bar{x} \in \overline{X}_j \cap \overline{X}_\beta$, define $j' = \beta_\lambda(j, \bar{x})$ to be that index for which $\beta_\lambda(\bar{x}) \in \overline{X}_{j'}$. A coherent set of signatures $\sum_\alpha = \{\sigma_{x^\alpha} : x^\alpha \in S^\alpha\}$ on the $C^{1+a'}$ 1-complex S^α ([6]) induces in the obvious way a set of signatures $\{\sigma(\bar{x}^\alpha) : \bar{x}^\alpha \in \overline{X}^\alpha_\beta\}$ on $\overline{X}^\alpha_\beta$. Define the signatures $\{\sigma_\alpha(\bar{x}) : \bar{x} \in \overline{X}_\beta\}$ on $\overline{X}_\beta$ by the rule $\sigma_\alpha(\bar{x}) = \sigma(A(\bar{x}))$.

Let L be a complex Lie group and $f : \overline{X}_\beta \to L$ a Hölder continuous map. f is said to be *compatible with the welding data* if for every $\lambda \in \Lambda$ and every $\bar{x} \in \sigma^1_\lambda$ $f(\bar{x}) = f(\alpha_\lambda(\bar{x}))$; if so, f induces the Hölder continuous map $f^\alpha : S^\alpha \to L$ by the rule $f(\bar{x}) = f^\alpha(\chi(\bar{x}))$. f is said to be *compatible with* $\sum_\alpha$ if $f^\alpha(x^\alpha)^{|\sigma x^\alpha|} \equiv e$ on S^α, e being the unit of L.

§ 4. The Haseman Problem

Given

1) a Riemann surface X, a C^{1+a} 1-complex S on X, welding data $\{\sigma_\lambda^1, \alpha_\lambda : \lambda \in \Lambda\}$ as in chapter I satisfying condition (1) and corresponding welding signatures $\{\sigma(\bar{x}) : \bar{x} \in \overline{X}\}$, and a coherent set of signatures $\sum_\alpha$ on S^α;

2) a complex Lie group L which acts as a complex automorphism group on the complex space F, where F has an antiholomorphic involution ι;

3) a Hölder continuous map $f : \overline{X}_\beta \to L$ which is compatible with both $\sum_\alpha$ and the welding data,

the Haseman problem is to find holomorphic maps $s_j : \mathrm{Int}\,\overline{X}_j \to F$ for each j such that

a) s_j has a continuous extension to $\overline{X}_j$;

b) for every $\lambda \in \Lambda$ and every $x \in \sigma_\lambda^1$,

$$\iota^{\tilde{\sigma}(\alpha_\lambda(x))} \circ s_{\beta_\lambda(j,x)}(\alpha_\lambda(x)) = f(x)^{\sigma_\alpha(x)}\, \iota^{\tilde{\sigma}(x)} \circ s_j(x). \tag{4}$$

(4.1) THEOREM. *If the welding signatures have constant value $\tilde{\sigma}(j)$ on each $\overline{X}_j$ there is then a holomorphic fibre bundle $\hat{\mathfrak{F}} \to X^\alpha$ over the welded Riemann surface X^α with structure group L and fibre F and an isomorphism from the space of global holomorphic sections in $\hat{\mathfrak{F}}$ onto the space of solutions of the above Haseman problem. This isomorphism is functorial in the obvious sense.*

Proof. Let $s_j^\alpha = \iota^{\tilde{\sigma}(j)} \circ s_j \circ A^{-1}$ for every j. $s_j^\alpha : \overline{X}_j^\alpha \to \mathrm{F}$ is continuous, $s_j^\alpha \mid \mathrm{Int}\,\overline{X}_j^\alpha$ is holomorphic, and for every $x^\alpha \in \overline{X}_\beta^\alpha$

$$s^\alpha_{\beta^\alpha(j, x^\alpha)}(\beta^\alpha x^\alpha) = f^\alpha(\mathrm{pr}^\alpha x^\alpha)^{\sigma(x^\alpha)}\, s_j^\alpha(x^\alpha). \tag{5}$$

Clearly there is an isomorphism from the space of solutions of the Haseman problem (4) onto the space of solutions of the Riemann-Privalov problem (5). The theorem now follows from the results in [6].

When L is the general linear group $\mathrm{GL}(q, \mathbf{C})$ and $F = \mathbf{C}^q$ another form of the Haseman problem can be solved. Given

1) a Riemann surface X, a simple closed C^{1+a} curve S on X together with a C_0^{1+a} homeomorphism α' of S onto itself. α' induces the C_0^{1+a} homeomorphism $\alpha : \overline{X}_2 \cap \cap \overline{X}_\beta \to \overline{X}_1 \cap \overline{X}_\beta$ in the obvious way ($\overline{X}_1$ and $\overline{X}_2$ are the two connected components of $\overline{X}$) for which there are welding signatures $\tilde{\sigma}(1)$ and $\tilde{\sigma}(2)$;

2) the involution $\iota : \mathbf{C}^q \to \mathbf{C}^q$ is the map γ which sends each entry of $z \in \mathbf{C}^q$ into its complex conjugate;

3) a continuous map $f_2 : S \to \mathrm{GL}(q, \mathbf{C})$, Hölder continuous $g_2 : S \to \mathrm{GL}(q, \mathbf{C})$ and $q \times q$ matrices of functions f_1, f_3, g_1, g_3 for which $\gamma^{\tilde{\sigma}(1)} \circ f_1$, $\gamma^{\tilde{\sigma}(1)} \circ f_3$, $\gamma^{\tilde{\sigma}(2)} \circ g_1$, $\gamma^{\tilde{\sigma}(2)} \circ g_3$ are meromorphic matrices in an open neighborhood U of S whose determinants are not identically zero,

the problem is to find holomorphic maps $s_j: \mathrm{Int}\,\overline{X}_j \to \mathbf{C}^q$ which have angular boundary values almost everywhere on $\overline{X}_j \cap \overline{X}_\beta$ (i.e., boundary values taken along nontangential paths to $\overline{X}_j \cap \overline{X}_\beta$) and satisfy for almost all $\bar{x} \in \overline{X}_2 \cap \overline{X}_\beta$ the eq ' tion

$$(f_1 f_2 f_3)(\mathrm{pr}\,\alpha(\bar{x}))\,\gamma^{\tilde{\sigma}(1)} \circ s_1(\alpha(\bar{x})) = (g_1 g_2 g_3)(\mathrm{pr}\,\bar{x})\,\gamma^{\tilde{\sigma}(2)} \circ s_2(\bar{x}). \tag{6}$$

As before, we will associate with (6) a holomorphic cocycle with values in $\mathrm{GL}(q, \mathbf{C})$ on the welded Riemann surface X^α by considering an associated problem on X^α of the Riemann-Privalov type. In order to avoid inconvenient notation this associated problem will also be given on X. Given

1) $X, S, \alpha, \tilde{\sigma}(1), \tilde{\sigma}(2)$ as above;

2) $\iota = \gamma$ as above;

3) f_2, g_2 as above and f_ν, g_ν $(\nu = 1, 3)$ are $q \times q$ matrices of functions satisfying

i) there is an open neighborhood U of S such that f_ν (resp. g_ν) is a meromorphic matrix on $\mathrm{pr}\,\overline{X}_1 \cap U \backslash S$ (resp. $\mathrm{pr}\,\overline{X}_2 \cap U \backslash S$) whose determinant is not identically zero;

ii) $\gamma^{\tilde{\sigma}(1)} \circ f_\nu \circ \mathrm{pr} \circ \chi_1^{-1}$ and $\gamma^{\tilde{\sigma}(2)} \circ g_\nu \circ \mathrm{pr} \circ \chi_2^{-1}$ extend to meromorphic matrices in an open neighborhood of S^α,

the problem is to find s_1, s_2 as above so that for almost all $\bar{x} \in \overline{X}_2 \cap \overline{X}_\beta$

$$(f_1 f_2 f_3)(\mathrm{pr}\,\bar{x}) \cdot s_1(\beta\bar{x}) = (g_1 g_2 g_3)(\mathrm{pr}\,\bar{x}) \cdot s_2(\bar{x}). \tag{7}$$

Since S is a simple closed C^{1+a} curve there is an open neighborhood V of S which is schlichtartig, and so by the Koebe mapping theorem there is a biholomorphic map v of V into a domain in the 1 dimensional projective space $\mathbf{P}^1$. Clearly this can be done so that $v(S)$ does not contain the point at infinity. Moreover, $v(S)$ is a simple closed C^{1+a} curve in $\mathbf{P}^1$. Let G'_j be a continuous map of $v(V \cap \mathrm{pr}\,\overline{X}_j)$ into $\mathrm{GL}(q, \mathrm{C})$ $(j = 1, 2)$ whose restriction to $\mathrm{Int}\,v(V \cap \mathrm{pr}\,\overline{X}_j)$ is holomorphic and such that for all $t \in v(S)$, $G'_1(t) = g_2(v^{-1}(t)) \cdot G'_2(t)$. The existence of such maps is known ([6] or [11], § 127). Since $G'_1(t)^{-1} f_2(v^{-1}(t))\, G'_1(t)$ is continuous, it is known ([10], [12]) that there are holomorphic maps $\overline{X}'_j: \mathrm{Int}\,v(V \cap \mathrm{pr}\,\overline{X}_j) \to \mathrm{GL}(q, \mathbf{C})$ with angular boundary values almost everywhere on $v(S)$ which are in $L_p(S)$ for every $p > 1$ and satisfy almost everywhere on $v(S)$ the equation $X'_2(t) = G'_1(t)^{-1} \cdot f_2(v^{-1}(t)) \cdot G'_1(t)\, X'_1(t)$. Let $G_j = G'_j \circ v$, $X''_j = X'_j \circ v$.

Two cases must be considered. Either (1) α is the boundary value of a biholomorphic map $\alpha: \mathrm{Int}\,\mathrm{pr}\,\overline{X}_2 \cap V$ or (2) it is not. In the first case f_1 extends to a meromorphic matrix in an open neighborhood of S. Let $V' \subseteq V$ be an open neighborhood of S so that on $V' \backslash S f_1, f_3, g_1$, and g_3 are holomorphic matrices with values in $\mathrm{GL}(q, \mathbf{C})$. Define $Y_2 = f_1 \cdot X''_2$ if α is as in (1) and $Y_2 = X''_2$ otherwise. Let $Y_1 = f_3^{-1} \cdot X''_1$, $F_1 = G_1$ and $F_2 = g_3^{-1} \cdot G_2$, and let $\mathrm{Int}\,\mathrm{pr}\,\overline{X}_j = U_j$, $V' = U_3$. Then $g_{13} = F_1 \cdot Y_1$, $g_{23} = F_2 \cdot Y_2$ is a holomorphic $U = \{U_\nu : \nu = 1, 2, 3\}$-cocycle with values in $\mathrm{GL}(q, \mathbf{C})$ and defines the holomorphic vectorbundle $\hat{W} \to X$.

(4.2) THEOREM. *Suppose that* $g_1=1$ *and*

(1) *if* α *is the boundary value of a biholomorphic map* $\alpha:\operatorname{Int}\operatorname{pr}\bar{X}_2\cap V\to X$ *for some open neighborhood* V *of* S, *then* $f_\nu G_1=G_1 f_\nu$ $(\nu=1,3)$;

(2) *if* α *is not as in* (1), *then* $f_1=1$ *and* $f_3G_1=G_1f_3$.

Then there is an injection from the space of global holomorphic (resp. meromorphic) sections in $\hat{W}$ *to the space of holomorphic (resp. meromorphic) solutions of problem* (7). *If* f_1, f_3 *and* g_3 *have values in* $\mathrm{GL}(q, \mathbf{C})$ *and it is required that the boundary values of a solution* $\{s_1, s_2\}$ *of* (7) *satisfy*

$$s_j\circ\chi_j^{-1}\circ v^{-1}\mid v(S)\in L_p(v(S))$$

for all $p>1$ *and* $j=1, 2$, *this map is bijective.*

Proof. The proof is the same as that of theorem (4.1) in [6], using known properties of X_1'' and X_2''.

Suppose now that f_2 is Hölder continuous. It may then be required that the solutions s_1, s_2 satisfy

(A_1) s_1 and s_2 have continuous extensions to $\bar{X}_1$ and $\bar{X}_2$ respectively.

We will also require the following property.

(A_2) Let $\{x_1,\dots, x_k\}$ be the smallest set of points on S (in the sense of inclusion) such that f_1, f_3 and g_3 have values in $\mathrm{GL}(q, \mathbf{C})$ on $S\setminus\{x_1,\dots, x_k\}$. Then $f_1(\operatorname{pr}\bar{x})f_2(x_\mu)f_3(\operatorname{pr}\bar{x})\,s_1(\bar{x})$ and $g_1(\operatorname{pr}\bar{x})\,g_2(x_\mu)\,g_3(\operatorname{pr}\bar{x})\,s_2(\bar{x})$ should have continuous extensions to $\bar{X}_1\cap\operatorname{pr}^{-1}(U_\mu)$ and $\bar{X}_2\cap\operatorname{pr}^{-1}(U_\mu)$ respectively for an open neighborhood U_μ of x_μ, $\mu=1,\dots, k$.

Just as in theorem (4.2), the correspondence between the space of global holomorphic sections in $\hat{W}$ and the space of holomorphic solutions of problem (7) is not in general surjective. For example, let $X=\mathbf{P}^1$, $S=\{w:|w|=1\}$, $\alpha=$identity, $q=1, f_1=g_1=f_2=g_2=1$ and $f_3=g_3=(w-1)^{-1}$. Then $s_1=s_2=1$ is a solution of (7), but s_1, s_2, $s_3=(w-1)^{-1}$ is a meromorphic, not a holomorphic, section in $\hat{W}$. However, unlike theorem (4.2) the correspondence is also not in general injective. To see this, take all data as in the above example except for f_3 and g_3 and set $f_3=w-1$, $g_3=1$. Then $\{s_1=(w-1)^{-1}, s_2=s_3=1\}$ is a holomorphic section in $\hat{W}$, but $\{s_1, s_2\}$ is not a solution of (7) in the above sense because it does not satisfy (A_1). This situation is rectified by adjusting the cocycle which defines $\hat{W}$.

Let $n_{\mu,\varrho}$ be the integers such that for any meromorphic section $\{s_1, s_2, s_3\}$ in $\hat{W}$, the ϱth component of s_3 has a singularity of order $\geqslant n_{\mu,\varrho}$ at x_μ for every $\varrho=1,\dots, q$ if and only if s_1 and s_2 have continuous extensions to $\bar{X}_1\cap\operatorname{pr}^{-1}U_\mu$ and $\bar{X}_2\cap\operatorname{pr}^{-1}U_\mu$ respectively for some open neighborhood U_μ of x_μ. The $n_{\mu,\varrho}$ are easily computed in terms of the orders of the singularities of the components of $\gamma^{\tilde{\sigma}(1)}\circ f_\nu\circ\operatorname{pr}\circ\chi_1^{-1}$ and $\gamma^{\tilde{\sigma}(2)}\circ g_3\circ\operatorname{pr}\circ\chi_2^{-1}$, $\nu=1, 3$. Let h_ϱ be a meromorphic function on U_3 with divisor

$\sum_{\mu=1}^{k} n_{\mu,\varrho} x_\mu$, $h = h_1 \oplus \cdots \oplus h_q$ and $\tilde{g}_{13} = g_{13} h$, $\tilde{g}_{23} = g_{23} h$. The cocycle $\tilde{g}_{ij}$ defines the holomorphic vectorbundle $\tilde{W} \to X$.

(4.3) COROLLARY. *Suppose that in problem* (7) f_ν *and* g_ν $(\nu = 1, 3)$ *satisfy the hypothesis of* (4.2) *and furtheremore that* f_2 *is Hölder continuous. Then there is an isomorphism from the space of global holomorphic (resp. meromorphic) sections in the vectorbundle* $\tilde{W} \to X$ *onto the space of holomorphic (resp. meromorphic) solutions of problem* (7) *which satisfy the conditions* (A_1) *and* (A_2) *above.*

§ 5. The Carleman Problem

Given

1) a bordered Riemann surface X whose boundary is a C^{1+a} 1-complex with simplicial structure $\{\sigma_\lambda : \lambda \in \Lambda\}$;

2) a homeomorphism $\alpha : bdy X \to bdy X$ which preserves this simplicial structure and for which $\alpha_\lambda = \alpha \mid \mathrm{Int}\,\sigma_\lambda^1$ is a C_0^{1+a} map for every $\lambda \in \Lambda$ and satisfies the *Carleman condition* $\alpha \circ \alpha = \text{identity}$. If α_λ is orientation preserving (resp. reversing) we write $\alpha_\lambda = \alpha_\lambda^+$ (resp. $\alpha_\lambda = \alpha_\lambda^-$);

3) a complex Lie group L with antiholomorphic involution $\iota : L \to L$ which acts as a complex automorphism group on the abelian complex Lie group L_1, which also has an antiholomorphic involution $\iota_1 : L_1 \to L_1$, such that ι and ι_1 are automorphisms satisfying

$$\iota_1(\ell \cdot \ell_1) = \iota(\ell) \cdot \iota_1(\ell_1) \quad \text{for all} \quad \ell \in L, \ell_1 \in L_1;$$

4) a Hölder continuous map $f : bdy X \to L$ satisfying the *Carleman condition*

$$\begin{aligned} &(+) \quad f(x) = \iota_1 \circ f(\alpha(x))^{-1} \quad \text{for all} \quad x \in \sigma_\lambda^1 \quad \text{if} \quad \alpha_\lambda = \alpha_\lambda^+ \\ &(-) \quad f(x) = f(\alpha(x))^{-1} \quad \text{for all} \quad x \in \sigma_\lambda^1 \quad \text{if} \quad \alpha_\lambda = \alpha_\lambda^-, \end{aligned}$$

the Carleman problem is to find a holomorphic map $s : \mathrm{Int} X \to L_1$ which has a continuous extension to X and satisfies

$$\begin{aligned} s(\alpha(x)) &= f(x)\, \iota_1 \circ s(x) \quad \text{for all} \quad x \in \sigma_\lambda^1 \quad \text{if} \quad \alpha_\lambda = \alpha_\lambda^+ \\ s(\alpha(x)) &= f(x)\, s(x) \quad \text{for all} \quad x \in \sigma_\lambda^1 \quad \text{if} \quad \alpha_\lambda = \alpha_\lambda^-, \end{aligned} \tag{8}$$

$\lambda \in \Lambda$.

Let $X_1 = X$, $X_2 = X$ as sets but with opposite structures and $\bar{X}$ equal the disjoint union $X_1 \cup X_2$. The simplicial structure $\{\sigma_\lambda^1 : \lambda \in \Lambda\}$ on $bdy X_1$ defines a similar structure on $bdy\, X_2$ in the obvious way and thereby one on $\bar{X}_\beta = bdy\, X_1 \cup \cup\, bdy X_2$ which will be denoted by $\{\sigma_\varrho^1 : \varrho \in P\}$. There is a natural involution γ of $\bar{X}$ onto itself which maps X_1 onto X_2 and X_2 onto X_1, is antiholomorphic on $\bar{X} \backslash \bar{X}_\beta$, and continuous on $\bar{X}$. Define the bijections $\varphi : \Lambda \to \Lambda$ by $\alpha(\sigma_\lambda^1) = \sigma_{\varphi(\lambda)}^1$ and $\psi : P \to P$ by

1. if there exists $\lambda \in \Lambda$ for which $\sigma_\lambda^1 = \sigma_\varrho^1$, then $\gamma \circ \alpha(\sigma_\varrho^1) = \sigma_{\psi(\varrho)}^1$;

2. if there exists $\lambda\in\Lambda$ for which $\sigma_\lambda^1=\gamma(\sigma_\varrho^1)$, then $\alpha^{-1}\circ\gamma(\sigma_\varrho^1)=\sigma_{\psi(\varrho)}^1$.

Correspondingly, for each $\varrho\in P$ define the C_0^{1+a} homeomorphism $\alpha_\varrho:\sigma_\varrho^1\to\sigma_{\psi(\varrho)}^1$ by the rules

1'. if there exists $\lambda\in\Lambda$ for which $\sigma_\lambda^1=\sigma_\varrho^1$, then $\alpha_\varrho=\gamma\circ\alpha_\lambda$;

2'. if there exists $\lambda\in\Lambda$ for which $\sigma_\lambda^1=\gamma(\sigma_\varrho^1)$, then $\alpha_\varrho=\alpha_\lambda^{-1}\circ\gamma$.

Define the Hölder continuous map $\tilde{f}:\overline{X}_\beta\to L$ as follows:

1°. if for σ_ϱ^1 there is $\sigma_\lambda^1=\sigma_\varrho^1$, then

$$\tilde{f}(\bar{x})=\begin{cases}\iota\circ f(\bar{x}) & \text{for all } \bar{x}\in\sigma_\varrho^1 \text{ if } \alpha_\lambda=\alpha_\lambda^+\\ f(\bar{x}) & \text{for all } \bar{x}\in\sigma_\varrho^1 \text{ if } \alpha_\lambda=\alpha_\lambda^-;\end{cases}$$

2°. if for σ_ϱ^1 there is $\sigma_\lambda^1=\iota(\sigma_\varrho^1)$, then

$$\tilde{f}(\bar{x})=\begin{cases}\iota\circ f(\alpha_\varrho(\bar{x})) & \text{for all } \bar{x}\in\sigma_\varrho^1 \text{ if } \alpha_\lambda=\alpha_\lambda^+\\ f(\alpha_\varrho(\bar{x})) & \text{for all } \bar{x}\in\sigma_\varrho^1 \text{ if } \alpha_\lambda=\alpha_\lambda^-.\end{cases}$$

Define the signatures $\{\sigma(\bar{x}):\bar{x}\in\overline{X}_\beta\}$ by the rule

$$\sigma(\bar{x})=\begin{cases}1, & \bar{x}\in bdy\,X_1\\ -1, & \bar{x}\in bdy\,X_2\end{cases}$$

and the welding signatures $\{\tilde{\sigma}(\bar{x}):\bar{x}\in\overline{X}_\beta\}$ by

$$\tilde{\sigma}(\bar{x})=\begin{cases}0, & \bar{x}\in bdy\,X_1 \text{ or } \bar{x}\in bdy\,X_2\cap\sigma_\varrho^1 \text{ and } \alpha_\varrho=\alpha_\varrho^+\\ 1, & \bar{x}\in bdy\,X_2\cap\sigma_\varrho^1 \text{ and } \alpha_\varrho=\alpha_\varrho^-;\end{cases}$$

$\tilde{\sigma}(x)$ is well-defined since it is constant on each connected component of $\overline{X}_\beta$.

Finally, let $\beta(1)=2$ and $\beta(2)=1$ and consider the Haseman problem to find holomorphic maps $s_\nu:\mathrm{Int}X_\nu\to L_1$ with continuous extensions to X_ν $(\nu=1,2)$ such that for every $x\in\overline{X}_\beta$

$$\iota_1^{\tilde{\sigma}(\alpha_\varrho(x))}\circ s_{\beta(\nu)}(\alpha_\varrho(x))=\tilde{f}(x)^{\sigma(x)}\,\iota_1^{\tilde{\sigma}(x)}\circ s_\nu(x),\quad\text{where}\quad x\in\sigma_\varrho^1. \tag{9}$$

(5.1) THEOREM. *Suppose that α is such that either $\alpha_\lambda=\alpha_\lambda^+$ for all $\lambda\in\Lambda$ or $\alpha_\lambda=\alpha_\lambda^-$ for all $\lambda\in\Lambda$. Then every solution s of the Carleman problem* (8) *defines a solution of the Haseman problem* (9), *namely $\{s_1=s, s_2=\iota_1\circ s\circ\gamma\}$. On the other hand, if $\{s_1, s_2\}$ is a solution of* (9), *then $s=s_1+\iota_1\circ s_2\circ\gamma$ is a solution of* (8).

Proof. Verification.

Let $\mathscr{L}_1$ (resp. $\mathscr{L}_2$) be the space of solutions of problem (8) (resp. (9)) and $\chi_1:\mathscr{L}_1\to\mathscr{L}_2$, $\chi_2:\mathscr{L}_2\to\mathscr{L}_1$ the maps described in (5.1). χ_1 is an injection. χ_2 is not bijective, and so there is in general nothing equivalent to (4.1) for the Carleman problem. However, the existence of global holomorphic sections in the fibre bundle $\hat{\mathfrak{F}}\to X^\alpha$ described by problem (9) implies the existence of solutions of the Carleman problem (8).

If $L_1 = \mathbf{C}^q$ and L is a complex Lie subgroup of the complex affine group $GA(q, \mathbf{C})$, it makes sense to define χ_2 by the rule $\{s_1, s_2\} \to \frac{1}{2}(s_1 + \iota_1 \circ s_2 \circ \gamma)$. Theorem (5.1) remains valid and moreover, $\chi_2 \circ \chi_1 = \text{identity}$. If $\hat{W} \to X^\alpha$ is the holomorphic vector-bundle associated with the holomorphic affine bundle $\hat{\mathfrak{F}} \to X^\alpha$ which corresponds to the homogeneous Carleman problem (see [6]) and $\mathcal{O}(\hat{W})$ is the sheaf of germs of holomorphic sections in $\hat{W}$, then from (5.1) and [6], theorem (5.1) one has

(5.2) THEOREM. *$\mathcal{L}_1$ is either empty or* $\dim_{\mathbf{R}} \mathcal{L}_1 = \dim_{\mathbf{C}} H^\circ(X^\alpha, \mathcal{O}(\hat{W}))$.

Remarks. 1. It is clear that one can study Carleman problems corresponding to the various forms of problem (6) discussed in § 4. Moreover, because of the results in § 3 and [6], the Haseman and Carleman problems can also be analysed when given on a holomorphic family of Riemann surfaces with appropriate S, welding data (i.e., shifts) and transmission map f.

2. As in [6], known properties of the space of global holomorphic sections in a global holomorphic fibre bundle can be used to describe the solution spaces of the various problems discussed here. The results in [6] carry over to the Haseman and Carleman problems when modified in the obvious ways to take into account the fact that the correspondences in the various theorems of § 4 and § 5 are not necessarily bijective.

REFERENCES

[1] L. BERS, *Riemann Surfaces*, Lecture notes, New York University (1957–58).

[2] C. BLANC, *Les surfaces de Riemann des fonctions méromorphes*, Comment. Math. Helv. *9*, (1934), 193–216; 335–368.

[3] O. D. KELLOGG, *Harmonic functions and Greens Integral*, Trans. Amer. Math. Soc. *13* (1912), 109–132.

[4] K. OIKAWA, *Welding of Polygons and the Type of Riemann Surfaces*, Kodai Math. Sem. Rep. *13* (1961), 37–52.

[5] D. ORTH, *Conformal Mappings Depending Holomorphically on a Parameter*, Tech. Rep. under Contract Nonr 2216 (28) (NR-043-332).

[6] H. RÖHRL, *Über das Riemann-Privalovsche Randwertproblem*. Math. Ann. *151* (1963), 365–423.

[7] ———, *Ω-degenerate Singular Integral Equations and Holomorphic Affine Bundles over Compact Riemann Surfaces*, I, Comment. Math. Helv. *38* (1963) 84–120.

[8] L. I. VOLKOVISKII, *Quasiconformal Mapping and the Problem of Conformal Pasting*. Ukrain. Mat. Z. *3* (1951), 39–51.

[9] S. WARSCHAWSKI, *Über einen Satz von O. D. Kellogg*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klasse *1932*, 73–86.

[10] G. F. MANDZHAVIDZE and B. V. KHVEDELIDZE, *On the Problem of Riemann-Privalov with Continuous Coefficient*. Dokl. Akad. Nauk. Mat. *123* (1958), 791–794.

[11] N. I. MUSKHELISHVILI, *Singular Integral Equations* (Noordhoff, Groningen 1953).

[12] I. B. SIMONENKO, *The Boundary Problem of Riemann with Continuous Coefficients*, Dokl. Akad. Nauk. Math. *124* (1959), 278–281.

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