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## On a Special Class of Hamiltonian Graphs

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One of the most basic questions asked about a graph (finite, undirected, without loops or multiple edges) is whether its structure is such that it can be traversed or traced in a certain manner. Undoubtedly, the two most important classes of graphs dealing with traversability are the eulerian graphs and the hamiltonian graphs. A graph  $G$  is *eulerian* if it has a closed path (called an eulerian path) containing every edge of  $G$  exactly once and every vertex of  $G$  at least once, while  $G$  is *hamiltonian* if it has a closed path containing every vertex of  $G$  exactly once, i.e., if it has a hamiltonian cycle.

A graph  $G$  is said to be *randomly eulerian from a vertex  $v$*  if the following procedure always results in an eulerian path. Begin at the given vertex  $v$  and traverse any incident edge. On arriving at a vertex, choose any incident edge which has not yet been traversed. When no new edges are available the procedure terminates. These graphs have also been referred to as arbitrarily traversable from  $v$  and arbitrarily traceable from  $v$  and have been investigated by BÄBBLER [1], HARARY [3], and ORE [4].

This suggests the following concept. We define a graph  $G$  to be *randomly hamiltonian from the vertex  $v$*  if the following procedure always results in a hamiltonian cycle. Begin at the vertex  $v$  and proceed to any adjacent vertex. On arriving at a vertex, select any adjacent vertex not previously encountered. When no new vertices remain, then an edge exists between the final vertex chosen and  $v$ , and the procedure terminates. Thus in a graph  $G$  which is randomly hamiltonian from a vertex  $v$ , any path beginning at  $v$  can be extended to a hamiltonian cycle. Graphs which are randomly hamiltonian from every vertex were characterized in [2] and are called simply randomly hamiltonian graphs.

It is the object of this article to present a characterization of graphs which are randomly hamiltonian from a vertex, and thereby provide a classification of all such graphs.

It is convenient to introduce notation for several types of graphs which are encountered throughout the course of this article. The complete graph with  $p$  vertices is denoted by  $K_p$ , while  $C_p$  represents the cycle with  $p \geq 3$  vertices. The complete bipartite graph  $K(m, n)$  is the graph with  $p = m + n$  vertices whose vertex set  $V$  can be partitioned as  $V_1 \cup V_2$  such that  $|V_1| = m$ ,  $|V_2| = n$ , and vertices  $u$  and  $v$  are adjacent if and only if  $u \in V_i$  and  $v \in V_j$ ,  $i \neq j$ . It was shown in [2] that a graph  $G$  with  $p \geq 3$  vertices is randomly hamiltonian if and only if it is one of the graphs  $K_p$ ,  $C_p$ , and  $K(p/2, p/2)$ .

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We express a graph  $G$  as  $H+v$  provided  $v$  is a vertex of  $G$  adjacent to all other vertices of  $G$ , where then  $H$  is the graph obtained from  $G$  by the removal of  $v$  and all edges incident with  $v$ . For example, the graph  $C_n+v$  is often referred to as the wheel  $W_n$ . The graphs  $K(3,3)+v$  and  $W_5=C_5+v$  are illustrated in Figure 1. In each case, the graph is randomly hamiltonian from  $v$ .

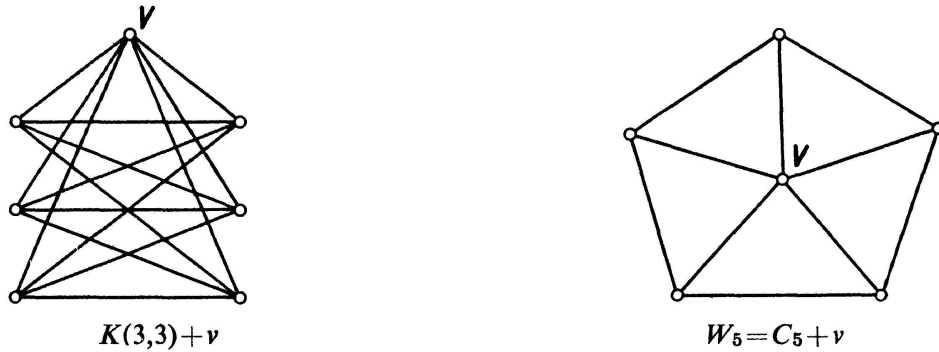


Figure 1

Two graphs which are randomly hamiltonian from the vertex  $v$

Of course, if a graph  $G$  is randomly hamiltonian from a vertex, then  $G$  is hamiltonian and therefore has a hamiltonian cycle. Thus whenever we have a graph  $G$  with  $p$  vertices which is randomly hamiltonian from a vertex we assume the existence of a hamiltonian cycle  $C$  whose vertices are labeled cyclically  $v_1, v_2, \dots, v_p$ . Each edge of  $G$  then either belongs to  $C$ , and is called a *cycle edge* of  $G$ , or joins two non-consecutive vertices of  $C$  and is called a *diagonal*.

If  $G$  is a graph which is randomly hamiltonian from some vertex (and which contains a hamiltonian cycle  $C$  labeled as earlier indicated), then any cycle of  $G$  containing exactly one diagonal of  $G$  is called an *outer cycle* of  $G$ . An *outer  $n$ -cycle* has length  $n$ , and an outer 3-cycle is also referred to as an *outer triangle*.

We now present the main result of the paper.

**THEOREM.** *A graph  $G$  is randomly hamiltonian from a vertex  $v$  if and only if  $G$  is randomly hamiltonian or  $G=H+v$ , where  $H$  is randomly hamiltonian.*

*Proof.* If  $G$  is a randomly hamiltonian graph containing a vertex  $v$  or if  $G$  is expressible as  $H+v$ , where  $H$  is randomly hamiltonian, then it is easily observed that  $G$  is randomly hamiltonian from  $v$ .

Conversely, let  $G$  be a graph with  $p$  vertices which is randomly hamiltonian from the vertex  $v$ . Thus  $G$  contains a hamiltonian cycle  $V$  whose vertices we label cyclically as  $v=v_1, v_2, \dots, v_p$ .

Suppose that  $G$  is not randomly hamiltonian so that  $G$  is none of the graphs  $K_p, C_p, K(p/2, p/2)$ . In particular, this implies that  $G$  contains diagonals so that  $G$  necessarily contains outer cycles. Hence the vertex  $v$  belongs to one or more outer cycles.

Let  $n$  be the length of the smallest outer cycle containing  $v$ . We first show that there exists an outer  $n$ -cycle containing  $v$  but in which  $v$  is not the endpoint of the associated diagonal. Suppose that the vertices of an outer  $n$ -cycle are  $v_1, v_2, \dots, v_n$ . Consider the path which commences at  $v_1$ , proceeds to  $v_n$  along the diagonal  $v_1 v_n$ , and encounters in succession the vertices  $v_{n+1}, v_{n+2}, \dots, v_p$ . Since  $G$  is randomly hamiltonian from  $v=v_1$  and  $v$  belongs to no outer  $k$ -cycle,  $k < n$ , the diagonal  $v_p v_{n-1}$  must be present in  $G$ . Hence  $v$  belongs to the outer  $n$ -cycle whose vertices are  $v_p, v_1, \dots, v_{n-1}$ . In a similar way, one can show that if  $v_{p-n+2} v_1$  is a diagonal of  $G$ , then  $v_{p-n+3} v_2$  is a diagonal of  $G$ .

Thus we may assume the existence of an outer  $n$ -cycle whose vertices are  $v_m, v_{m+1}, \dots, v_p, v_1, \dots, v_{k-1}$ , where  $m=p-n+k$  and  $3 \leq k \leq n$ . We now show that the diagonals  $v_{m-1} v_{k-2}$  and  $v_{m+1} v_k$  are present in  $G$  in addition to  $v_m v_{k-1}$ . We begin a path at  $v=v_1$  and proceed along  $C$  to  $v_p, v_{p-1}, \dots, v_m$ . Following along the diagonal  $v_m v_{k-1}$  to  $v_{k-1}$  and then taking  $v_k, v_{k+1}, \dots, v_{m-1}$ , we see that  $v_{m-1} v_{k-2}$  is a diagonal of  $G$  since  $G$  is randomly hamiltonian from  $v$  and  $v$  belongs to no outer  $t$ -cycle,  $t < n$ . Similarly, by applying the preceding arguments to the path  $v_1, v_2, \dots, v_{k-1}, v_m, v_{m-1}, v_{m-2}, \dots, v_k$ , we observe that  $v_{m+1} v_k$  is a diagonal of  $G$ .

We now prove that  $n < 5$ , for suppose, to the contrary, that  $n \geq 5$ . We have already seen that there exists an outer  $n$ -cycle whose vertices are  $v_m, v_{m+1}, \dots, v_p, v_1, \dots, v_{k-1}$ , where  $m=p-n+k$  and  $3 \leq k \leq n$ , and, in addition, the edges  $v_{m-1} v_{k-2}$  and  $v_{m+1} v_k$  belong to  $G$ . Furthermore, since  $n \geq 5$ ,  $v_1$  is not adjacent to both  $v_m$  and  $v_{k-1}$ . Let us say that  $v_1$  is not adjacent to  $v_{k-1}$ , the other case being handled analogously. We now construct a path which begins at  $v=v_1$  and takes in succession  $v_p, v_{p-1}, \dots, v_{m+1}$ . We then proceed to  $v_k$  via the diagonal  $v_{m+1} v_k$  and move along  $C$  in the order  $v_{k+1}, v_{k+2}, \dots, v_{m-1}$ . On reaching  $v_{m-1}$ , we next take  $v_{k-2}$  (which is different from  $v$ ),  $v_{k-1}$ , and then  $v_m$ . Since  $G$  is randomly hamiltonian from  $v$ , there exists either a vertex not yet encountered which is adjacent to  $v_m$  or the edge  $v_m v$  which completes a hamiltonian cycle. In either case, there exists an edge  $v_m u$ , where  $u$  is one of the vertices  $v_{m+2}, v_{m+3}, \dots, v_p, v_1, \dots, v_{k-3}$ , which determines an outer cycle containing  $v$  having length less than  $n$ , and this is a contradiction.

We now show that  $n \neq 4$ . To prove this, we assume  $n=4$  so that  $v$  belongs to an outer 4-cycle but not an outer triangle. From what we have shown above, we may assume, without loss of generality, that  $v_p, v_1, v_2, v_3$  are the vertices of an outer 4-cycle. Since  $G$  is randomly hamiltonian from  $v=v_1$ , the path  $v_1, v_2, v_3, v_p, v_{p-1}, v_{p-2}, \dots, v_4$ , which contains all vertices of  $G$ , implies that  $v_1 v_4$  is an edge of  $G$ . The path  $v_1, v_4, v_3, v_p, v_{p-1}, v_{p-2}, \dots, v_5$  contains all the vertices of  $G$  with the exception of  $v_2$ ; hence  $v_2 v_5$  is an edge of  $G$ . Next the path  $v_1, v_2, v_5, v_4, v_3, v_p, v_{p-1}, v_{p-2}, \dots, v_6$  contains all the vertices of  $G$  and, as such, implies that  $v_1 v_6$  is an edge of  $G$ . Continuing inductively, it is now easily verified that all edges of the type  $v_1 v_{2m}$  belong to  $G$  as do all edges of the type  $v_2 v_{2m+1}$ . From this it now follows that every two vertices  $v_\alpha$

and  $v_\beta$ , where  $\alpha$  and  $\beta$  are of opposite parity, are adjacent. To see this, let  $v_{2r}$  and  $v_{2s+1}$  be two non-consecutive vertices of  $C$ , where  $v_{2r}$  is different from  $v_2$  and  $v_{2s+1}$  is not  $v_1$ . There are two cases to consider according to whether the path  $v_{2r}, v_{2r+1}, \dots, v_{2s+1}$  does or does not contain the vertex  $v$ . We treat here only the latter case, the former case being handled in a similar manner. We construct a path which begins at  $v_1$ , proceeds along a diagonal to  $v_{2s}$ , then along  $C$  to the vertices  $v_{2s-1}, v_{2s-2}, \dots, v_{2r+1}$ , from where we move to  $v_2$  by way of the diagonal  $v_2 v_{2r+1}$ . Next we proceed to  $v_{2r-1}$  via the diagonal  $v_2 v_{2r-1}$  and then take  $v_{2r-2}, v_{2r-3}, \dots, v_3, v_p, v_{p-1}, \dots, v_{2s+1}$ , which produces a path failing only to contain  $v_{2r}$ . Since  $G$  is randomly hamiltonian from  $v$ , the edge  $v_{2r} v_{2s+1}$  must be present in  $G$ . Finally, we show that if  $\alpha$  and  $\beta$  are of the same parity, then  $v_\alpha$  and  $v_\beta$  are not adjacent. We consider here only the case where  $\alpha$  and  $\beta$  are odd, the other case following similarly. Assume, to the contrary, that the vertices  $v_{2r+1}$  and  $v_{2s+1}$  are adjacent, where  $2r+1 < 2s+1$ , say. The path  $v_1, v_{2s+2}, v_{2s+3}, \dots, v_p, v_3, v_4, \dots, v_{2r+1}, v_{2s+1}, v_{2s}, \dots, v_{2r+2}$  fails only to contain the vertex  $v_2$ ; thus  $v_2 v_{2r+2}$  is an edge of  $G$ . From this we see that the path  $v_1, v_p, v_{p-1}, \dots, v_{2r+2}, v_2, v_{2r+1}, v_{2r}, \dots, v_3$ , which contains all vertices of  $G$ , implies that  $v_1 v_3$  is a diagonal of  $G$ . However, this contradicts the fact that  $v_1$  belongs to no outer triangle. Hence,  $v_\alpha$  and  $v_\beta$  are adjacent if and only if  $\alpha$  and  $\beta$  are of opposite parity. This implies that  $p$  is even since  $v_1 v_p$  is an edge of  $G$ . Furthermore, by letting  $V_1 = \{v_{2n} \mid n=1, 2, \dots, p/2\}$  and  $V_2 = \{v_{2n-1} \mid n=1, 2, \dots, p/2\}$ , we see that  $G$  is the graph  $K(p/2, p/2)$ , which, as noted earlier, is randomly hamiltonian. However, this is a contradiction since it is contrary to our assumption that  $G$  is not randomly hamiltonian.

We now arrive at the conclusion that the only possible value is  $n=3$ ; thus  $v$  belongs to an outer triangle. From methods similar to those we have already employed, it is immediately established that  $G$  contains the edges  $v_1 v_3, v_2 v_p$ , and  $v_1 v_{p-1}$ . Thus  $v_1$  is adjacent to each of the vertices  $v_2, v_3$ , and  $v_p$ . However,  $v_1$  is necessarily adjacent to all other vertices of  $G$ , for if  $v_1 v_k$  is an edge of  $G$ ,  $3 \leq k < p-1$ , then so too is  $v_1 v_{k+1}$  an edge of  $G$  since the path  $v_1, v_k, v_{k+1}, \dots, v_2, v_p, v_{p-1}, \dots, v_{k+1}$  contains all vertices of  $G$  and therefore  $v_1$  is adjacent to  $v_{k+1}$ . The result then follows by induction.

Hence we may express  $G$  as  $H+v$ . The only remaining detail now is to verify that  $H$  is randomly hamiltonian. In order to prove this, it is necessary to show that any path  $u_1, u_2, \dots, u_k$  of  $H$  can be extended to a hamiltonian cycle of  $H$ . Since  $v$  is adjacent to  $u_1$ :  $v, u_1, u_2, \dots, u_k$  is a path of  $G$  and can be extended to a hamiltonian cycle  $v, u_1, u_2, \dots, u_k, u_{k+1}, \dots, u_{p-1}, v$  of  $G$ . However,  $v$  is also adjacent to  $u_2$ ; thus  $v, u_2, u_3, \dots, u_{p-1}$  is also a path of  $G$  and can be extended to a hamiltonian cycle of  $G$ . This implies that  $u_{p-1} u_1$  is an edge of  $G$  so that  $u_1, u_2, \dots, u_{p-1}, u_1$  is a hamiltonian cycle of  $H$ . Hence  $H$  is randomly hamiltonian, completing the proof.

The preceding theorem now indicates that the only graphs with  $p$  vertices which are randomly hamiltonian from some vertex  $v$  are  $C_p, K_p, K(p/2, p/2), C_{p-1} + v$ , and  $K((p-1)/2, (p-1)/2) + v$ . As one final observation, we state the following.

COROLLARY. *The number of vertices in a graph  $G$  with  $p$  vertices from which  $G$  is randomly hamiltonian is either 0, 1, or  $p$ .*

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