

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 43 (1968)  
  
**Artikel:** Canonical Vector Fields on Spheres.  
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**DOI:** <https://doi.org/10.5169/seals-32927>

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# Canonical Vector Fields on Spheres

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## § 1. Introduction

We are interested in norm-preserving bilinear forms

$$M: R^r \otimes R^n \rightarrow R^n,$$

where  $\otimes = \otimes_R$  and by norm preserving we mean  $\|M(u \otimes v)\| = \|u\| \cdot \|v\|$ . Such a form implies the existence of  $r-1$  mutually orthonormal vector fields on  $S^{n-1}$  (see 1.2 below). Given  $n$ , the question of finding the largest  $r$  so that such a form exists was solved in 1923 by RADON [5], by HURWITZ [3], and again in 1942 by ECKMANN [2]. The methods of RADON and HURWITZ yield complicated iterative schemes for actually constructing the forms, which have recently been simplified by ADAMS, LAX, and PHILLIPS [1]. We now give a still simpler construction and prove certain relevant properties of the "canonical" vector fields thus obtained. In particular, they are closed under the intrinsic join operations of JAMES [4] (cf. Prop. 4.4).

Let  $M$  be a form as above and let  $e_0, \dots, e_{r-1}$  be an orthonormal basis for  $R^r$ . Then one obtains  $r$  orthogonal transformations  $M_0, \dots, M_{r-1} \in O(n)$  by defining

$$M_i(v) = M(e_i \otimes v), \quad 0 \leq i \leq r-1, \quad v \in R^n.$$

Conversely,  $M$  is defined by the  $M_i$  using the formula

$$M(u \otimes v) = \sum \alpha_i M_i(v), \quad \text{where } u = \sum \alpha_i e_i \quad \text{and} \quad i = 0, \dots, r-1.$$

1.1. THEOREM: *The following are equivalent*

A:  $M$  is norm-preserving,

B:  $\langle M_i(v), M_j(v) \rangle = \delta_{ij} \|v\|^2 \quad \forall 0 \leq i, j \leq r-1 \text{ and } v \in R^n$ ,

C:  $M_i \in O(n)$  and  $M_i^t M_j + M_j^t M_i = 0, i \neq j$ .

This theorem has been used in one form or another by most of the above authors, and its proof is omitted.

One can assume without loss of generality that  $M_0 = \text{id}$ , by following  $M$  with  $M_0^{-1}$  if necessary. Then from (B) it follows that  $\langle v, M_i(v) \rangle = 0, 1 \leq i \leq r-1$ , and hence if we restrict  $v$  to  $S^{n-1}$ , i.e.  $\|v\| = 1$ , we obtain

1.2. COROLLARY:  $M_1(v), \dots, M_{r-1}(v)$  define a family of  $r-1$  orthonormal vector fields on  $S^{n-1}$ .

Furthermore, using (C) together with  $M_0 = \text{id}$  and  $M_i^t M_i = 1$ , we obtain

1.3. COROLLARY:  $M_i + M_i^t = 0, M_i^2 = -1, M_i M_j + M_j M_i = 0, 1 \leq i, j \leq r-1$ .

1.4. DEFINITION: A norm preserving form  $M: R^r \otimes R^n \rightarrow R^n$  is orthogonal to the identity if  $\langle v, M(u \otimes v) \rangle = 0 \forall u \in R^r, v \in R^n$ .

From the above remarks such a form is clearly equivalent to the existence of a norm preserving form  $M'_0 = \text{id}$  and  $M'_i = M_{i-1}$ ,  $i \geq 1$ . Furthermore,  $M$  then defines  $r$  orthonormal vector fields on  $S^{n-1}$  and  $M_i, M_j$  satisfy 1.3,  $0 \leq i, j \leq r-1$ .

We will use the notation  $M_u = M(u \otimes -): R^n \rightarrow R^n$ ,  $u \in R^r$ . Clearly  $M_u/\|u\| \in O(n)$ , and if  $M$  is orthogonal to  $\text{id}$  then  $M_u$  is antisymmetric. In all cases one has the following identity:

$$\langle M(u \otimes v_1), M(u \otimes v_2) \rangle = \langle M_u v_1, M_u v_2 \rangle = \|u\|^2 \left\langle \frac{M_u}{\|u\|} v_1, \frac{M_u}{\|u\|} v_2 \right\rangle = \|u\|^2 \langle v_1, v_2 \rangle.$$

## § 2. Tensor Products of Inner Product Spaces

Let  $V, W$  be inner product spaces over a field  $F$ . Then  $V \otimes_F W$  is an inner product space, where

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle.$$

In case  $V = R^m$  and  $W = R^n$ , with their usual products, it is not hard to see that the resulting inner product on  $R^{m \cdot n}$  is also the usual one.

The following lemma will be exceedingly useful in the proof of Theorem 3.1.

2.1. ORTHOGONALITY LEMMA: Let  $V, W$  be inner product spaces with commutative inner products and suppose  $A: V \rightarrow V$  and  $B: W \rightarrow W$  are endomorphisms such that

- (i)  $A$  is orthogonal to  $\text{id}_V$ , that is  $\langle v, Av \rangle = 0 \forall v \in V$ , or  $B$  is orthogonal to  $\text{id}_W$
- (ii)  $A$  is symmetric and  $B$  antisymmetric, or vice-versa.

Then the two endomorphisms  $\phi = A \otimes 1$  and  $\psi = 1 \otimes B$  of  $V \otimes W$  are orthogonal, that is  $\langle \phi a, \psi a \rangle = 0 \forall a \in V \otimes W$ .

*Proof:* Let  $a = \sum_i v_i \otimes w_i$ . Then

$$\begin{aligned} \langle \phi a, \psi a \rangle &= \left\langle \sum_i A v_i \otimes w_i, \sum_j v_j \otimes B w_j \right\rangle \\ &= \sum_{i,j} \langle A v_i, v_j \rangle \langle w_i, B w_j \rangle. \end{aligned}$$

Now (i) clearly implies that the terms where  $i=j$  vanish. Then supposing  $A^t = A$ ,  $B^t = -B$ , we have

$$\begin{aligned} \langle \phi a, \psi a \rangle &= \sum_{i < j} (\langle A v_i, v_j \rangle \langle w_i, B w_j \rangle + \langle A v_j, v_i \rangle \langle w_j, B w_i \rangle) \\ &= \sum_{i < j} (\langle v_j, A v_i \rangle \langle B w_j, w_i \rangle + \langle v_j, A v_i \rangle \langle -B w_j, w_i \rangle) \\ &= 0. \end{aligned}$$

REMARK: The representation  $a = \sum_{i=1}^t v_i \otimes w_i$  is of course not unique. One can,

however, always choose it so that  $v_1, \dots, v_t$  form a given basis of  $V$ , or similarly for the  $w_i$  (but not both).

### § 3. The Basic Construction

Let  $C: R^8 \otimes R^8 \rightarrow R^8$  be the Cayley multiplication. Let  $i: R^7 \rightarrow R^8$  be inclusion into the last seven co-ordinates, then  $C \circ (i \otimes 1): R^7 \otimes R^8 \xrightarrow{C_1} R^8$  is a norm preserving multiplication orthogonal to the identity. Now define a form  $N: R^7 \otimes R^{16} \rightarrow R^{16}$  by the composition

$$R^7 \otimes R^{16} \xrightarrow{\approx} R^7 \otimes R^8 \oplus R^7 \otimes R^8 \xrightarrow{C_1 \otimes (-C_1)} R^8 \oplus R^8 \xrightarrow{\approx} R^{16}.$$

Clearly  $N$  is norm preserving, orthogonal to id, and for  $0 \leq i \leq 6$  each  $N_i$  is antisymmetric. Furthermore,  $N_i$  has the form

$$N_i = \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix}, \quad B_i \in O(8).$$

3.1. THEOREM: Let  $M: R^r \otimes R^n \rightarrow R^n$ ,  $n$  even, be a norm-preserving form such that (a)  $M$  is orthogonal to id

(b)  $M_i U = -U M_i$ ,  $0 \leq i \leq r-1$ , where  $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O(n)$

Then the form  $\bar{M}$  defined by the composition below is norm preserving and also satisfies (a), (b), (relative to  $r+8$  and  $16n$ ):

$$\begin{array}{ccc} R^{r+8} \otimes R^{16n} & & \\ \downarrow \approx & \searrow & \\ (R^r \oplus R^7 \oplus R^1) \otimes R^n \otimes R^{16} & & \\ \downarrow \approx & & \\ (R^r \otimes R^n) \otimes R^{16} \oplus R^n \otimes (R^7 \otimes R^{16}) \oplus R^n \otimes R^{16} & \xrightarrow{\bar{M}} & R^{16n} \\ \downarrow M \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + U \otimes N + 1 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \\ R^n \otimes R^{16} & \xrightarrow{\approx} & R^{16n} \end{array}$$

*Proof:* Condition (b) follows readily from the fact that  $\bar{M}_i = \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix}$ ,  $0 \leq i < r+7$ , while  $\bar{M}_{r+7} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Let  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T$ ,  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = V \in O(16)$ .

From (b) it follows that  $M_u U = -U M_u \forall u \in R^r$ . Then  $U M_u$  is symmetric. Similarly, since  $N_i = \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix}$  satisfies (b),  $0 \leq i \leq 6$ , one sees that  $V N_u$  is antisymmetric  $\forall u \in R^7$  and  $T N_u$  antisymmetric. Also,  $V T$  is symmetric.

Now, starting with  $u \otimes v \in R^{r+8} \otimes R^{16n}$ , let  $u = u_1 \oplus u_2 \oplus u_3 \in R^r \oplus R^7 \oplus R^1$  and  $v = \sum_i v'_i \otimes v''_i \in R^n \otimes R^{16}$ . Then  $\bar{M}(u \otimes v) = a + b + c$ , where

$$\begin{aligned} a &= \sum_j M(u_1 \otimes v'_j) \otimes T v''_j, \\ b &= \sum_k U v'_k \otimes N(u_2 \otimes v''_k), \\ c &= u_3 \sum_l v'_l \otimes V v''_l. \end{aligned}$$

To prove (a), we show  $\langle v, a \rangle = \langle v, b \rangle = \langle v, c \rangle = 0$ .  $\langle v, a \rangle = \sum_{i,j} \langle v'_i, M_{u_1}(v'_j) \rangle \langle v''_i, T v''_j \rangle$ .

Choosing  $v''_i = e_i$ , the standard basis for  $R^{16}$ ,  $\langle v''_i, T v''_j \rangle = \pm \delta_{ij}$  and  $\langle v, a \rangle = \sum_i \pm \langle v'_i, M_{u_1}(v'_j) \rangle = 0$  since  $M$  is orthogonal to the identity.  $\langle v, b \rangle =$

$\sum_{i,l} \langle v'_i, U v'_l \rangle \langle v''_i, N_{u_2} v''_l \rangle = 0$  by the orthogonality lemma.  $\langle v, c \rangle = u_3 \sum_{i,l} \langle v'_i, v'_l \rangle$

$\langle v''_i, V v''_l \rangle = 0$  by choosing  $\{v'_i\}$  orthonormal and noticing that  $V$  is orthogonal to id.

To show that  $\bar{M}$  is norm preserving, we first prove that  $\langle a, b \rangle = \langle a, c \rangle = \langle b, c \rangle = 0$ .

$$\begin{aligned} \langle a, b \rangle &= \sum_{j,k} \langle M_{u_1}(v'_j), U v'_k \rangle \langle T v''_j, N_{u_2}(v''_k) \rangle \\ &= \sum_{j,k} \langle U M_{u_1}(v'_j), v'_k \rangle \langle v''_j, T N_{u_2}(v''_k) \rangle. \end{aligned}$$

Choosing  $v''_i = e_i$  as before, one has  $\langle T v''_i, N_{u_2}(v''_i) \rangle = \pm \langle v''_i, N_{u_2}(v''_i) \rangle = 0$ . Thus one need only consider the terms where  $j \neq k$ , which sum to zero since  $U M_{u_1}$  is symmetric and  $T N_{u_2}$  antisymmetric. The other two orthogonality relations are proved quite analogously, where in  $\langle b, c \rangle$  one takes  $\{v'_i\}$  to be the standard basis for  $R^n$  to insure that the  $(i, i)$  terms vanish. Thus

$$\|\bar{M}(u \otimes v)\|^2 = \|a\|^2 + \|b\|^2 + \|c\|^2.$$

Choosing  $v''_i = e_i$ , one easily sees that the individual terms in  $a, b, c$  are mutually orthogonal, being already orthogonal in the second factor. Then, since  $T, U$ , and  $V$  are all orthogonal transformations,

$$\begin{aligned} \|\bar{M}(u \otimes v)\|^2 &= \sum_i \|u_1\|^2 \|v'_i\|^2 \|v''_i\|^2 + \sum_i \|v'_i\|^2 \|u_2\|^2 \|v''_i\|^2 + u_3^2 \sum_i \|v'_i\|^2 \|v''_i\|^2 \\ &= (\|u_1\|^2 + \|u_2\|^2 + u_3^2) \sum_i \|v'_i\|^2 \|v''_i\|^2 \\ &= \|u\|^2 \|v\|^2. \end{aligned}$$

3.2. COROLLARY: If  $n = s \cdot 2^{4a+b}$ ,  $s$  odd,  $0 \leq b \leq 3$ , then  $S^{n-1}$  admits  $8a + 2^b - 1$  orthonormal vector fields.

*Proof:* If  $n = s$  one has a trivial form  $R^0 \otimes R^s \xrightarrow{0} R^s$ . Applying the theorem " $a$ " times gives a norm preserving form orthogonal to the identity

$$R^{8a} \otimes R^{s \cdot 16^a} \rightarrow R^{s \cdot 16^a}$$

(the fact that  $n$  is odd on the first iteration causes no trouble since  $r=0$  there). This is the case  $b=0$ . For  $b=1, 2, 3$  one need only apply the theorem once more and observe that  $\bar{M}(\mu R^{8a+2^b-1} \otimes \mu R^{s \cdot 16^a \cdot 2^b}) \subset \mu R^{s \cdot 16^a \cdot 2^b}$ , where  $\mu$  denotes the generic inclusion of  $R^m$  into the first  $m$  co-ordinates of  $R^{m+k}$  for any  $m, k$ . This is so because  $N(\mu R^{2^b-1} \otimes \mu R^{2^b}) \subset \mu R^{2^b}$ ,  $b=1, 2, 3$ , corresponding to the complex numbers, quaternions, and Cayley numbers respectively.

REMARK:  $\varrho(n) = 8a + 2^b$  is called the Radon-Hurwitz function.

#### § 4. Definition and Properties of Canonical Vector Fields

Let  $R^\infty = \lim_{\rightarrow} R^m$ . It is clear, using the definition of  $\bar{M}$ , that the following commutes:

$$\begin{array}{ccc} R^r \otimes R^n & \xrightarrow{M} & R^m \\ \downarrow \mu \otimes \mu & & \downarrow \mu \\ R^{r+8} \otimes R^{16n} & \xrightarrow{\bar{M}} & R^{16n} \end{array}$$

Starting with  $R^0 \otimes R^1 \xrightarrow{0} R^1$ , we now iterate Theorem 3.1 and pass to the limit, obtaining a norm preserving multiplication orthogonal to the identity

$$\mathbf{M}: R^\infty \otimes R^\infty \rightarrow R^\infty.$$

Let  $\mu_i: R^m \hookrightarrow R^\infty$  be the inclusion of  $R^m$  into the  $i$ 'th block of  $m$  co-ordinates,  $0 \leq i$ . Thus, for  $i \leq n$ , one has a commutative diagram

$$\begin{array}{ccc} R^m & \xrightarrow{\mu_i} & R^\infty \\ \downarrow (\cdot) \otimes e_i & & \uparrow \mu \\ R^m \otimes R^n & \xrightarrow{\approx} & R^{mn}. \end{array}$$

Also,  $\mu_0 = \mu$ .

The following theorem says that  $\mathbf{M}$  in effect gives a maximal family of orthonormal vector fields on  $S^{n-1}$  for every  $n$ .

4.1. THEOREM: If  $r \leq \varrho(n) - 1$  then, for any  $i \geq 0$ ,

$$\mathbf{M}(\mu R^r \otimes \mu_i R^n) \subset \mu_i R^n.$$

*Proof:* This property will certainly hold for  $n$  if it is true for some divisor of  $n$ , the same  $r$ , and all  $i$ . Letting  $n = s \cdot 2^{4a+b}$ ,  $s$  odd,  $0 \leq b \leq 3$ , it will thus suffice to prove the theorem for  $2^{4a+b}$ , since also  $\varrho(2^{4a+b}) = \varrho(n)$ . In other words, we can assume without loss of generality that  $n = 2^{4a+b}$ . Furthermore, if the result holds for  $r = \varrho(n) - 1$  it will certainly hold for smaller  $r$ , so we also take  $r = \varrho(n) - 1 = 8a + 2^b - 1$ .

First consider  $b=0$  and let  $e_0, e_1, \dots$  be the usual basis for  $R^\infty$ . We shall prove that if the result holds for  $0 \leq i \leq 16^m - 1$  then it also holds for  $0 \leq i \leq 16^{m+1} - 1$ , giving an inductive proof of the theorem for the case  $b=0$  (clearly  $m=0$  furnishes a base for the induction). Write  $i = t \cdot 16^m + s$ , where  $0 \leq s \leq 16^{m+a+1}$  and  $0 \leq t \leq 15$ . The inclusion  $R^n = R^{16^a} \xrightarrow{\mu_i} R^{16^{m+a+1}}$  corresponds to the composition

$$R^{16^a} \xrightarrow{\mu_s} R^{16^{m+a}} \xrightarrow{(\cdot) \otimes e^t} R^{16^{m+a}} \otimes R^{16} \xrightarrow{\approx} R^{16^{m+a+1}}.$$

Then in the passage from  $M$  to  $\bar{M}$ , i.e., from  $R^{8(m+a)-1} \otimes R^{16^{m+a}}$  to  $R^{8(m+a+1)-1} \otimes R^{16^{m+a+1}}$ , we have a commutative diagram

$$\begin{array}{ccc} R^{8(m+a+1)-1} \otimes R^{16^{m+a+1}} & \xrightarrow{\approx} & R^{8(m+a)-1} \oplus R^8 \otimes R^{16^{m+a}} \otimes R^{16} \\ \uparrow \mu \otimes \mu_i & \nearrow (\mu, 0) \otimes \mu_s \otimes e_t & \downarrow \approx \\ R^{8a-1} \otimes R^{16^a} & \xrightarrow{(\mu \otimes \mu_s) \otimes e_t, 0} & (R^{8(m+a)-1} \otimes R^{16^{m+a}}) \otimes R^{16} \oplus (R^{16^{m+a}} \otimes (R^8 \otimes R^{16})) \end{array}$$

Performing the multiplications and applying the inductive hypothesis, we find

$$\mathbf{M}(\mu R^{8a-1} \otimes \mu_i R^{16^a}) \subset \mathbf{M}(\mu R^{8a} \otimes \mu_s R^{16^a}) \otimes e_t \subset \mu_s R^{16^a} \otimes e_t,$$

and the latter corresponds to  $\mu_i R^{16^a}$  under the isomorphism

$$R^{16^{(m+a)}} \otimes R^{16} \approx R^{16^{m+a+1}}.$$

This completes the proof for  $b=0$ . A similar method works for  $b=1, 2, 3$ . For example, if  $b=2$ , we use the existence of quaternions to establish the cases  $0 \leq i \leq 3$  (similar to the proof of Cor. 3.2) as base for the induction, then pass from  $0 \leq i \leq 4 \cdot 16^m - 1$  to  $0 \leq i \leq 4 \cdot 16^{m+1} - 1$ .

**4.2. COROLLARY:** *If  $r \leq \varrho(n) - 1$  then the following composition defines a norm preserving multiplication orthogonal to id:*

$$R^r \otimes R^n \xrightarrow{\mu \otimes \mu_i} R^\infty \otimes R^\infty \xrightarrow{\mathbf{M}} R^\infty \xrightarrow{\mu_i^{-1}} R^n.$$

Denoting this multiplication “ $\mathbf{M}_{i,r}^n$ ”, let us call the resultant  $r$  orthonormal vector fields on  $S^{n-1}$  “ $f_{i,r}^n$ ”. More precisely,  $f_{i,r}^n: S^{n-1} \rightarrow V_{n,r+1}$  is the cross section of the fibration  $V_{n,r+1} \rightarrow S^{n-1}$  such that

$$f_{i,r}^n(x) = \begin{pmatrix} x \\ x_0 \\ \vdots \\ x_{r-1} \end{pmatrix} \in V_{n,r+1}, \quad \text{where } x_j = \mathbf{M}_{i,r}^n(e_j \otimes x).$$

These are our “canonical” vector fields.

4.3. DEFINITION: Let  $M: R^r \otimes R^m \rightarrow R^m$  and  $N: R^r \otimes R^n \rightarrow R^n$  be norm preserving forms orthogonal to the identity. Then their intrinsic join  $M*N$  is the composition

$$R^r \otimes (R^m \oplus R^n) \xrightarrow{\sim} R^r \otimes R^m \oplus R^r \otimes R^n \xrightarrow{M \oplus N} R^m \oplus R^n \xrightarrow{\sim} R^{m+n}.$$

Clearly  $M*N$  is also norm preserving and orthogonal to id. If  $f: S^{m-1} \rightarrow V_{m,r+1}$  and  $g: S^{n-1} \rightarrow V_{n,r+1}$  are the corresponding cross sections, then their intrinsic join  $f*g$  is defined as the composition

$$S^{m+n-1} \xleftarrow[\varphi]{\cong} S^{m-1} * S^{n-1} \xrightarrow{f*g} V_{m,r+1} * V_{n,r+1} \xrightarrow{\varphi} V_{m+n,r+1},$$

$\varphi$  being the intrinsic join map of JAMES [4]. One easily sees that  $f*g$  corresponds to  $M*N$ , and it is then clear that the canonical vector fields can be joined together in many ways to give other canonical fields. A typical example is the formula

$$f_{0,3}^4 * f_{1,3}^4 = f_{0,3}^8.$$

More generally, one can easily establish the following.

$$4.4. \text{ PROPOSITION: } f_{i,m,r}^n * f_{i,m+1,r}^n * \dots * f_{i,m+m-1,r}^n = f_{i,r}^{m \cdot n}$$

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Received November 29, 1967