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## An Approximate Reciprocity Formula for Some Exponential Sums

by K. CHANDRASEKHARAN and RAGHAVAN NARASIMHAN

§ 1. The Dedekind zeta-function  $\zeta_K(s, \mathfrak{C})$  of an ideal class  $\mathfrak{C}$  in a quadratic field  $K = \mathcal{Q}(\sqrt{d})$  can be represented by the Dirichlet series  $\sum a_m m^{-s}$  in the half-plane  $\text{Re } s > 1$ , where  $a_m$  is the number of non-zero integral ideals of class  $\mathfrak{C}$  with norm  $m$ . In a recent paper [4] we showed that

$$\sum_{m \leq T} a_m \exp(2\pi i m x) = o(T), \quad \text{as } T \rightarrow \infty, \quad (1.1)$$

for any irrational  $x$ , and used this fact to prove that  $\zeta_K(\frac{1}{2} + it, \mathfrak{C})$  vanishes for an infinity of real values of  $t$ . This result goes through also for zeta-functions with grössencharacters associated with quadratic fields.

We shall here prove the following

**THEOREM 1.** *If  $\lambda = |d|^{1/2}$ , and  $0 < x \leq \lambda$ , we have*

$$\sum_{m \leq X} a_m \exp(2\pi i m x / \lambda) = \frac{c_1}{x} \sum_{m \leq X x^2} a_m \exp(-2\pi i m / \lambda x) + O(X^{1/2} \log X), \quad (1.2)$$

*provided that  $X x^2 \geq 1/\lambda > 0$ . Here  $c_1 = 1$ , or  $i$ , according as the field  $K$  is real or imaginary.*

*If  $0 < X x^2 < 1/\lambda$ , then*

$$\sum_{m \leq X} a_m \exp(2\pi i m x / \lambda) = \frac{c_2}{2\pi i x} (\exp(2\pi i x X / \lambda) - 1) + O(X^{1/3}), \quad (1.3)$$

*where  $c_2$  is a constant which depends on the field. (If  $\kappa$  denotes the residue of  $\zeta_K(s, \mathfrak{C})$  at  $s=1$ , then  $c_2 = \kappa\lambda$ , or  $\kappa\lambda/2\pi$ , according as  $K$  is real or imaginary). The error-terms in (1.2) and (1.3) are uniform with respect to  $x$ .*

From this we deduce the following

**COROLLARY.** *If  $x$  is rational, and  $x = 1/(kd)$ , where  $k$  is an integer, then*

$$\sum_{m \leq X} a_m \exp(2\pi i m x) = c_k X + O(X^{1/2} \log X), \quad \text{as } X \rightarrow \infty, \quad (1.4)$$

*where  $c_k$  is a constant which depends on  $k$  and the field.*

When  $a_m = d(m)$ , the number of divisors of  $m$ , formulas of the type (1.2) and (1.3) were first given by J. R. WILTON [6], who used them to study the order of magnitude of sums of the type

$$\sum_{m \leq X} d(m) \cos(2\pi m x),$$

for  $x$  belonging to various classes of numbers. Such a study was originated by HARDY

and LITTLEWOOD, and later carried on by a number of authors. References to the literature can be found in WILTON's paper [6].

The proof of Theorem 1 requires, among other things, properties of convergence of a class of infinite series of Bessel functions with coefficients  $a_m$ . The study of such series when  $a_m = d(m)$ , or  $r(m)$  (the number of representations of  $m$  as a sum of two squares), originated with VORONOI, HARDY, and LANDAU [5]. In two of our earlier papers [1, 2] we gave a general method for attacking the convergence problem for a wide class of such series. While the proof of Theorem 1 can be effected *without* using the sharpest known results on such series, it seems possible, by using them, to obtain a general summation formula for  $a_m$ . This is given in Theorem 2, § 5, and includes the VORONOI-HARDY-LANDAU formula when  $K = Q(\sqrt{-4})$ . Our proof goes through also in the case  $d = 1$  of equation (2.1), when  $\zeta_K$  is the square of Riemann's zeta-function, so that Voronoi's summation formula for the divisor function is also included.

We are indebted to Professor C. L. SIEGEL for critically reading a first version of this paper and making several helpful comments. He pointed out to us that in the case of an imaginary quadratic field, the constant  $c_1 = i$ , and that an additional term  $f(0)/w$  appears in (5.6). He has given an alternative method for sharpening the Corollary to Theorem 1, which we quote, with his permission, in § 6.

§ 2. We shall first prove Theorem 1 for a real quadratic field  $K = Q(\sqrt{d})$ ,  $d > 0$ . In that case, we have

$$\zeta_K(s, \mathfrak{C}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} = \zeta_K(s, \tilde{\mathfrak{C}}), \quad \operatorname{Re} s > 1,$$

where  $\tilde{\mathfrak{C}}$  is the ideal class conjugate to  $\mathfrak{C}$ , and the following functional equation holds:

$$\left(\frac{\sqrt{d}}{\pi}\right)^s \Gamma^2\left(\frac{s}{2}\right) \zeta_K(s, \mathfrak{C}) = \left(\frac{\sqrt{d}}{\pi}\right)^{1-s} \Gamma^2\left(\frac{1-s}{2}\right) \zeta_K(1-s, \mathfrak{C}).$$

This is of the form

$$\Gamma^2\left(\frac{s}{2}\right) \phi(s) = \Gamma^2\left(\frac{1-s}{2}\right) \phi(1-s), \quad (2.1)$$

where

$$\phi(s) = \sum_{m=1}^{\infty} \frac{a_m}{\mu_m^s}, \quad \mu_m = \frac{m\pi}{\lambda}, \quad \lambda = \sqrt{d}. \quad (2.2)$$

Let

$$A_{\mu}^q(x) = \frac{1}{\Gamma(q+1)} \sum'_{\mu_n \leq x} a_n (x - \mu_n)^q,$$

for  $x > 0$ ,  $\varrho \geq 0$ , the dash denoting that the last term has to be multiplied by  $\frac{1}{2}$ , if  $\varrho = 0$  and  $x = \mu_n$ . By the standard convention that an empty sum is zero,  $A_\mu^\varrho(x) = 0$  for  $0 \leq x < \mu_1$ ,  $\varrho \geq 0$ .

According to a previous result of ours [2, p. 116], equation (2.1) implies the existence of an identity, the precise form of which is given by the following

LEMMA 1. *Equation (2.1) implies the identity*

$$A_\mu^\varrho(x) - P_\varrho(x) = -2^{-\varrho} \sum_{m=1}^{\infty} a_m \left( \frac{x}{\mu_m} \right)^{(1+\varrho)/2} F_{1+\varrho} \{4(x \mu_m)^{1/2}\}, \quad (2.3)$$

where  $x > 0$ ,  $F_\nu(x) = Y_\nu(x) + (-1)^{\nu-1} (2/\pi) K_\nu(x)$ ,  $Y_\nu$  and  $K_\nu$  being the well-known Bessel functions,  $\varrho$  is an integer, such that  $\varrho \geq 0$ ,  $\varrho \geq 2\beta - \frac{5}{2}$ , where  $\beta = 1 + \varepsilon > 1$ , and

$$P_r(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{\Gamma(s) \varphi(s) x^{s+r} ds}{\Gamma(s+r+1)},$$

where  $\mathcal{C}_r$  is a curve which encloses all the singularities of the integrand. The series in (2.3) converges uniformly in any interval of the positive real axis in which the function on the left is continuous. If  $\varrho = 0$ , it converges boundedly in any interval  $0 < x_1 \leq x \leq x_2 < \infty$ .

This lemma is a consequence of Theorem 7.1(c) of [2], since the condition given there, namely

$$\sup_{0 \leq h \leq 1} \left| \sum_{n^2 < \mu_m < (n+h)^2} a_m \mu_m^{1/2-\beta} \right| = o(1),$$

as  $n \rightarrow \infty$ , is fulfilled, because of the known estimate [2, p. 128]

$$\sum_{m \leq x} a_m = \kappa x + O(x^{1/3}), \quad (2.4)$$

where  $\kappa$  is the residue of  $\zeta_K(s, \mathbb{C})$  at its only pole  $s = 1$ , and of the fact that  $\beta = 1 + \varepsilon > 1$ . The method of proof is that of equiconvergent trigonometric integrals used by us in [1]. The assumption that  $\varrho$  is an integer is not necessary; but our applications do not require more.

In the case of a real quadratic field, it is known that

$$P_0(x) = \frac{2 \log \eta}{\pi} x,$$

$\eta > 1$  being the fundamental unit in  $K$ , while

$$P_1(x) = -\zeta_K(-1, \mathbb{C}) \frac{\pi}{\sqrt{d}} + \frac{\log \eta}{\pi} x^2.$$

Thus, if we take  $\varrho = 0$ , and write  $\pi y$  for  $x$ , we get

$$A(y) = \sum'_{\lambda_m \leq y} a_m = P_0(\pi y) - \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^{1/2}} y^{1/2} F_1[4\pi(\lambda_m y)^{1/2}], \quad (2.5)$$



where now

$$\lambda_m = \frac{m}{\lambda}, \lambda = \sqrt{d}, P_0(\pi y) = 2 \log \eta \cdot y = c \cdot y, \quad \text{say.} \quad (2.6)$$

we define for  $y > 0, \varrho \geq 0$ ,

$$A^\varrho(y) = A_\lambda^\varrho(y) = \frac{1}{\Gamma(\varrho + 1)} \sum'_{\lambda_m \leq y} a_m (y - \lambda_m)^\varrho.$$

If we set

$$I_\nu(y) = -y^{(\nu+1)/2} F_{\nu+1}(4\pi y^{1/2}), \quad (2.7)$$

then we can rewrite (2.5) as

$$A(y) = c y + \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m} I_0(\lambda_m y), \quad \lambda_m = \frac{m}{\lambda}. \quad (2.8)$$

Because of Lemma 1, with  $\varrho=0$ , the infinite series on the right-hand side of (2.8) converges boundedly for  $0 < \alpha \leq y \leq \alpha' < \infty$ . But this result is *not* necessary for the proof of Theorem 1.

The case  $\varrho=1$  of Lemma 1 gives

$$A^1(y) = \frac{1}{2} c y^2 + \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} I_1(\lambda_m y), \quad (2.9)$$

since

$$\begin{aligned} \frac{d}{dx} [x^{\nu/2} F_\nu \{4\pi(\lambda_m x)^{1/2}\}] &= 2\pi \lambda_m^{1/2} x^{(\nu-1)/2} F_{\nu-1} [4\pi(\lambda_m x)^{1/2}] \\ &= -2\pi \lambda_m^{1/2} I_{\nu-2}(\lambda_m x), \end{aligned}$$

or

$$\frac{d}{dx} [I_{\nu-1}(\lambda_m x)] = 2\pi \lambda_m \cdot I_{\nu-2}(\lambda_m x), \quad (2.10)$$

and the series in (2.9) is uniformly convergent for  $y \geq 0$ . This result is much easier to establish than the convergence of the series in (2.8).

If  $f$  is a function which is twice continuously differentiable in  $[0, \infty)$ , then by Abel's Lemma on partial summation, we have

$$\sum_{\lambda_m \leq X} a_m f(\lambda_m) = A(X) f(X) - \int_{\lambda_1}^X A(t) f'(t) dt,$$

where  $f'$  is the derivative of  $f$ . If we integrate by parts, once more, we have

$$\sum_{\lambda_m \leq X} a_m f(\lambda_m) = A(X) f(X) - A^1(X) f'(X) + \int_0^X A^1(t) f''(t) dt.$$

If we choose  $f(t) = \exp(2\pi i x t)$ ,  $x > 0$ , and use (2.9) and (2.4) we get

$$\begin{aligned} \sum_{\lambda_m \leq X} a_m f(\lambda_m) &= c X f(X) - \frac{1}{2} c X^2 f'(X) + \int_0^X \frac{1}{2} c t^2 f''(t) dt + O(X^{1/3}) \\ &\quad - \frac{f'(X)}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} I_1(\lambda_m X) \\ &\quad + \int_0^X \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} I_1(\lambda_m t) f''(t) dt, \end{aligned}$$

the series occurring in the last integral being uniformly convergent. The error-term  $O(X^{1/3})$  comes from formula (2.4). Hence

$$\begin{aligned} \sum_{\lambda_m \leq X} a_m f(\lambda_m) &= c \int_0^X f(t) dt + O(X^{1/3}) \\ &\quad + \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} \left[ \int_0^X I_1(\lambda_m t) f''(t) dt - I_1(\lambda_m X) f'(X) \right], \end{aligned}$$

and the last term equals

$$\frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} \left\{ [I_1(\lambda_m t) f'(t)]_0^X - \int_0^X (2\pi \lambda_m) I_0(\lambda_m t) f'(t) dt - I_1(\lambda_m X) f'(X) \right\}.$$

The first term in the curly brackets vanishes at  $t=0$  because of (2.9). Taking  $f(t) = \exp(2\pi i x t)$  we get the formula

$$\sum_{\lambda_m \leq X} a_m \exp(2\pi i \lambda_m x) = c \int_0^X \exp(2\pi i x t) dt + O(X^{1/3}) + \sum_{m=1}^{\infty} a_m r_m, \quad (2.11)$$

where

$$r_m = - \frac{2\pi i x}{\lambda_m} \int_0^X I_0(\lambda_m t) \exp(2\pi i x t) dt. \quad (2.12)$$

We shall prove Theorem 1 by estimating the sum  $\sum_{m=1}^{\infty} a_m r_m$  in (2.11) by means of the following lemma due to J. R. WILTON [6].

LEMMA 2 (WILTON). *Let*

$$J_m = \int_0^x I(\lambda_m t) \exp(2\pi i x t) dt, \quad I(t) = \frac{d}{dt} [I_0(t)], \quad (2.13)$$

and

$$N = X x^2.$$

Let

$$R_m = J_m - \frac{c_1}{x} \exp - (2\pi i \lambda_m/x), \quad \text{if } \lambda_1 \leq \lambda_m \leq N, \quad (2.14)$$

where  $c_1$  is defined as in (1.2), and

$$R_m = J_m - \frac{1}{\lambda_m} I_0(\lambda_m X) \exp(2\pi i x X), \quad \text{if } \lambda_m > N \geq c' > 0. \quad (2.14')$$

Then we have the estimates:

$$R_m = O(x^{-1/2} \lambda_m^{-1/4} N^{-1/4}) + O\left(\frac{x}{\lambda_m^2}\right), \quad \text{if } \lambda_m < (1 - \varepsilon) N, \quad 0 < \varepsilon < 1; \quad (2.15)$$

$$R_m = O(x^{-1/2} N^{1/2} |N - \lambda_m|^{-1}), \quad \text{if } c_2 N < \lambda_m < c_3 N, \quad c_2 > 0, \quad \lambda_m \neq N; \quad (2.16)$$

$$R_m = O\left(\frac{1}{x}\right), \quad \text{if } N - c_4 \sqrt{N x} < \lambda_m < N + c_5 \sqrt{N x}, \quad c_4, c_5 > 0; \quad (2.17)$$

$$R_m = O(x^{-1/2} N^{3/4} \lambda_m^{-5/4}), \quad \text{if } \lambda_m > (1 + \varepsilon) N > N. \quad (2.18)$$

The  $O$ 's are uniform in  $x$ , for  $0 < x \leq A$ , for any constant  $A$ ,  $\varepsilon$  being fixed.

An integration by parts applied to (2.12) yields the relation

$$r_m = J_m - \frac{1}{\lambda_m} I_0(\lambda_m X) \exp(2\pi i x X), \quad (2.19)$$

where  $J_m$  is defined as in (2.13). Therefore, by (2.14'),

$$r_m = R_m, \quad \text{if } \lambda_m > N \geq c' > 0, \quad (2.20)$$

and, by (2.19) and (2.14), we have

$$r_m = \frac{c_1}{x} \exp - (2\pi i \lambda_m/x) + R_m - \frac{1}{\lambda_m} I_0(\lambda_m X) \exp(2\pi i x X), \quad \text{if } \lambda_1 \leq \lambda_m \leq N. \quad (2.21)$$

Combining (2.20) and (2.21) we get

$$\sum_{m=1}^{\infty} a_m r_m = \frac{c_1}{x} \sum_{\lambda_m \leq N} a_m \exp - (2\pi i \lambda_m/x) - \sum_{\lambda_m \leq N} \frac{a_m}{\lambda_m} I_0(\lambda_m X) \exp(2\pi i x X) + \sum_{m=1}^{\infty} a_m R_m. \quad (2.22)$$

It is known that  $I_0(t) = O(t^{1/4})$ , as  $t \rightarrow \infty$  [2, p. 96], so that

$$\sum_{\lambda_m \leq N} \frac{a_m}{\lambda_m} I_0(\lambda_m X) \exp(2\pi i x X) = O\left(X^{1/4} \sum_{\lambda_m \leq N} \frac{a_m}{\lambda_m^{3/4}}\right) = O(X^{1/2}), \quad (2.23)$$

because of (2.4).

In order to estimate the sum  $\sum_{m=1}^{\infty} a_m R_m$ , we split it up into three parts. If  $0 < \varepsilon < 1$ , we write

$$\begin{aligned} \sum_{m=1}^{\infty} a_m R_m &= \sum_{\lambda_m \leq (1-\varepsilon)N} + \sum_{(1-\varepsilon)N < \lambda_m < (1+\varepsilon)N} + \sum_{\lambda_m \geq (1+\varepsilon)N} \\ &= \sum_1 + \sum_2 + \sum_3, \quad \text{say.} \end{aligned} \quad (2.24)$$

If we use (2.15) in  $\sum_1$ , we get

$$\begin{aligned} \left| \sum_1 \right| &\leq c \cdot \sum_{\lambda_m \leq (1-\varepsilon)Xx^2} \frac{a_m}{\lambda_m^{1/4}} X^{-1/4} x^{-1} \\ &\leq c \cdot (Xx^2)^{3/4} X^{-1/4} x^{-1} \leq c \cdot X^{1/2}. \end{aligned} \quad (2.25)$$

(N.B. Here, as elsewhere,  $c$  is a constant with possibly different values at different occurrences).

If we use (2.18) in  $\sum_3$ , we get

$$\begin{aligned} \left| \sum_3 \right| &\leq c \cdot \sum_{\lambda_m > Xx^2} a_m \lambda_m^{-5/4} (Xx^2)^{3/4} x^{-1/2} \\ &\leq c \cdot (Xx^2)^{-1/4+3/4} x^{-1/2} \leq c \cdot X^{1/2}. \end{aligned} \quad (2.26)$$

Finally we consider  $\sum_2$ . Here we have, by (2.17),

$$\begin{aligned} \left| \sum_{N-\sqrt{N}x \leq \lambda_m \leq N+\sqrt{N}x} a_m R_m \right| &\leq c \cdot \frac{1}{x} \sum_{N-\sqrt{N}x \leq \lambda_m \leq N+\sqrt{N}x} a_m = \frac{c}{x} [c\sqrt{Nx} + O(N^{1/3})] \\ &= O(X^{1/2}) + O(X^{1/3} x^{-1/3}) = O(X^{1/2}), \end{aligned} \quad (2.27)$$

if  $Nx \geq c > 0$ , and  $0 < x \leq \lambda$ . If  $Nx$  is so small that  $\lambda\sqrt{Nx} < \frac{1}{2}$ , then the sum on the right-hand side reduces to a single term  $a_m$ , and the estimate is

$$O\left(\frac{N^\varepsilon}{x}\right) = O(X^{1/2}),$$

since  $N \geq 1/\lambda > 0$ . On the other hand, we have, because of (2.16),

$$\left| \sum_{N+\sqrt{N}x \leq \lambda_m \leq (1+\varepsilon)N} a_m R_m \right| \leq c \sum_{N+\sqrt{N}x \leq \lambda_m \leq (1+\varepsilon)N} \frac{a_m}{\lambda_m - N} x^{-1/2} N^{1/2}.$$

Now

$$c \sum_{N+\sqrt{N}x \leq \lambda_m \leq (1+\varepsilon)N} \frac{a_m}{\lambda_m - N} = O\left(\int_{N+\sqrt{N}x}^{(1+\varepsilon)N} \frac{dA(t)}{t - N}\right),$$

and, by (2.4),  $A(t) = \kappa t + E(t)$ , where  $E(t) = O(t^{1/3})$ . Thus

$$\left| \sum_{N+\sqrt{N}x \leq \lambda_m \leq (1+\varepsilon)N} a_m R_m \right| = O(X^{1/2} \log X), \quad (2.28)$$

and similarly

$$\left| \sum_{(1-\varepsilon)N \leq \lambda_m \leq N-\sqrt{N}x} a_m R_m \right| = O(X^{1/2} \log X). \quad (2.29)$$

Hence (2.27), (2.28), and (2.29) together give

$$|\sum_2| = O(X^{1/2} \log X). \quad (2.30)$$

On combining (2.24), (2.25), (2.26), and (2.30), we get

$$\sum_{m=1}^{\infty} a_m R_m = O(X^{1/2} \log X). \quad (2.31)$$

If we use this, together with (2.23), in (2.22), then we get

$$\sum_{m=1}^{\infty} a_m r_m = \frac{c_1}{x} \sum_{\lambda_m \leq N} a_m \exp(-2\pi i \lambda_m / x) + O(X^{1/2} \log X). \quad (2.32)$$

Further

$$\left| \int_0^x \exp(2\pi i x t) dt \right| \leq \frac{c}{x} = O(X^{1/2}), \quad (2.33)$$

since, by hypothesis,  $N = Xx^2 \geq 1/\lambda > 0$ .

Formula (2.11), combined with (2.32) and (2.33), gives (1.2) in the case of a real quadratic field, provided that  $N \geq 1/\lambda$ .

If  $N < 1/\lambda$ , then (2.20) gives

$$\sum_{m=1}^{\infty} a_m r_m = \sum_{m=1}^{\infty} a_m R_m,$$

in which case we can use (2.18), so that

$$\left| \sum_{m=1}^{\infty} a_m R_m \right| \leq \frac{c(Xx^2)^{3/4}}{x^{1/2}} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^{5/4}} \leq c X^{3/4} x \leq c X^{1/4},$$

since  $Xx^2 = N < c$ . Hence, by formula (2.11),

$$\sum_{\lambda_m \leq X} a_m \exp(2\pi i \lambda_m x) = \frac{c}{2\pi i x} (\exp(2\pi i x X) - 1) + O(X^{1/3}),$$

uniformly for  $0 < x \leq \lambda$ , which proves (1.3) in the case of a real quadratic field, and thereby completes the proof of Theorem 1 in that case. We note that the term  $O(X^{1/3})$  comes from (2.4).

§ 3. The proof of Theorem 1 for an imaginary quadratic field  $K = \mathcal{Q}(\sqrt{-d})$ ,  $d > 0$ , is similar. Here we have a functional equation with a simple gamma factor, namely

$$\left(\frac{\sqrt{d}}{2\pi}\right)^s \Gamma(s) \zeta_K(s, \mathfrak{C}) = \left(\frac{\sqrt{d}}{2\pi}\right)^{1-s} \Gamma(1-s) \zeta_K(1-s, \mathfrak{C}),$$

which takes the form

$$\Gamma(s) \varphi(s) = \Gamma(1-s) \varphi(1-s), \quad (3.1)$$

where

$$\varphi(s) = \sum_{m=1}^{\infty} \frac{a_m}{\mu_m^s}, \mu_m = \frac{2m\pi}{\lambda}, \lambda = \sqrt{d}.$$

We define  $A_\mu^e(x)$  as before. As in Lemma 1, equation (3.1) again implies the existence of an identity with the Bessel function  $J_\nu(x)$  – not to be confused with the integral  $J_m$  in Lemma 2 – in place of the special function  $F_\nu(x)$ :

$$A_\lambda^e(x) - Q_e(x) = (2\pi)^{-e} \sum_{m=1}^{\infty} a_m \left(\frac{x}{\lambda_m}\right)^{(1+e)/2} J_{1+e}\{4\pi(x\lambda_m)^{1/2}\}, \quad (3.2)$$

where  $\lambda_m = m/\lambda$ ,  $e \geq 2\beta - \frac{5}{2}$ ,  $\beta = 1 + \varepsilon > 1$ . The series converges absolutely for  $e > \frac{1}{2}$ . Further, it converges boundedly in any interval of the real axis when  $e = 0$ , and uniformly whenever the function on the left-hand side is continuous [1, Th. III]. The asymptotic behaviour of  $J_\nu(x)$  is similar to that of  $Y_\nu(x)$ :  $J_\nu(x) = O(x^{-1/2})$ , as  $x \rightarrow \infty$ . The analogue of (2.4) is true; there is an additive constant which is absorbed by the term  $O(x^{1/3})$ . The proof of Lemma 2 goes through with these changes; in fact, it becomes simpler than WILTON's original proof. In (2.14) we now have  $c_1 = i$ , instead of  $c_1 = 1$ . Estimate (2.15) holds without the error-term  $O(x/\lambda_m^2)$ . The rest of the proof of (1.2) proceeds on the same lines as before, and we omit details.

If we take  $x = 1/k\lambda$ , where  $k$  is an integer, in (1.2), we get

$$\begin{aligned} \sum_{m \leq X} a_m \exp(2\pi i m/k \lambda^2) &= \frac{c_1}{x} \sum_{m \leq Xx^2} a_m \exp(-2\pi i m k) + O(X^{1/2} \log X) \\ &= \frac{c_1}{x} \sum_{m \leq Xx^2} a_m + O(X^{1/2} \log X) \\ &= c_k X + O(X^{1/2} \log X), \quad c_k = \frac{c_1}{k\lambda}, \end{aligned}$$

which is the Corollary to Theorem 1.

§ 4. The estimate in (1.1) for irrational  $x$  was based on purely arithmetical considerations [4], though the proof that  $\zeta_K(\frac{1}{2} + it)$  vanishes for an infinity of values of  $t$  made use of the functional equation satisfied by  $\zeta_K(s)$ . The proof of the reciprocity formula, and of the estimate in (1.4) for *rational*  $x$ , was based on the functional equation for  $\zeta_K(s)$ , and the consequent existence of an arithmetical identity like (2.8). This fact enables us to extend the formula to zeta-functions with Grössencharacters  $\zeta(s, \chi)$ . The functional equation in that case, the analogue of identity (2.8), namely

$$A_\mu^\varrho(x) - P_\varrho(x) = \sum_{m=1}^{\infty} \frac{b_m}{\mu_m^{1+\varrho}} I_\varrho(\mu_m x),$$

and of the estimate (2.4), were considered by us in [2, p. 128]. The properties of  $I_\varrho(x)$  are known [3, p. 33], and enable us to uphold the validity of Wilton's lemma. Thus Theorem 1 is valid for zeta-functions with Grössencharacters associated with quadratic fields.

§ 5. We shall now make use of the relatively difficult case  $\varrho=0$  of Lemma 1 to obtain an exact summation formula instead of the asymptotic formula in (2.11).

Let  $a_n$ ,  $\lambda_n$ , and  $A_\lambda^\varrho(x)$  be defined as hitherto, and  $f$  a function which is twice continuously differentiable in  $[0, \infty)$ . By Abel's lemma on partial summation, we have, if  $0 < \alpha < \lambda_1$ ,

$$\sum_{\lambda_m \leq x} a_m f(\lambda_m) = A(x) f(x) - \int_{\alpha}^x A(t) f'(t) dt. \quad (5.1)$$

If we first confine ourselves to the case of a real quadratic field, then because of the case  $\varrho=0$  of Lemma 1, we have

$$\int_{\alpha}^x A(t) f'(t) dt = \int_{\alpha}^x c t f'(t) dt + \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m} \int_{\alpha}^x I_0(\lambda_m t) f'(t) dt. \quad (5.2)$$

The first term on the right-hand side is continuous in  $0 \leq \alpha \leq 1$ ; so is the second, for by partial integration, we have

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m} \int_{\alpha}^x I_0(\lambda_m t) f'(t) dt &= \left[ \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} I_1(\lambda_m t) f'(t) \right]_{\alpha}^x \\ &\quad - \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_{\alpha}^x I_1(\lambda_m t) f''(t) dt. \end{aligned} \quad (5.3)$$

Lemma 1, with  $\varrho=1$ , shows that this is also continuous in  $0 \leq \alpha \leq 1$ . Hence from (5.1) we obtain

$$\begin{aligned} \sum_{\lambda_m \leq x} a_m f(\lambda_m) &= A(x)f(x) - \int_0^x A(t)f'(t) dt \\ &= cxf(x) - \int_0^x ctf'(t) dt + \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m} \left[ I_0(\lambda_m x)f(x) \right. \\ &\quad \left. - \int_0^x I_0(\lambda_m t)f'(t) dt \right] \end{aligned} \quad (5.4)$$

An integration by parts gives

$$\sum_{\lambda_m \leq x} a_m f(\lambda_m) = c \int_0^x f(t) dt + \sum_{m=1}^{\infty} a_m \int_0^x I(\lambda_m t)f(t) dt, \quad (5.5)$$

where

$$I(y) = \frac{d}{dy} [I_0(y)] = -2\pi \left( Y_0(4\pi\sqrt{y}) - \frac{2}{\pi} K_0(4\pi\sqrt{y}) \right),$$

and  $c = \kappa\sqrt{d}$ , where  $\kappa$  is the residue of  $\zeta_K(s, \mathfrak{C})$  at  $s=1$ .

In the case of an imaginary quadratic field  $K = \mathbb{Q}(\sqrt{-d})$ ,  $d > 0$ , the term  $Q_0(x)$  in (3.2) gives

$$Q_0(x) = \zeta_K(0, \mathfrak{C}) + \frac{x}{w} = -\frac{1}{w} + \frac{x}{w},$$

where  $w$  is the number of roots of unity in  $K$ , so that (5.5) becomes

$$\sum_{\lambda_m \leq x} a_m f(\lambda_m) + \frac{f(0)}{w} = c' \int_0^x f(t) dt + \sum_{m=1}^{\infty} a_m \int_0^x I(\lambda_m t)f(t) dt, \quad (5.6)$$

where  $c' = (\kappa\sqrt{d/2w})$ , where  $\kappa$  is the residue of  $\zeta_K(s, \mathfrak{C})$  at  $s=1$ .

If we make the convention that  $\lambda_0 = 0$ , and  $a_0 = 0$ , or  $1/w$ , according as the field  $K$  is real or imaginary, we obtain the following

**THEOREM 2.** *If  $f$  has a continuous second derivative in  $[0, \infty)$ , then for any  $X > 0$ , we have*

$$\sum_{\lambda_m \leq X}^* a_m f(\lambda_m) = c \int_0^X f(t) dt + \sum_{m=1}^{\infty} a_m \int_0^X I(\lambda_m t)f(t) dt, \quad (5.7)$$

where the star  $*$  indicates the convention regarding  $a_0$ , and  $c = \kappa\lambda$ , or  $\kappa\lambda/2\pi$ , according as  $K$  is real or imaginary,  $\kappa$  being the residue of  $\zeta_K(s, \mathfrak{C})$  at  $s=1$ , and  $\lambda = |d|^{1/2}$ .



We can take  $f(t) = \exp(2\pi i x t)$ ,  $x > 0$ , and obtain an exact formula in place of (2.11). The case  $d = +1$  goes back to VORONOI; the case  $d = -4$  is Satz 559 in [5]. The condition that  $f$  be twice continuously differentiable in  $[0, \infty)$  is *not* fulfilled if we take  $f(t) = \exp(2\pi i t^{1/2} x)$ . To accommodate this case, which is of interest in several applications, we shall formulate more general assumptions on  $f$  near the origin.

We remark, first of all, that if  $0 < a \leq b$ , and  $f$  has a continuous derivative in  $(0, \infty)$ , then

$$\sum_{a < \lambda_m \leq b} a_m f(\lambda_m) = c \int_a^b f(t) dt + \sum_{m=1}^{\infty} a_m \int_a^b I(\lambda_m t) f(t) dt. \quad (5.8)$$

In order to be able to replace  $a$  by 0, we now assume that  $f$  is twice continuously differentiable in  $(0, \infty)$ , and satisfies, near the origin, the condition

$$\int_0^1 t^p |f''(t)| dt < \infty, \quad (5.9)$$

for some  $p < 1$ . We then have

$$\begin{aligned} t^p |f'(t)| - c_3 &\leq t^p \int_t^1 |f''(u)| du \leq \int_t^1 u^p |f''(u)| du \\ &\leq \int_0^1 u^p |f''(u)| du < \infty, \end{aligned}$$

where  $c_3$  is a constant, so that  $tf'(t) \rightarrow 0$ , as  $t \rightarrow 0$ , and  $\int_0^1 t |f'(t)| dt < \infty$ . We again have (5.1) and (5.2). The first term on the right-hand side of (5.2) tends to  $\int_0^x c t f'(t) dt$ , as  $\alpha \rightarrow 0$ , since

$$\int_0^x |c t f'(t)| dt < \infty,$$

because of assumption (5.9), while the second term, by partial integration, leads to (5.3), namely

$$\left[ \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} I_1(\lambda_m t) f'(t) \right]_{\alpha}^x - \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_{\alpha}^x I_1(\lambda_m t) f''(t) dt. \quad (5.10)$$

Because of identity (2.9), we have

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} I_1(\lambda_m t) = O(t), \quad (5.11)$$

as  $t \rightarrow 0$ . Hence, if we let  $\alpha \rightarrow 0$ , the first term in (5.10) gives just

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} I_1(\lambda_m x) f'(x),$$

because of assumption (5.9). To consider the second term in (5.10), we take the series

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_0^1 |I_1(\lambda_m t) \cdot f''(t)| dt, \quad (5.12)$$

and define

$$e_m = \sup_{0 \leq t \leq 1} \frac{|I_1(\lambda_m t)|}{t^p},$$

where  $p < 1$ , given as in (5.9). If  $\lambda_m t < 1$ , then, since  $I_1(t) = O(t^{3/2})$ , as  $t \rightarrow 0$ , we have

$$t^{-p} |I_1(\lambda_m t)| = O(\lambda_m^{3/2} t^{3/2-p}) = O(\lambda_m^p),$$

while, if  $\lambda_m t \geq 1$ , then, since  $I_1(t) = O(t^{3/4})$ , as  $t \rightarrow \infty$ , we have

$$t^{-p} |I_1(\lambda_m t)| = O(\lambda_m^{3/4} t^{3/4-p}) = O(\lambda_m^{3/4} + \lambda_m^p),$$

since  $0 \leq t \leq 1$ . Hence

$$e_m = O(\lambda_m^q), \quad (5.13)$$

where  $q = \max(\frac{3}{4}, p) < 1$ . Thus, by assumption (5.9), we have

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_0^1 |I_1(\lambda_m t) f''(t)| dt < c \sum_{m=1}^{\infty} \frac{a_m e_m}{\lambda_m^2} < \infty.$$

Hence the series

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_{\alpha}^x I_1(\lambda_m t) f''(t) dt$$

is uniformly convergent in  $0 < \alpha \leq 1$ , and we have

$$\lim_{\alpha \rightarrow 0} \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_{\alpha}^x I_1(\lambda_m t) f''(t) dt = \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_0^x I_1(\lambda_m t) f''(t) dt \quad (5.14)$$

Thus in (5.10), and so also in (5.2), we can pass to the limit as  $\alpha \rightarrow 0$ . The rest of the argument is the same as in the case where  $f$  has a continuous second derivative in  $[0, \infty)$ . Thus we obtain

THEOREM 3. If  $f$  has a continuous second derivative in  $(0, \infty)$ , and satisfies the condition

$$\int_0^1 t^p |f''(t)| dt < \infty,$$

for some  $p < 1$ , then formula (5.7) is valid.

§ 6. Professor SIEGEL's treatment of the Corollary to Theorem 1, referred to in § 1, runs as follows.

"I start from the Gaussian sum

$$G = \sum_{p, q \pmod{dk}} \exp(2\pi i(a p^2 + b p q + c q^2)/dk) = \begin{cases} k d \sqrt{d}, \\ i k d \sqrt{d}, \end{cases}$$

where  $b^2 - 4ac = +d > 0$  in the first case, and  $b^2 - 4ac = -d < 0$ ,  $a > 0$ , in the second case. In the second case the expression

$$S = \sum_{ap^2 + bpq + cq^2 \leq X} \exp 2\pi i(a p^2 + b p q + c q^2)/dk$$

has to be evaluated. The area of the ellipse  $ap^2 + bpq + cq^2 \leq X$  is  $2\pi X/\sqrt{d}$ , and therefore the number of lattice points with any given residue classes of  $p$  and  $q$  modulo  $dk$  is  $(dk)^{-2} 2\pi X/\sqrt{d} + O(X^{1/2})$ . Hence

$$S = (dk)^{-2} \frac{2\pi X}{\sqrt{d}} G + O(X^{1/2}),$$

$$S = \frac{2\pi i}{k d} X + O(X^{1/2}).$$

In the first case the ellipse is replaced by a sector of a hyperbola. It is clear how to improve the error-term  $O(X^{1/2})$ . The same idea goes through if the ratio  $1/dk$  is replaced by any rational number. The result is a little more general than in your text, since  $d$  need not be the discriminant of a quadratic number field; even a square number is allowed."

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