Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 43 (1968)

Artikel: On the Uniform Approximation of Analytic Functions by Means of Interpolation Polynomials.

Autor: Kövari, T.

DOI: https://doi.org/10.5169/seals-32918

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 05.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

On the Uniform Approximation of Analytic Functions by Means of Interpolation Polynomials

T. KÖVARI

1. Introduction

Let D be a domain of the complex plane bounded by a smooth closed Jordan curve Γ.

Let $\{z_k^{(n)}\}$, $1 \leqq k \leqq n$, $n=1, 2, 3, \ldots$ be a triangular matrix of points of Γ ($z_i^{(n)} \neq z_k^{(n)}$ for $i \neq k$), and consider the fundamental polynomials of the Lagrange interpolation:

$$l_j^{(n)}(z) = \prod_{k \neq j} \frac{z - z_k}{z_j - z_k} \tag{1.1}$$

which have the property that $l_j(z_j)=1$ and $l_j(z_k)=0$ for $k \neq j$. We will say that $\{z_k^{(n)}\}$ is a *regular point system* if the polynomials $l_j^{(n)}(z)$ are uniformly bounded in $\bar{D}$, i.e. if

$$|l_j^{(n)}(z)| \leqq M \quad \text{for} \quad z \in \bar{D}, \quad 1 \leqq j \leqq n, \quad n = 1, 2, \ldots \tag{1.2}$$

for some M. In particular a system of *Fekete points*[1]) is a regular point system; in fact in this case (1.2) holds with $M=1$.

If $f(z)$ is a function regular in D, and continuous in $\bar{D}$, it can be uniformly approximated in $\bar{D}$ by polynomials. In this paper I shall construct a sequence of polynomials which converges uniformly to $f(z)$ and, at the same time interpolates $f(z)$ at a regular point system.

In 1942 ERDÖS proved the following result about real interpolation [3]. Let $\{x_k^{(n)}\}$ be a regular point system in $[-1, +1]$. Then to every continuous function $f(x)$ and $\eta > 0$ there exists a sequence of polynomials $p_n(x)$ such that, 1) the degree of $p_n(x)$ is $\leqq n(1+\eta)$, 2) $p_n(x_i^{(n)}) = f(x_i^{(n)})$, $1 \leqq i \leqq n$, $n=1, 2, \ldots$, 3) $p_n(x) \to f(x)$ uniformly in $[-1, +1]$.

Adapting Erdös's proof, I shall prove the following

THEOREM: *Let D be a domain bounded by a closed Jordan curve Γ which satisfies Alper's smoothness condition* (2.1). *Let $\{z_k^{(n)}\} \in \Gamma$ be a regular point system for $\bar{D}$. Then, to every function $f(z)$ regular in D and continuous in $\bar{D}$ and every $\eta > 0$, there exists a sequence of polynomials $p_n(z)$ such that*

---

[1]) The Fekete points $\{w^{(n)}{}_k\}$ are defined by the property that for each n, they maximise the discriminant $\prod_{\substack{1 \leqq k, j \leqq n \\ k \neq j}} |w_k - w_j|$.

1) *the degree of* $p_n(z) \leqq n(1+\eta)$
2) $p_n(z_k^{(n)}) = f(z_k^{(n)})$, $1 \leqq k \leqq n$, $n = 1, 2, \ldots$
3) $p_n(z) \to f(z)$ *uniformly in* $\bar{D}$
4) $p_n = T_n(f)$ *is a linear operator, given explicitly.*

2. Preliminary results

Let $\vartheta(s)$ denote the angle between the tangent to Γ and the positive real axis (as a function of the arc length parameter s). Let $\omega(h)$ denote the modulus of continuity of the function $\vartheta(s)$. In the papers [1,2] S. Y. ALPER introduced the class of domains whose boundary Γ satisfies the condition

$$\int_0^h \frac{\omega(x)}{x} |\log x| \, dx < +\infty \tag{2.1}$$

Assuming that condition (2.1) is satisfied, we are able to estimate certain fundamental polynomials.

Let

$$z = \psi(\zeta) = \beta\zeta + a_0 + \frac{a_1}{\zeta} + \cdots$$

map $|\zeta| > 1$ conformally onto the exterior of Γ. We can assume without loss of generality that $\beta = 1$. By a classical result $\psi(\zeta)$ is continuous for $|\zeta| \geqq 1$. Thus, the regular point system $\{z_k^{(n)}\}$ can be written in the form: $z_k^{(n)} = \psi(e^{i\vartheta_k^{(n)}})$, where we can assume that $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_n$, $\vartheta_n - \vartheta_1 < 2\pi$. In another paper [4, Theorem 3] POMMERENKE and I proved the following result:

LEMMA 2.1. *If* Γ *satisfies the condition* (2.1) *and* $\psi(e^{i\vartheta_k^{(n)}})$ is a regular point system, then:

$$\vartheta_{k+\nu}^{(n)} - \vartheta_k^{(n)} > \frac{c\nu}{n}, \quad k = 1, 2, \ldots n, \quad \nu = 1, 2, \ldots n-1 \quad (\vartheta_{n+j}^{(n)} = \vartheta_j^{(n)} + 2\pi) \tag{2.2}$$

where the constant $c > 0$ does not depend on n, k or ν. To avoid an excess of indices, we shall now drop the index: (n).

Let $w_{k,j}^{(m)} = \psi(e^{i(\vartheta_k + 2\pi j/m)})$, $1 \leqq k \leqq n$, $0 \leqq j < m$, $w_{k,0}^{(m)} = z_k^{(n)}$ and let $\tilde{l}_k^{(m)}(z)$ denote the fundamental polynomials:

$$\tilde{l}_k^{(m)}(z) = \prod_{j=1}^{m-1} \frac{z - w_{k,j}^{(m)}}{z_k - w_{k,j}^{(m)}}.$$

In the above quoted paper [4, (4.3), (4.6)] it was shown, that if (2.1) is satisfied, one has the following estimates

LEMMA 2.2

(i) $$|\tilde{l}_k^{(m)}(\psi(e^{i\vartheta}))| < B \tag{2.3}$$

(ii) $$|\tilde{l}_k^{(m)}(\psi(e^{i\vartheta}))| < \frac{\pi B}{m|\vartheta - \vartheta_k|} \tag{2.4}$$

for $-\pi \leqq \vartheta - \vartheta_k \leqq +\pi$, *where the constant B depends only on the domain D.*

From lemmas 2.1 and 2.2 we deduce

LEMMA 2.3

$$\sum_{k=1}^{n} |\tilde{l}_k^{(m)}(z)|^2 \leqq C_1 + C_2 \frac{n^2}{m^2}, \quad \text{for} \quad z \in \bar{D} \tag{2.5}$$

where C_1 *and* C_2 *does not depend on n or m.*

Proof: Since the sum on the left hand side is subharmonic, it is sufficient to prove (2.5) for $z \in \Gamma$, i.e. $z = \psi(e^{i\vartheta})$. Without loss of generality we can also assume that $\vartheta_n - 2\pi < \vartheta \leqq \vartheta_1$. Then, applying Lemma 2.2. and Lemma 2.1,

$$\begin{aligned}
\sum_{k=1}^{n} |\tilde{l}_k^{(m)}(\psi(e^{i\vartheta}))|^2 &= |\tilde{l}_1^{(m)}(\psi(e^{i\vartheta}))|^2 \\
&+ \sum_{j=1}^{[(n+1)/2]-1} |\tilde{l}_{1+j}^{(m)}(\psi(e^{i\vartheta}))|^2 + \sum_{j=1}^{[n/2]-1} |\tilde{l}_{n-j}^{(m)}(\psi(e^{i\vartheta}))|^2 \\
&+ |\tilde{l}_n^{(m)}(\psi(e^{i\vartheta}))|^2 \\
&\leqq B^2 + \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[(n+1)/2]-1} \frac{1}{(\vartheta - \vartheta_{1+j})^2} \\
&+ \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[n/2]-1} \frac{1}{(\vartheta - \vartheta_{n-j} + 2\pi)^2} + B^2 \\
&\leqq 2B^2 + \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[(n+1)/2]-1} \frac{1}{(\vartheta_1 - \vartheta_{1+j})^2} \\
&+ \frac{\pi^2 B^2}{m^2} \sum_{j=1}^{[n/2]-1} \frac{1}{(\vartheta_n - \vartheta_{n-j})^2} \\
&\leqq 2B^2 + \frac{2\pi^2 B^2 n^2}{c^2 m^2} \sum_{1 \leqq j \leqq n/2} \frac{1}{j^2} \\
&\leqq 2B^2 + \frac{\pi^4 B^2}{c^2} \frac{n^2}{m^2}.
\end{aligned}$$

which proves (2.5).

Remark. It follows from (2.3), that if condition (2.1) is satisfied, the system:

$$z_k^{(n)} = \psi(e^{i(\alpha_n + 2\pi i k/n)}) \tag{2.6}$$

provides another example for *regular* point systems. Here the real numbers $\alpha_1, \alpha_2, \alpha_3, \ldots$ are completely arbitrary. We call the system (2.6) a system of *Fejèr-points.*

3. Proof of the theorem

Let $\{q_n(z)\}$ be a sequence of polynomials with the property that

(i) $q_n(z)$ is of degree n

(ii) $q_n(z) \to f(z)$ uniformly in $\bar{D}$.

By Walsh's classical result, such a sequence certainly exists. We will specify the choice of $\{q_n(z)\}$ later. We have that

$$\varepsilon_n = \max_{z \in \bar{D}} |f(z) - q_n(z)| \to 0 \tag{3.1}$$

We now write $m = [\frac{1}{2}\eta n]$, and

$$p_{n-1}(z) = q_{n-1}(z) + \sum_{k=1}^{n} \{f(z_k) - q_{n-1}(z_k)\}\, l_k(z)\, [\tilde{l}_k^{(m)}(z)]^2 \tag{3.2}$$

Clearly:

$$p_{n-1}(z_\nu) = q_{n-1}(z_\nu) + f(z_\nu) - q_{n-1}(z_\nu) = f(z_\nu)$$

for $\nu = 1, 2, \ldots n$. Further

$$|f(z) - p_{n-1}(z)| \leqq |f(z) - q_{n-1}(z)| + \sum_{k=1}^{n} |f(z_k) - q_{n-1}(z_k)|\, |l_k(z)|\, |\tilde{l}_k^{(m)}(z)|^2 .$$

Applying (3.1), (1.2), and (2.5):

$$|f(z) - p_{n-1}(z)| \leqq \varepsilon_n + M\varepsilon_n \sum_{k=1}^{n} |\tilde{l}_k^{(m)}(z)|^2$$

$$\leqq M\varepsilon_n\left(1 + C_1 + C_2 \frac{n^2}{m^2}\right) \leqq M\varepsilon_n\left(1 + C_1 + C_2 \frac{4}{\eta^2}\right) \quad \text{for} \quad n \geqq \frac{2}{\eta}.$$

Hence

$$\max_{z \in \bar{D}} |f(z) - p_{n-1}(z)| \to 0 \quad \text{as} \quad n \to \infty .$$

Thus we have proved Theorem 3 with the exception of the last assertion. To complete the proof it only remains to specify the choice of the polynomials $q_n(z)$.

The polynomial

$$s_{n-1}(z) = \frac{1}{2\pi i} \int_{|t|=1} f(\psi(t)) \left[\sum_{m=0}^{n-1} \left(1 - \frac{m}{n}\right) \frac{F_m(z)}{t^{m+1}}\right] dt$$

(where $F_m(z)$ is the m-th Faber-polynomial of the domain D) represents the arithmetic mean of the partial sums of the Faber-expansion of $f(z)$. It is known that for every $f(z)$ regular in D and continuous in $\bar{D}$,

$$s_n(z) \to f(z)$$

uniformly in $\bar{D}$ (cf. [1]). Thus we can choose

$$q_n(z) = s_n(z).$$

REFERENCES

[1] S. Y. ALPER, *On the uniform approximation of functions of a complex variable in closed domains* (in Russian), Izv. Akad. Nauk SSSR, Mat. *19* (1955), 423–44.

[2] S. Y. ALPER, *On the convergence of Lagrange interpolation polynomials in the complex domain* (in Russian), Uspekhi Mat. Nauk *XI*, 5, 44–49.

[3] P. ERDÖS, *On some convergence properties of the interpolation polynomials*, Annals of Math. *44*, No. 2 (1943), 330–337.

[4] T. KÖVARI and Ch. POMMERENKE, *On the distribution of Fekete points*, Mathematika (to be published).

Imperial College, London S.W. 7

Received October 18, 1967