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# The Real Cohomology of Differentiable Fibre Bundles

PAUL BAUM<sup>1)</sup> and LARRY SMITH<sup>2)</sup>

Throughout algebraic topology one very often studies fibre bundles  $\xi = (E, p, B, G/H, G)$  where  $G$  is a compact connected Lie group and  $H \subset G$  is a closed connected subgroup,  $E$  and  $B$  are differentiable manifolds and  $p: E \rightarrow B$  is a differentiable map. Typically one tries to compute the cohomology of the total space from a knowledge of the cohomology of the base  $B$ , the fibre  $G/H$  and some invariant of the bundle. The usual procedure involves calculating with the Serre spectral sequence. However this does not take full advantage of the fact that  $\xi$  is a fibre bundle, for we have a classifying diagram

$$\begin{array}{ccc} G/H & = & G/H \\ \downarrow & & \downarrow \\ E & \rightarrow & B_H \\ p\downarrow & & \downarrow e \\ B & \xrightarrow{f} & B_G \end{array}$$

where  $\xi(G, H) = (B_H, \varrho, B_G, G/H, G)$  is a universal bundle. Using techniques of EILENBERG and MOORE [8] we shall show

**THEOREM:** *If  $B$  is a Riemannian symmetric space [5] and  $R$  is the field of real numbers then  $H^*(E; R)$  and  $\text{Tor}_{H^*(B_G; R)}(H^*(B; R), H^*(B_H; R))$  are isomorphic as algebras.*

This extends results of BOREL [3] and CARTAN [6]. BOREL [3] further shows how the map  $\varrho^*: H^*(B_G; R) \rightarrow H^*(B_H; R)$  can be computed from information on the Weyl groups of  $G$  and  $H$ .

It is well known [4], [13], [15] that  $H^*(B_G; R)$  is a polynomial algebra (over  $R$ ) on even dimensional generators. Therefore for the above result to be of use we must have available a fairly simple technique for computing  $\text{Tor}_A(B, A)$  when  $A$  is a polynomial algebra. This is the objective of the first section. The second section gives a proof of the above result. The final section gives an example to show that the technical assumption that  $B$  is a Riemannian symmetric space is essential.

We shall assume that the reader is familiar with the material of [1] or [8] or [13] or [16]. Our notation will be that of [12].

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### 1. The Two Sided Koszul Complex

Throughout this section the ground ring will be a fixed field  $k$ .  $\otimes$  will always mean  $\otimes_k$ .

Suppose that

$$\Lambda = P[x_1, \dots, x_n].$$

Of course if the characteristic of  $k$  is not 2 then of necessity  $\deg(x_i)$  will be even. Denote by

$$\mu: \Lambda \otimes \Lambda \rightarrow \Lambda$$

the multiplication map of  $\Lambda$ . Note that  $\mu$  is onto.

LEMMA 1.1:  $\ker \mu = (x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n)$ .

*Proof:* Let

$$I = (x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n).$$

Then clearly  $I \subset \ker \mu$ . Thus there is a natural map of algebras

$$\alpha: \frac{\Lambda \otimes \Lambda}{I} \rightarrow \frac{\Lambda \otimes \Lambda}{\ker \mu} = \Lambda.$$

Let  $[x_i \otimes 1], [1 \otimes x_j]$  denote  $x_i \otimes 1$  and  $1 \otimes x_j$  as elements of  $\Lambda \otimes \Lambda / I$ . Then the monomials in  $[x_1 \otimes 1], \dots, [x_n \otimes 1], [1 \otimes x_1], \dots, [1 \otimes x_n]$  generate  $\Lambda \otimes \Lambda / I$  as a  $k$ -module. Since  $[x_i \otimes 1] = [1 \otimes x_i]$   $i = 1, \dots, n$  it follows that the monomials in  $[x_1 \otimes 1], \dots, [x_n \otimes 1]$  generate  $\Lambda \otimes \Lambda / I$  as a  $k$ -module.

Next recall that the monomials in  $x_1, \dots, x_n$  are a  $k$ -basis for  $\Lambda$ . Since  $\alpha([x_i \otimes 1]) = x_i, i = 1, \dots, n$  and  $\alpha$  is a map of algebras it follows that  $\alpha$  maps a  $k$ -generating set for  $\Lambda \otimes \Lambda / I$  in a one-one-onto fashion to a  $k$ -basis for  $\Lambda$ . Hence  $\alpha$  must be an isomorphism.

Since everything in sight is of finite type it follows that in each degree  $I$  and  $\ker \mu$  have the same dimension (finite) as vector spaces over  $k$ . Since  $I \subset \ker \mu$  it follows that  $I = \ker \mu$ .  $\square$

Now note that  $x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n$  is an ESP-sequence in  $\Lambda \otimes \Lambda$  generating the ideal  $\ker \mu$ . (See [16], also called an  $E$ -sequence in [1], or an  $S$ -sequence in [10]). Therefore we have the Koszul complex [1], [10], [12], [16], [18]

$$\begin{aligned} \mathcal{E}^2 &= \Lambda \otimes E[u_1, \dots, u_n] \otimes \Lambda \\ d(a \otimes u_i \otimes b) &= a x_i \otimes 1 \otimes b - a \otimes 1 \otimes x_i b, \quad i = 1, \dots, n \\ d(a \otimes 1 \otimes b) &= 0 \quad d \text{ a derivation} \end{aligned}$$

$\mathcal{E}^2$  is given a bigraded structure by requiring that

$$\deg u_i = (-1, \deg x_i), \quad i = 1, \dots, n, \deg a = (0, \deg a) \quad \text{all } a \in \Lambda.$$

We then have [10; 7], [16; § 2.1]

$$H^0(\mathcal{E}^2) = \Lambda \otimes \Lambda / \ker \mu = \Lambda, H^p(\mathcal{E}^2) = 0, \quad p \neq 0.$$

Thus  $\mathcal{E}^2$  is a  $\Lambda \otimes \Lambda$  resolution of  $\Lambda$ . We will refer to  $\mathcal{E}^2$  as the two sided Koszul complex by analogy with the two sided bar construction.

**PROPOSITION 1.2:** *If  $A$  is any  $\Lambda$ -module then  $\mathcal{E}^2 \otimes_{\Lambda} A$  is a free resolution of  $A$  as a  $\Lambda$ -module.*

*Proof:* Since  $\mathcal{E}^2$  is a free  $\Lambda$ -module we have a spectral sequence (see [12; page 400])  $E^r \Rightarrow H(\mathcal{E}^2 \otimes_{\Lambda} A)$ ,  $E^2 = \text{Tor}_{\Lambda}(H(\mathcal{E}^2), A) = \text{Tor}_{\Lambda}(\Lambda, A) = A$  i.e.  $E_{p,*}^2 = 0$   $p \neq 0$  which implies

$$H^0(\mathcal{E}^2 \otimes_{\Lambda} A) = A, H^p(\mathcal{E}^2 \otimes_{\Lambda} A) = 0 \quad p \neq 0.$$

Since  $\mathcal{E}^2 \otimes_{\Lambda} A$  is obviously a free  $\Lambda$ -module the result follows.  $\square$

**COROLLARY 1.3:** *If  $(B_{\Lambda}, {}_{\Lambda}A)$  is given then*

$$\text{Tor}_{\Lambda}(B, A) = H(B \otimes E[u_1, \dots, u_n] \otimes A; d) \quad \text{where}$$

$$d(b \otimes 1 \otimes a) = 0, \quad d(b \otimes u_i \otimes a) = b x_i \otimes 1 \otimes a - b \otimes 1 \otimes x_i a, \\ \text{deg}(u_i) = (-1, \text{deg } x_i). \quad \square$$

**ACKNOWLEDGMENT:** The existence of the two sided Koszul complex was suggested to us by Prof. J. P. MAY.

We shall have occasion to consider the case where  $A$  is a differential  $\Lambda$ -module. In this case we shall need:

**PROPOSITION 1.4:** *If  $A$  is a differential  $\Lambda$ -module then  $\mathcal{E}^2 \otimes_{\Lambda} A$  is a proper projective resolution ([12], [16]) of  $A$  as a differential  $\Lambda$ -module.*

*Proof:* We must show the following

- (i)  $\mathcal{E}^2 \otimes_{\Lambda} A$  is a proper projective  $\Lambda$ -module.
- (ii)  $\mathcal{E}^2 \otimes_{\Lambda} A$  is a resolution of  $A$ .
- (iii) If  $d_A$  denotes the differential in  $A$  then

$$Z_{\Lambda}(\mathcal{E}^2 \otimes_{\Lambda} A) \quad \text{is a resolution of} \quad Z(A).$$

$$H_{\Lambda}(\mathcal{E}^2 \otimes_{\Lambda} A) \quad \text{is a resolution of} \quad H(A).$$

To see (i) observe that  $\mathcal{E}^2 \otimes_{\Lambda} A = \Lambda \otimes E[u_1, \dots, u_n] \otimes A$  as a  $\Lambda$ -module. Since  $k$  is a field it follows that  $E^2 \otimes_{\Lambda} A$  is a proper projective  $\Lambda$ -module [13], [16]. (MOORE does not use the adjective proper.)

(ii) is just Proposition 1.2.

To obtain (iii) we note that there is a decomposition of vector spaces,

$$A = R \oplus P \oplus Q,$$

with  $d_A$  given by  $d^n: Q^n \approx R^{n+1}$  (see [12; page 398]) and so we see

$$\begin{aligned} Z_A(\mathcal{E}^2 \otimes_A A) &= Z_A(\Lambda \otimes E[u_1, \dots, u_n] \otimes A) = Z_A(\Lambda \otimes E[u_1, \dots, u_n] \otimes (R \oplus P \oplus Q)) \\ &= \Lambda \otimes E[u_1, \dots, u_n] \otimes (R \oplus P) = \Lambda \otimes E[u_1, \dots, u_n] \otimes Z(A) = \mathcal{E}^2 \otimes_A Z(A). \end{aligned}$$

which is a resolution of  $Z(A)$  by Proposition 1.2.

Finally since  $k$  is a field the Kunneth theorem gives

$$H_A(\mathcal{E}^2 \otimes_A A) = H(\Lambda \otimes E[u_1, \dots, u_n] \otimes A) = \Lambda \otimes E[u_1, \dots, u_n] \otimes H(A) = \mathcal{E}^2 \otimes_A H(A)$$

which is a resolution of  $H(A)$  by Proposition 1.2.  $\square$

We can now proceed in the obvious fashion to compute  $\text{Tor}_A(B, A)$  when  $B, A$  are differential  $A$ -modules.

### 2. Differentiable Fibre Bundles

Suppose that  $\xi = (E, p, B, G/H, G)$  is a differentiable fibre bundle with classifying diagram

$$\begin{array}{ccc} G/H & = & G/H \\ \downarrow & & \downarrow \\ E & \rightarrow & B_H \\ \downarrow & & \downarrow \\ B & \rightarrow & B_G \end{array}$$

Let us assume that  $G$  is a compact connected Lie group and  $H \subset G$  is a closed connected subgroup. In addition assume that  $B$  is a compact Riemannian symmetric space. (We recall that a compact Riemannian symmetric space  $M$  is an analytic manifold with a fixed Riemannian metric such that each point  $x \in M$  is a fixed point of some involutive isometry of  $M$ .)

Throughout this section the ground field  $k$  will be the field of real numbers  $R$ . If  $X$  is a topological space we shall write  $H^*(X)$  for  $H^*(X; R)$ . Our goal is to prove

**THEOREM 2.1:** *Under the above conditions there is an isomorphism of algebras*

$$H^*(E) \cong \text{Tor}_{H^*(B_G)}(H^*(B), H^*(B_H)).$$

The proof of Theorem 2.1 will be accomplished with the use of deRham cohomology for manifolds modeled on separable Hilbert spaces (see [7], [9], [14]). For the convenience of the reader we will recall some of the important facts that we shall use.

If  $M$  is a Riemannian manifold modeled on a separable Hilbert space then  $R^\#(M)$  denotes the deRham cochain algebra of  $M$ . The differential (exterior derivative) is denoted by  $d$ . We then have [7] that the algebras  $H^*(M)$  and  $H^*(R_\#(M), d)$  are naturally isomorphic.

If  $M$  is a compact Riemannian manifold then the Riemannian metric  $g$  on  $M$  induces an inner product in  $R^\#(M)$  by

$$(\alpha, \beta) = \int_M \alpha \wedge \beta^*, \quad \text{deg } \alpha = \text{deg } \beta$$

The adjoint of  $d$  relative to this inner product is called the coderivative and is denoted by  $\delta$ .

DEFINITION: A form  $\alpha \in R^\#(M)$  is said to be

$$\begin{aligned} \text{closed iff} & \quad d(\alpha) = 0 \\ \text{coclosed iff} & \quad \delta(\alpha) = 0 \\ \text{harmonic iff} & \quad d(\alpha) = 0 = \delta(\alpha). \end{aligned}$$

THEOREM 2.2 (HODGE): If  $M$  is a compact Riemannian manifold then each cohomology class  $a \in H^*(M)$  contains a unique harmonic form  $\alpha \in R^\#(M)$ .

Let  $M$  be a Riemannian manifold and denote by  $I(M)$  the group of isometries of  $M$ . Then  $I(M)$  is a Lie group and acts on the algebra  $R^\#(M)$  of differential forms on  $M$ .

THEOREM 2.3 (E. CARTAN [5]): If  $M$  is a compact Riemannian symmetric space then the harmonic forms on  $M$  are precisely the  $I(M)$  invariant forms. Therefore the  $\wedge$  product of two harmonic forms is again harmonic.

Proof of Theorem 2.1: Let

$$\begin{array}{ccc} G/H = G/H & & \\ \downarrow & \downarrow & \\ E \rightarrow B_H & & \\ p \downarrow & \downarrow^e & \\ B \xrightarrow{f} B_G & & \end{array}$$

be the classifying diagram for  $\xi$ . Following EELLS in [7] we may assume that  $B_H$  and  $B_G$  are differentiable manifolds modeled on separable Hilbert space. By differentiable approximation we may then assume that all the maps are differentiable.

Following [8] (see also [1], [16]) we then have a natural isomorphism of algebras  $H^*(E) \cong \text{Tor}_{R^\#(B_G)}(R^\#(B), R^\#(B_H))$ .

Now we know [3]  $H^*(B_G) = P[x_1, \dots, x_n] \quad n = \text{rank } G,$

$$H^*(B_H) = P[y_1, \dots, y_m] \quad m = \text{rank } H.$$

Choose representative cocycles  $\alpha_1, \dots, \alpha_n \in R^\#(B_G)$  for  $x_1, \dots, x_n$ . Since the multiplication in  $R^\#(B_G)$  is commutative the map  $x_i \rightarrow \alpha_i \quad i=1, \dots, n$  extends to a unique map of algebras  $\alpha: H^*(B_G) \rightarrow R^\#(B_G)$ . If we think of  $H^*(B_G)$  as a differential algebra with zero differential then  $\alpha$  is a map of differential algebras inducing an isomorphism in homology.

In a similar manner we construct a map  $\beta: H^*(B_H) \rightarrow R^\#(B_H)$ . Consider the diagram

$$\begin{array}{ccccc}
 R^\#(B_H) & \xleftarrow{\varrho^\#} & R^\#(B_G) & \xrightarrow{f^\#} & R^\#(B) \\
 \beta \uparrow & & \uparrow \alpha & & \\
 H^*(B_H) & \xleftarrow{\varrho^\#} & H^*(B_G) & \xrightarrow{f^\#} & H^*(B)
 \end{array}$$

Figure A

We do not claim that the left hand square commutes. However using this diagram we can make  $R^\#(B_H)$  into an  $H^*(B_G)$  module in two different ways, i.e. by means of the maps  $\beta\varrho^\#$  and  $\varrho^\#\alpha$ . We can also make  $R^\#(B)$  into an  $H^*(B_G)$  module by means of the map  $f^\#\alpha$ .

Hence there are two different torsion products which we shall denote by

$$\begin{aligned}
 & \beta\varrho^\# \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\
 & \varrho^\#\alpha \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H))
 \end{aligned}$$

We claim that these two torsion products are isomorphic. To see this set  $\beta\varrho^\#(x_i) = \eta_i$ ,  $\varrho^\#\alpha(x_i) = \eta'_i$ ,  $f^\#\alpha(x_i) = \zeta_i$ . Let  $d_B$  denote the boundary in  $R^\#(B)$  and  $d_H$  the boundary in  $R^\#(B_H)$ . Then using the two sided Koszul complex of the previous section we see

$$\beta\varrho^\# \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) = H(R^\#(B) \otimes E[u_1, \dots, u_n] \otimes R^\#(B_H))$$

where

$$\begin{aligned}
 d(\alpha \otimes 1 \otimes \beta) &= d_B \alpha \otimes 1 \otimes \beta + \alpha \otimes 1 \otimes d_H \beta \\
 d(1 \otimes u_i \otimes 1) &= \zeta_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \eta_i
 \end{aligned}$$

and similarly

$$\varrho^\#\alpha \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) = H(R^\#(B) \otimes E[v_1, \dots, v_n] \otimes R^\#(B_H))$$

where

$$\begin{aligned}
 d(\alpha \otimes 1 \otimes \beta) &= d_B \alpha \otimes 1 \otimes \beta + \alpha \otimes 1 \otimes d_H \beta \\
 d(1 \otimes v_i \otimes 1) &= \zeta_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \eta'_i
 \end{aligned}$$

Now since Figure A certainly commutes when we pass to homology it follows that for each  $i$  we can choose  $\lambda_i \in R^\#(B_H)$  so that  $\eta'_i = \eta_i + d_H \lambda_i$ .

Define a map

$$T: R^\#(B) \otimes E[u_1, \dots, u_n] \otimes R^\#(B_H) \rightarrow R^\#(B) \otimes E[v_1, \dots, v_n] \otimes R^\#(B_H)$$

by  $T(\alpha \otimes 1 \otimes \beta) = \alpha \otimes 1 \otimes \beta$

$$T(1 \otimes u_i \otimes 1) = 1 \otimes v_i \otimes 1 - 1 \otimes 1 \otimes \lambda_i$$

and requiring that  $T$  be a map of algebras. A direct computation shows that  $T$  is a map of complexes. As  $T^{-1}$  is readily defined we see that  $T$  gives an isomorphism of algebras

$$T^*: {}_{\beta} q^* \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \rightarrow {}_q \alpha^* \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)).$$

We then have algebra isomorphisms

$$\begin{aligned} & \text{Tor}_{R^\#(B_G)}(R^\#(B), R^\#(B_H)) \\ & \quad \approx \uparrow \quad \text{Tor}_\alpha(1, 1) \\ & {}_q \alpha^* \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\ & \quad \approx \uparrow \quad T \\ & {}_{\beta} q^* \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\ & \quad \approx \uparrow \quad \text{Tor}_1(1, \beta) \\ & \text{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H)) \end{aligned}$$

Recall now that we assumed  $B$  to be a compact Riemannian symmetric space. Define a map  $\theta: H^*(B) \rightarrow R^\#(B)$  by  $a \rightarrow$  the unique harmonic form contained in  $a$ . It follows from the results of Hodge and Cartan stated above that  $\theta$  is a map of algebras inducing an isomorphism in homology. Consider now the diagram

$$\begin{array}{ccc} R^\#(B_G) & \xrightarrow{f} & R^\#(B) \\ \alpha \downarrow & & \downarrow \theta \\ H^*(B_G) & \rightarrow & H^*(B) \end{array}$$

As above this leads to two torsion products

$$\begin{aligned} & f^* \alpha^* \text{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H)) \\ & \theta f^* \text{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H)) \end{aligned}$$

which are seen to be isomorphic by an argument analogous to the one above. This gives us a string of algebra isomorphisms

$$\begin{aligned} H^*(E) & \cong \text{Tor}_{R^\#(B_G)}(R^\#(B), R^\#(B_H)) \\ & \quad \uparrow \text{Tor}_\alpha(1, 1) \\ & {}_q \alpha^* \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\ & \quad \uparrow T \\ & {}_{\beta} q^* \text{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\ & \quad \uparrow \text{Tor}_1(1, \beta) \\ & f^* \alpha^* \text{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H)) \\ & \quad \uparrow T' \\ & \theta f^* \text{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H)) \\ & \quad \uparrow \text{Tor}_1(\theta, 1) \\ & \text{Tor}_{H^*(B_G)}(H^*(B), H^*(B_H)) \end{aligned}$$

which completes the proof.  $\square$

If in Theorem 2.1 we set  $B = \text{point}$  then we obtain a result of CARTAN [6] as restated by BAUM in [2]. If we set  $H = 1$  in Theorem 2.1 then we obtain a result of BOREL and HIRSCH [4].

### 3. An Example

Of all the hypotheses of Theorem 2.1 probably the least satisfying is the assumption that  $B$  be a Riemannian symmetric space. However this is an essential assumption as the following example will show.

Let  $Y = S^2 \vee S^2 \vee S^2$ . Let  $f, g, h \in \Pi_2(Y)$  represent the homotopy classes of the inclusions

$$\begin{aligned} S^2 &\xrightarrow{f} S^2 \vee * \vee * \subset Y \\ S^2 &\xrightarrow{g} * \vee S^2 \vee * \subset Y \\ S^2 &\xrightarrow{h} * \vee * \vee S^2 \subset Y \end{aligned}$$

Let  $t: S^4 \rightarrow Y$  represent the Whitehead product  $[f, [g, h]] \in \Pi_4(Y)$  and let  $X = Y U_t e^5$  where  $e^5$  is a five cell. MASSEY and UEHARA [11] have shown that there are indecomposable elements  $z_1, z_2, z_3 \in H^2(X; Z)$  and  $w \in H^5(X; Z)$  with the triple product  $\langle z_1, z_2, z_3 \rangle$  defined and

$$\langle z_1, z_2, z_3 \rangle = w \neq 0 \in H^*(X, Z)/H^*(X, Z) z_1 + z_3 H^*(X; Z)$$

Also from [11] we shall need

LEMMA 3.1: *Suppose that  $f: A \rightarrow B$  is a continuous map. Let  $u, v, w \in H^*(B; Z)$  such that*

(i)  $uv = 0 = vw$ , (ii)  $f^*(u) = 0 = f^*(w)$  then

$$\langle u, v, w \rangle \in \ker(f^*: H^*(B; Z) \rightarrow H^*(A, Z)).$$

*Proof:* See [11] Lemma 5 on page 369.  $\square$

Now  $X$  is a 5-dimensional simplicial complex and so we can imbed  $X$  in  $R^{11}$ . Let  $B$  be the double of a regular neighborhood of  $X$  in  $R^{11}$ . Then  $B$  is a smooth manifold, but not a Riemannian symmetric space.  $X$  is a retract of  $B$ . Thus there are classes  $x_1, x_2, x_3 \in H^2(B; Z)$  and  $y \in H^5(B; Z)$  with  $\langle x_1, x_2, x_3 \rangle$  defined and

$$\langle x_1, x_2, x_3 \rangle = y \neq 0 \in H^*(B, Z)/H^*(B, Z) x_1 + x_3 H^*(B, Z).$$

We now construct an  $S^1 \times S^1$  bundle over  $B$  as follows. Choose maps

$$f_i: B \rightarrow K(Z, 2) = CP^\infty = B_{S^1} \quad i = 1, 3$$

representing the classes  $x_1, x_3$ . Form the diagram

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\quad \quad \quad} & S^1 \times S^1 \\ \downarrow & & \downarrow \\ E & \xrightarrow{\quad \quad \quad} & E_{S^1 \times S^1} \\ \downarrow p & \searrow f_1 \times f_3 & \downarrow \\ B & \xrightarrow{\quad \quad \quad} & B_{S^1 \times S^1} \end{array}$$

which is the classifying diagram of a principal  $S^1 \times S^1$  bundle  $\xi$  over  $B$ .

**PROPOSITION 3.2:**  $H^*(E; k)$  and  $\text{Tor}_{H^*(B_{S^1 \times S^1}; k)}(H^*(B; k), k)$  are not isomorphic as vector spaces for any field  $k$ .

*Proof:* Consider the Eilenberg-Moore spectral sequence [1], [8], [16]  $\{E_r, d_r\}$  of the above diagram with  $k$  as coefficients. It has

$$E_r \Rightarrow H^*(E; k)$$

$$E_2 = \text{Tor}_{H^*(B_{S^1 \times S^1}; k)}(H^*(B; k), k).$$

Clearly it suffices to show that  $E_2 \neq E_\infty$ .

By direct computation we have

$$E_2^{0,*} = H^*(B; k)/H^*(B; k)x_1 + x_3H^*(B; k).$$

Now the map  $p^*: H^*(B; k) \rightarrow H^*(E; k)$  is given by the composition

$$H^*(B; k) \rightarrow H^*(B; k)/(x_1, x_3) = E_2^{0,*} \xrightarrow{\varepsilon} E_\infty^{0,*} \subset H^*(E; k).$$

Now we claim that  $p^*(y) = 0$ . For we know that  $y = \langle x_1, x_2, x_3 \rangle$  and  $p^*(x_1) = 0 = p^*(x_3)$  and so by Lemma 3.1  $p^*(y) = 0$ .

But  $y \neq 0 \in H^*(B; k)/(x_1, x_3)$  and hence the map  $\varepsilon: E_2^{0,*} \rightarrow E_\infty^{0,*}$  is not a monomorphism. Therefore  $E_2 \neq E_\infty$ .  $\square$

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