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The Real Cohomology of Differentiable Fibre Bundles

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Throughout algebraic topology one very often studies fibre bundles $\xi=(E, p, B, G/H, G)$ where G is a compact connected Lie group and $H\subset G$ is a closed connected subgroup, E and B are differentiable manifolds and $p: E\to B$ is a differentiable map. Typically one tries to compute the cohomology of the total space from a knowledge of the cohomology of the base B, the fibre G/H and some invariant of the bundle. The usual procedure involves calculating with the Serre spectral sequence. However this does not take full advantage of the fact that ξ is a fibre bundle, for we have a classifying diagram

$$\begin{array}{ccc} G/H & = & G/H \\ \downarrow & & \downarrow \\ E & \to & B_H \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \varrho} \\ B & \xrightarrow{f} & B_G \end{array}$$

where $\xi(G, H)=(B_H, \varrho, B_G, G/H, G)$ is a universal bundle. Using techniques of EILENBERG and MOORE [8] we shall show

THEOREM: *If B is a Riemannian symmetric space* [5] *and R is the field of real numbers then $H^*(E; R)$ and $\operatorname{Tor}_{H^*(B_G; R)}(H^*(B; R), H^*(B_H; R))$ are isomorphic as algebras.*

This extends results of BOREL [3] and CARTAN [6]. BOREL [3] further shows how the map $\varrho^*: H^*(B_G; R)\to H^*(B_H; R)$ can be computed from information on the Weyl groups of G and H.

It is well known [4], [13], [15] that $H^*(B_G; R)$ is a polynomial algebra (over R) on even dimensional generators. Therefore for the above result to be of use we must have available a fairly simple technique for computing $\operatorname{Tor}_\Lambda(B, A)$ when Λ is a polynomial algebra. This is the objective of the first section. The second section gives a proof of the above result. The final section gives an example to show that the technical assumption that B is a Riemannian symmetric space is essential.

We shall assume that the reader is familiar with the material of [1] or [8] or [13] or [16]. Our notation will be that of [12].

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1. The Two Sided Koszul Complex

Throughout this section the ground ring will be a fixed field k. $\otimes$ will always mean $\otimes_k$.

Suppose that

$$\Lambda = P[x_1, \ldots, x_n].$$

Of course if the characteristic of k is not 2 then of necessity $\deg(x_i)$ will be even. Denote by

$$\mu : \Lambda \otimes \Lambda \to \Lambda$$

the multiplication map of Λ. Note that μ is onto.

LEMMA 1.1: $\ker \mu = (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$.

Proof: Let

$$I = (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n).$$

Then clearly $I \subset \ker \mu$. Thus there is a natural map of algebras

$$\alpha : \frac{\Lambda \otimes \Lambda}{I} \to \frac{\Lambda \otimes \Lambda}{\ker \mu} = \Lambda .$$

Let $[x_i \otimes 1]$, $[1 \otimes x_j]$ denote $x_i \otimes 1$ and $1 \otimes x_j$ as elements of $\Lambda \otimes \Lambda / I$. Then the monomials in $[x_1 \otimes 1], \ldots, [x_n \otimes 1], [1 \otimes x_1], \ldots, [1 \otimes x_n]$ generate $\Lambda \otimes \Lambda / I$ as a k-module. Since $[x_i \otimes 1] = [1 \otimes x_i]$ $i = 1, \ldots, n$ it follows that the monomials in $[x_1 \otimes 1], \ldots, [x_n \otimes 1]$ generate $\Lambda \otimes \Lambda / I$ as a k-module.

Next recall that the monomials in $x_1, \ldots, x_n$ are a k-basis for Λ. Since $\alpha([x_i \otimes 1]) = x_i$, $i = 1, \ldots, n$ and α is a map of algebras it follows that α maps a k-generating set for $\Lambda \otimes \Lambda / I$ in a one-one-onto fashion to a k-basis for Λ. Hence α must be an isomorphism.

Since everything in sight is of finite type it follows that in each degree I and $\ker \mu$ have the same dimension (finite) as vector spaces over k. Since $I \subset \ker \mu$ it follows that $I = \ker \mu$. □

Now note that $x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n$ is an ESP-sequence in $\Lambda \otimes \Lambda$ generating the ideal $\ker \mu$. (See [16], also called an E-sequence in [1], or an S-sequence in [10]). Therefore we have the Koszul complex [1], [10], [12], [16], [18]

$$\mathscr{E}^2 = \Lambda \otimes E[u_1, \ldots, u_n] \otimes \Lambda$$
$$d(a \otimes u_i \otimes b) = a x_i \otimes 1 \otimes b - a \otimes 1 \otimes x_i b, \quad i = 1, \ldots, n$$
$$d(a \otimes 1 \otimes b) = 0 \qquad d \quad \text{a derivation}$$

$\mathscr{E}^2$ is given a bigraded structure by requiring that

$$\deg u_i = (-1, \deg x_i), \quad i = 1, \ldots, n, \deg a = (0, \deg a) \quad \text{all} \quad a \in \Lambda .$$

We then have [10; 7], [16; § 2.1]

$$H^0(\mathscr{E}^2) = \Lambda \otimes \Lambda/\ker\mu = \Lambda\,,\; H^p(\mathscr{E}^2) = 0\,, \quad p \neq 0\,.$$

Thus $\mathscr{E}^2$ is a $\Lambda\otimes\Lambda$ resolution of Λ. We will refer to $\mathscr{E}^2$ as the two sided Koszul complex by analogy with the two sided bar construction.

PROPOSITION 1.2: *If A is any Λ-module then $\mathscr{E}^2\otimes_\Lambda A$ is a free resolution of A as a Λ-module.*

Proof: Since $\mathscr{E}^2$ is a free Λ-module we have a spectral sequence (see [12; page 400]) $E^r\Rightarrow H(\mathscr{E}^2\otimes_\Lambda A)$, $E^2=\mathrm{Tor}_\Lambda(H(\mathscr{E}^2), A)=\mathrm{Tor}_\Lambda(\Lambda, A)=A$ i.e. $E^2_{p,*}=0\; p\neq 0$ which implies

$$H^0(\mathscr{E}^2 \otimes_\Lambda A) = A\,,\; H^p(\mathscr{E}^2 \otimes_\Lambda A) = 0 \quad p \neq 0\,.$$

Since $\mathscr{E}^2\otimes_\Lambda A$ is obviously a free Λ-module the result follows. □

COROLLARY 1.3: *If $(B_\Lambda, {}_\Lambda A)$ is given then*

$$\mathrm{Tor}_\Lambda(B, A) = H(B\otimes E[u_1,\ldots, u_n]\otimes A; d) \quad \textit{where}$$

$$d(b\otimes 1\otimes a) = 0\,, \quad d(b\otimes u_i\otimes a) = b\,x_i\otimes 1\otimes a - b\otimes 1\otimes x_i a\,,$$
$$\deg(u_i) = (-1\,, \deg x_i)\,. \;\square$$

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We shall have occasion to consider the case where A is a differential Λ-module. In this case we shall need:

PROPOSITION 1.4: *If A is a differential Λ-module then $\mathscr{E}^2\otimes_\Lambda A$ is a proper projective resolution* ([12], [16]) *of A as a differential Λ-module.*

Proof: We must show the following

(i) $\mathscr{E}^2\otimes_\Lambda A$ is a proper projective Λ-module.

(ii) $\mathscr{E}^2\otimes_\Lambda A$ is a resolution of A.

(iii) If d_A denotes the differential in A then

$$Z_A(\mathscr{E}^2 \otimes_\Lambda A) \quad \text{is a resolution of} \quad Z(A)\,.$$
$$H_A(\mathscr{E}^2 \otimes_\Lambda A) \quad \text{is a resolution of} \quad H(A)\,.$$

To see (i) observe that $\mathscr{E}^2\otimes_\Lambda A=\Lambda\otimes E[u_1,\ldots, u_n]\otimes A$ as a Λ-module. Since k is a field it follows that $E^2\otimes_\Lambda A$ is a proper projective Λ-module [13], [16]. (MOORE does not use the adjective proper.)

(ii) is just Proposition 1.2.

To obtain (iii) we note that there is a decomposition of vector spaces,

$$A = R\oplus P\oplus Q\,,$$

with d_A given by $d^n\colon Q^n\approx R^{n+1}$ (see [12; page 398]) and so we see

$$Z_A(\mathscr{E}^2 \otimes_\Lambda A) = Z_A(\Lambda \otimes E[u_1, \ldots, u_n] \otimes A) = Z_A(\Lambda \otimes E[u_1, \ldots, u_n] \otimes (R \oplus P \oplus Q))$$
$$= \Lambda \otimes E[u_1, \ldots, u_n] \otimes (R \otimes P) = \Lambda \otimes E[u_1, \ldots, u_n] \otimes Z(A) = \mathscr{E}^2 \otimes_\Lambda Z(A).$$

which is a resolution of $Z(A)$ by Proposition 1.2.

Finally since k is a field the Kunneth theorem gives

$$H_A(\mathscr{E}^2 \otimes_\Lambda A) = H(\Lambda \otimes E[u_1, \ldots, u_n] \otimes A) = \Lambda \otimes E[u_1, \ldots, u_n] \otimes H(A) = \mathscr{E}^2 \otimes_\Lambda H(A)$$

which is a resolution of $H(A)$ by Proposition 1.2. □

We can now proceed in the obvious fashion to compute $\operatorname{Tor}_\Lambda(B, A)$ when B, A are differential Λ-modules.

2. Differentiable Fibre Bundles

Suppose that $\xi = (E, p, B, G/H, G)$ is a differentiable fibre bundle with classifying diagram

$$\begin{array}{ccc} G/H & = & G/H \\ \downarrow & & \downarrow \\ E & \rightarrow & B_H \\ \downarrow & & \downarrow \\ B & \rightarrow & B_G \end{array}$$

Let us assume that G is a compact connected Lie group and $H \subset G$ is a closed connected subgroup. In addition assume that B is a compact Riemannian symmetric space. (We recall that a compact Riemannian symmetric space M is an analytic manifold with a fixed Riemannian metric such that each point $x \in M$ is a fixed point of some involutive isometry of M.)

Throughout this section the ground field k will be the field of real numbers R. If X is a topological space we shall write $H^*(X)$ for $H^*(X; R)$. Our goal is to prove

THEOREM 2.1: *Under the above conditions there is an isomorphism of algebras*

$$H^*(E) \cong \operatorname{Tor}_{H^*(B_G)}(H^*(B), H^*(B_H)).$$

The proof of Theorem 2.1 will be accomplished with the use of deRham cohomology for manifolds modeled on separable Hilbert spaces (see [7], [9], [14]). For the convenience of the reader we will recall some of the important facts that we shall use.

If M is a Riemannian manifold modeled on a separable Hilbert space then $R^\#(M)$ denotes the deRham cochain algebra of M. The differential (exterior derivative) is denoted by d. We then have [7] that the algebras $H^*(M)$ and $H^*(R_\#(M), d)$ are naturally isomorphic.

If M is a compact Riemannian manifold then the Riemannian metric g on M induces an inner product in $R^\#(M)$ by

$$(\alpha, \beta) = \int_M \alpha \wedge \beta^*, \qquad \deg\alpha = \deg\beta$$

The adjoint of d relative to this inner product is called the coderivative and is denoted by δ.

DEFINITION: *A form $\alpha \in R^{\#}(M)$ is said to be*

$$\begin{aligned} \textit{closed iff} \quad & d(\alpha) = 0 \\ \textit{coclosed iff} \quad & \delta(\alpha) = 0 \\ \textit{harmonic iff} \quad & d(\alpha) = 0 = \delta(\alpha). \end{aligned}$$

THEOREM 2.2 (HODGE): *If M is a compact Riemannian manifold then each cohomology class $a \in H^*(M)$ contains a unique harmonic form $\alpha \in R^{\#}(M)$.*

Let M be a Riemannian manifold and denote by $I(M)$ the group of isometries of M. Then $I(M)$ is a Lie group and acts on the algebra $R^{\#}(M)$ of differential forms on M.

THEOREM 2.3 (E. CARTAN [5]): *If M is a compact Riemannian symmetric space then the harmonic forms on M are precisely the $I(M)$ invariant forms. Therefore the $\wedge$ product of two harmonic forms is again harmonic.*

Proof of Theorem 2.1: Let

$$\begin{array}{ccc} G/H & = & G/H \\ \downarrow & & \downarrow \\ E & \rightarrow & B_H \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \varrho} \\ B & \xrightarrow{f} & B_G \end{array}$$

be the classifying diagram for ξ. Following EELLS in [7] we may assume that B_H and B_G are differentiable manifolds modeled on separable Hilbert space. By differentiable approximation we may then assume that all the maps are differentiable.

Following [8] (see also [1], [16]) we then have a natural isomorphism of algebras $H^*(E) \cong \mathrm{Tor}_{R^{\#}(B_G)}(R^{\#}(B), R^{\#}(B_H))$.

Now we know [3] $H^*(B_G) = P[x_1, \dots, x_n] \quad n = \operatorname{rank} G,$

$$H^*(B_H) = P[y_1, \dots, y_m] \quad m = \operatorname{rank} H.$$

Choose representative cocycles $\alpha_1, \dots, \alpha_n \in R^{\#}(B_G)$ for $x_1, \dots, x_n$. Since the multiplication in $R^{\#}(B_G)$ is commutative the map $x_i \to \alpha_i$ $i = 1, \dots, n$ extends to a unique map of algebras $\alpha: H^*(B_G) \to R_{\#}(B_G)$. If we think of $H^*(B_G)$ as a differential algebra with zero differential then α is a map of differential algebras inducing an isomorphism in homology.

In a similar manner we construct a map $\beta: H^*(B_H) \to R^\#(B_H)$.

Consider the diagram

$$\begin{array}{ccccc} R^\#(B_H) & \overset{\varrho^\#}{\leftarrow} & R_\#(B_G) & \overset{f^\#}{\rightarrow} & R^\#(B) \\ {}_\beta\uparrow & & \uparrow_\alpha & & \\ H^*(B_H) & \overset{\varrho^\#}{\leftarrow} & H^*(B_G) & \overset{f^\#}{\rightarrow} & H^*(B) \end{array} \qquad \text{Figure } A$$

We do not claim that the left hand square commutes. However using this diagram we can make $R^\#(B_H)$ into an $H^*(B_G)$ module in two different ways, i.e. by means of the maps $\beta\varrho^*$ and $\varrho^\#\alpha$. We can also make $R^\#(B)$ into an $H^*(B_G)$ module by means of the map $f^\#\alpha$.

Hence there are two different torsion products which we shall denote by

$$_{\beta\,\varrho^*}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H))$$

$$_{\varrho^\#\alpha}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H))$$

We claim that these two torsion products are isomorphic. To see this set $\beta\varrho^*(x_i)=\eta_i$ $\varrho^\#\alpha(x_i)=\eta_i'$ $f^\#\alpha(x_i)=\zeta_i$. Let d_B denote the boundary in $R^\#(B)$ and d_H the boundary in $R^\#(B_H)$. Then using the two sided Koszul complex of the previous section we see

$$_{\beta\,\varrho^*}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) = H(R^\#(B) \otimes E[u_1, \ldots, u_n] \otimes R^\#(B_H))$$

where

$$d(\alpha \otimes 1 \otimes \beta) = d_B\alpha \otimes 1 \otimes \beta + \alpha \otimes 1 \otimes d_H\beta$$

$$d(1 \otimes u_i \otimes 1) = \zeta_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \eta_i$$

and similarly

$$_{\varrho^\#\alpha}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) = H(R^\#(B) \otimes E[v_1, \ldots, v_n] \otimes R^\#(B_H))$$

where

$$d(\alpha \otimes 1 \otimes \beta) = d_B\alpha \otimes 1 \otimes \beta + \alpha \otimes 1 \otimes d_H\beta$$

$$d(1 \otimes v_i \otimes 1) = \zeta_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \eta_i'$$

Now since Figure A certainly commutes when we pass to homology it follows that for each i we can choose $\lambda_i \in R^\#(B_H)$ so that $\eta_i' = \eta_i + d_H\lambda_i$.

Define a map

$$T: R^\#(B) \otimes E[u_1, \ldots, u_n] \otimes R^\#(B_H) \to R^\#(B) \otimes E[v_1, \ldots, v_n] \otimes R^\#(B_H)$$

by $T(\alpha \otimes 1 \otimes \beta) = \alpha \otimes 1 \otimes \beta$

$$T(1 \otimes u_i \otimes 1) = 1 \otimes v_i \otimes 1 - 1 \otimes 1 \otimes \lambda_i$$

and requiring that T be a map of algebras. A direct computation shows that T is a map of complexes. As T^{-1} is readily defined we see that T gives an isomorphism of algebras

$$T^*: {}_{\beta\varrho^*}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \to {}_{\varrho^*\alpha}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)).$$

We then have algebra isomorphisms

$$\begin{array}{c} \mathrm{Tor}_{R^\#(B_G)}(R^\#(B), R^\#(B_H)) \\ \approx\uparrow \quad \mathrm{Tor}_\alpha(1,1) \\ {}_{\varrho^*\alpha}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\ \approx\uparrow \quad T \\ {}_{\beta\varrho^*}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\ \approx\uparrow \quad \mathrm{Tor}_1(1,\beta) \\ \mathrm{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H)) \end{array}$$

Recall now that we assumed B to be a compact Riemannian symmetric space. Define a map $\theta: H^*(B) \to R^\#(B)$ by $a \to$ the unique harmonic form contained in a. It follows from the results of Hodge and Cartan stated above that θ is a map of algebras inducing an isomorphism in homology. Consider now the diagram

$$\begin{array}{ccc} R^\#(B_G) & \xrightarrow{f} & R^\#(B) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\theta} \\ H^*(B_G) & \to & H^*(B) \end{array}$$

As above this leads to two torsion products

$${}_{f^*\alpha}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H))$$

$${}_{\theta f^*}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H))$$

which are seen to be isomorphic by an argument analogous to the one above. This gives us a string of algebra isomorphisms

$$\begin{array}{rl} H^*(E) \cong & \mathrm{Tor}_{R^\#(B_G)}(R^\#(B), R^\#(B_H)) \\ & \uparrow \mathrm{Tor}_\alpha(1,1) \\ & {}_{\varrho^*\alpha}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\ & \uparrow T \\ & {}_{\beta\varrho^*}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), R^\#(B_H)) \\ & \uparrow \mathrm{Tor}_1(1,\beta) \\ & {}_{f^*\alpha}\mathrm{Tor}_{H^*(B_G)}(R^\#(B), H^*(B_H)) \\ & \uparrow T' \\ & {}_{\theta f^*}\mathrm{Tor}_{H^*(B b)}(R^\#(B), H^*(B_H)) \\ & \uparrow \mathrm{Tor}_1(\theta,1) \\ & \mathrm{Tor}_{H^*(B_G)}(H^*(B), H^*(B_H)) \end{array}$$

which completes the proof. □

If in Theorem 2.1 we set B=point then we obtain a result of CARTAN [6] as restated by BAUM in [2]. If we set $H=1$ in Theorem 2.1 then we obtain a result of BOREL and HIRSCH [4].

3. An Example

Of all the hypotheses of Theorem 2.1 probably the least satisfying is the assumption that B be a Riemannian symmetric space. However this is an essential assumption as the following example will show.

Let $Y=S^2 \vee S^2 \vee S^2$. Let $f, g, h \in \Pi_2(Y)$ represent the homotopy classes of the inclusions

$$S^2 \xrightarrow{f} S^2 \vee * \vee * \subset Y$$
$$S^2 \xrightarrow{g} * \vee S^2 \vee * \subset Y$$
$$S^2 \xrightarrow{h} * \vee * \vee S^2 \subset Y$$

Let $t: S^4 \to Y$ represent the Whitehead product $[f, [g, h]] \in \Pi_4(Y)$ and let $X = Y\, U_t e^5$ where e^5 is a five cell. Massey and Uehara [11] have shown that there are indecomposable elements $z_1, z_2, z_3 \in H^2(X; Z)$ and $w \in H^5(X; Z)$ with the triple product $\langle z_1, z_2, z_3 \rangle$ defined and

$$\langle z_1, z_2, z_3 \rangle = w \neq 0 \in H^*(X, Z)/H^*(X, Z)\, z_1 + z_3 H^*(X; Z)$$

Also from [11] we shall need

Lemma 3.1: *Suppose that $f: A \to B$ is a continuous map. Let $u, v, w \in H^*(B; Z)$ such that*

(i) $uv = 0 = vw$, (ii) $f^*(u) = 0 = f^*(w)$ *then*

$$\langle u, v, w \rangle \in \ker(f^*: H^*(B; Z) \to H^*(A, Z)).$$

Proof: See [11] Lemma 5 on page 369. □

Now X is a 5-dimensional simplicial complex and so we can imbed X in R^{11}. Let B be the double of a regular neighborhood of X in R^{11}. Then B is a smooth manifold, but not a Riemannian symmetric space. X is a retract of B. Thus there are classes $x_1, x_2, x_3 \in H^2(B; Z)$ and $y \in H^5(B; Z)$ with $\langle x_1, x_2, x_3 \rangle$ defined and

$$\langle x_1, x_2, x_3 \rangle = y \neq 0 \in H^*(B, Z)/H^*(B, Z)\, x_1 + x_3 H^*(B, Z).$$

We now construct an $S^1 \times S^1$ bundle over B as follows. Choose maps

$$f_i: B \to K(Z, 2) = CP^\infty = B_{S^1} \qquad i = 1, 3$$

representing the classes x_1, x_3. Form the diagram

$$\begin{array}{ccc}
S^1 \times S^1 & = & S^1 \times S^1 \\
\downarrow & & \downarrow \\
E & \longrightarrow & E_{S^1 \times S^1} \\
p \downarrow & & \downarrow \\
B & \xrightarrow{f_1 \times f_3} & B_{S^1 \times S^1}
\end{array}$$

which is the classifying diagram of a principal $S^1 \times S^1$ bundle ξ over B.

PROPOSITION 3.2: *$H^*(E; k)$ and $\mathrm{Tor}_{H^*(B_{S^1 \times S^1}; k)}(H^*(B; k), k)$ are not isomorphic as vector spaces for any field k.*

Proof: Consider the Eilenberg-Moore spectral sequence [1], [8], [16] $\{E_r, d_r\}$ of the above diagram with k as coefficients. It has

$$E_r \Rightarrow H^*(E; k)$$
$$E_2 = \mathrm{Tor}_{H^*(B_{S^1} \times {S^1}: k)}(H^*(B: k), k).$$

Clearly it suffices to show that $E_2 \neq E_\infty$.

By direct computation we have

$$E_2^{0,*} = H^*(B; k)/H^*(B; k)\, x_1 + x_3 H^*(B; k).$$

Now the map $p^*: H^*(B; k) \to H^*(E; k)$ is given by the composition

$$H^*(B; k) \to H^*(B; k)/(x_1, x_3) = E_2^{0,*} \xrightarrow{\varepsilon} E_\infty^{0,*} \subset H^*(E; k).$$

Now we claim that $p^*(y) = 0$. For we know that $y = \langle x_1, x_2, x_3 \rangle$ and $p^*(x_1) = 0 = p^*(x_3)$ and so by Lemma 3.1 $p^*(y) = 0$.

But $y \neq 0 \in H^*(B; k)/(x_1, x_3)$ and hence the map $\in: E_2^{0,*} \to E_\infty^{0,*}$ is not a monomorphism. Therefore $E_2 \neq E_\infty$. □

REFERENCES

[1] P. BAUM, *Cohomology of Homogeneous Spaces*, Topology (to appear).
[2] P. BAUM, *Cohomology of Homogeneous Spaces*, Princeton University Thesis, 1963.
[3] A. BOREL, *Sur la cohomologie des espaces fibrés principeaux*..., Ann. of Math. *57* (1953), 115–207.
[4] A. BOREL, *Cohomologie des groupes de Lie compact*, Amer. J. of Math. *76* (1954), 273–342.
[5] E. CARTAN, Sur les invariants integraux des éspaces homogènes, Ann. soc. polon. math. *8* (1929), 181–225, = Selecta (Paris 1939), 203–233.
[6] H. CARTAN, a) *Notions d'Algèbre différentielle, applications aux*... b) *La transgression dans un group de Lie*..., Colloque de Topologie Bruxelles (1950).
[7] J. EELLS, *A Setting for Global Analysis*, Bulletin Amer. Math. Soc. *72* (1966), 751–807.
[8] S. EILENBERG and J. C. MOORE, *Homology and Fibrations*, I, II, Comment. Math. Helv. *40* (1966), 199–236, and to appear.
[9] S. LANG, *Introduction to Differentiable Manifolds*, Interscience Publ. Cie (1962).
[10] W. S. MASSEY and F. P. PETERSON, *Cohomology of Certain Fibre Spaces I*, Topology *4* (1965).
[11] W. S. MASSEY and H. UEHARA, *The Jacobi Identity for Whitehead Products*, Algebraic Topology and Geometry, Princeton University Press (1957).
[12] S. MACLANE, *Homology*, Academic Press-Springer Verlag (1963).
[13] J. MOORE, *Algèbre Homologique et des Espaces Classifiants*, Seminar Cartan et Moore 1959/1960 Exposé 7.
[14] G. de RHAM, *Variétés Différentiables*, Herman (1960).
[15] M. ROTHENBERG and N. STEENROD, *Cohomology of Classifying Spaces of H-Spaces*, (to appear).
[16] L. SMITH, *Homological Algebra and the Eilenberg-Moore Spectral Sequence*, Trans. of A.M.S. (to appear).
[17] N. STEENROD, *Topology of Fibre Bundles*, Princeton University Press (1951).
[18] J. TATE, *Homology of Noetherian Rings and Local Rings*, Illinois J. *1* (1957).

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