Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 42 (1967)

Artikel: The Index of a Tangent 2-Field.

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DOI: https://doi.org/10.5169/seals-32133

The Index of a Tangent 2-Field[1])

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Dedicated to Professor H. Hopf

1. Introduction

Let M be a connected, smooth, Riemannian manifold, and let k be a positive integer. By a *k-field* on M we mean an ordered set of k orthonormal tangent vector fields. We say that M has a k-field with *finite singularities* if there is a k-field on the manifold obtained from M by removing a finite number of points. Let $(X_1, \ldots, X_k)$ be such a k-field. Choose a triangulation of M such that each singular point of the k-field lies in the interior of a distinct m-simplex ($m = \dim M$). Let p be a singular point, say in the interior of the closed simplex σ. Suppose now that M is oriented. The tangent bundle of M restricted to σ is then isomorphic to the trivial bundle $\sigma \times R^m$, by an orientation preserving isomorphism, and this isomorphism can be chosen to be compatible with the standard Riemannian metric on $\sigma \times R^m$. Thus for each point q in $\sigma - \{p\}$ we can regard $(X_1(q), \ldots, X_k(q))$ as an orthonormal k-frame in R^m – that is, as a point in the Stiefel manifold $V_{m,k}$. Since M is oriented the boundary of σ, $\dot\sigma$, is then an oriented $(m-1)$-sphere. By the above remarks one sees that the k-field restricted to $\dot\sigma$ gives a map $\dot\sigma \to V_{m,k}$ and the homotopy class of this map is then an element of the homotopy group $\pi_{m-1}(V_{m,k})$. We define this homotopy class to be the *index* of the k-field at the singular point p (see HOPF [12], [13]), and write this Index $(X_1, \ldots, X_k)_p$. Now let $\{p_1, \ldots, p_r\}$ be the set of singular points of the k-field. We define

$$\text{Index}\,(X_1, \ldots, X_k) = \sum_i \text{Index}\,(X_1, \ldots, X_k)_{p_i} \in \pi_{m-1}(V_{m,k}).$$

One can show that this definition of the index agrees with the definition one obtains via obstruction theory. (See §§ 29–34 in [24].) This implies that the definition is independent of the choices made above; in particular it is independent of the orientation of M. Also, from obstruction theory it follows that $\text{Index}(X_1, \ldots, X_k) = 0$ iff there is a k-field without singularities on M which coincides with $(X_1, \ldots, X_k)$ on the $(m-2)$-skeleton of M. (See 34.2 of [24].)

A 1-field X on M is simply a field of unit tangent vectors. Since $V_{m,1} = (m-1)$-sphere and $\pi_{m-1}(V_{m,1}) = Z$, we may regard Index (X) as an integer. The celebrated theorem of H. HOPF [12] states that if X is a 1-field with finite singularities on a closed manifold[2]) M, then

$$\text{Index}\,(X) = \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M.

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[1]) Research supported by the National Science Foundation.

[2]) By using local coefficients one can define the index on a non-orientable manifold (See [24, §39.5].)

Let (X_1, X_2) be a 2-field with finite singularities on a closed oriented manifold M of dim m, with $m>4$. The index of (X_1, X_2) is then an element of the homotopy group $\pi_{m-1}(V_{m,2})$. This group depends on the parity of m as is shown below (see [8]):

$$\pi_{m-1}(_{m,2}) = \begin{cases} Z_2 , & \text{if } m \text{ odd} \\ Z \oplus Z_2, & \text{if } m \text{ even.} \end{cases}$$

Thus if m is odd we can regard Index (X_1, X_2) as an integer mod 2. If m is even we write

$$\text{Index}\,(X_1, X_2) = (\text{Index}_0(X_1, X_2), \text{Index}_2(X_1, X_2)),$$

where $\text{Index}_0(X_1, X_2) \in Z$, $\text{Index}_2(X_1, X_2) \in Z_2$. It is easily shown (see § 7 below) that $\text{Index}_0(X_1, X_2) = \chi(M)$. In a previous paper [27] we have proved: *If* $m \equiv 2$ or 3 mod 4, *and if* (X_1, X_2) *is a 2-field with finite singularities, then*

$$\begin{aligned} \text{Index}_2(X_1, X_2) &= 0, \quad \text{if} \quad m \equiv 2(4), \\ \text{Index}\ (X_1, X_2) &= 0, \quad \text{if} \quad m \equiv 3(4). \end{aligned}$$

The purpose of this paper is to consider 2-fields on m-manifolds where $m \equiv 0$, 1 mod 4.

The case of 4-manifolds has been completely solved by F. HIRZEBRUCH and H. HOPF [11]. For the rest of the section let M denote a closed oriented manifold of dim m, with $m>4$. Let $w_i M \in H^i(M; Z_2)$ denote the i^{th} Stiefel-Whitney class of M, $i \geq 1$. Recall (see § 39.1 in [24]) that if m is odd then M has a 2-field with finite singularities iff $w_{m-1} M = 0$, while if m is even then M has such a 2-field iff $\delta^* w_{m-2} M = 0$. (Here δ^* denotes the Bockstein coboundary from mod 2 coefficients to integer coefficients.) MASSEY [17] has shown that if m is even then one always has $\delta^* w_{m-2} M = 0$. Thus an orientable manifold of even dimension always has a 2-field with finite singularities.

Define

$$\chi^+ M = \sum_i \dim H_i(M; Z_2).$$

If $\chi^+ M$ is an even integer (as will be the case, for example, when m is odd), we define[3]) an integer mod 2 by

$$\hat{\chi}_2 M = \tfrac{1}{2} \chi^+ M \bmod 2.$$

We will prove the following result. (Recall that M is called a *spin* manifold if $w_2 M = 0$.)

THEOREM 1.1. *Let M be a closed spin manifold of* dim $4k+1$, $k>0$, *such that* $w_{4k} M = 0$. *If* (X_1, X_2) *is any 2-field with finite singularities, then*

$$\text{Index}\,(X_1, X_2) = \hat{\chi}_2 M.$$

As an immediate consequence we have

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[3]) See KERVAIRE, Math. Ann. *131* (1956) 220.

COROLLARY 1.2. *Let M be a closed spin manifold of* dim $4k+1$, $k>0$. *Then M has a 2-field without singularities if, and only if,*

$$w_{4k}M=0, \qquad \hat{\chi}_2 M = 0.$$

In case M is a π-manifold, this is given as part of Theorem 2 in [6].

The case $m \equiv 0 \bmod 4$ requires an additional hypothesis. Let M be a manifold of even dimension, say $2q$. We call M *symplectic* if, for all classes $u \in H^q(M; Z_2)$, $u^2=0$. We show below that if M is a spin manifold of dim $8k+4$, $k \geq 0$, then M is symplectic. Also, we will show that if M is symplectic then $w_{2q}M=0$, and so the Euler characteristic of M is an even integer. Therefore, by Poincaré duality, it follows that $\chi^+ M$ is also even and so $\hat{\chi}_2 M$ is defined. We will prove

THEOREM 1.3. *Let M be a closed spin manifold of* dim m, *where* $m \equiv 0 \bmod 4$ *and* $m>4$. *If* $m \equiv 0 \bmod 8$ *assume that M is symplectic. Then for any* 2-field (X_1, X_2) *with finite singularities.*

$$\mathrm{Index}_2(X_1, X_2) = \hat{\chi}_2 M .$$

Suppose that dim $M=4k$, $k>0$; set $d_i = \dim H_i(M; Z_2)$. By Poincaré duality,

$$\chi(M) = \sum_{i=0}^{2k-1} (-1)^i\, 2d_i + d_{2k},$$

$$\chi^+ M = \sum_{i=0}^{2k-1} 2d_i + d_{2k}.$$

Therefore,

$$\chi^+ M = \left(\sum_{i=0}^{2k-1} 2(1-(-1)^i)\, d_i\right) + \chi(M),$$

and so if $\chi(M)$ is even

$$\hat{\chi}_2 M = (\tfrac{1}{2}\chi(M)) \bmod 2 .$$

In particular

$$\hat{\chi}_2 M = 0 \quad \text{if, and only if,} \quad \chi(M) \equiv 0 \bmod 4 .$$

As a consequence we have

COROLLARY 1.4. *Let M be a closed spin manifold as in* 1.3. *Then M has a 2-field without singularities if, and only if,* $\chi(M)=0$.

Recall that a manifold M of even dimension $2q$ is said to have an *almost-complex* structure if there is a complex q-plane bundle ω over M such that the tangent bundle of M is equivalent to the real bundle underlying ω. Now this complex bundle ω has a complex 1-field with finite singularities, and the index of this 1-field is simply $\chi(M)$ [19, pp. 61, 65]. Moreover the complex 1-field determines a (real) 2-field on M also with finite singularities and for this 2-field (X_1, X_2), $\mathrm{Index}_2(X_1, X_2) = b w_{2q} M$, $b \in Z_2$. Thus by 1.3 and the computation given above for $\hat{\chi}_2 M$, we obtain

COROLLARY 1.5. *Let M be a closed spin manifold as in* 1.3. *If M admits an almost-complex structure, then the Euler characteristic of M is divisible by* 4.

This argument was originally used by HOPF [13] to show that S^4 and S^8 do not admit almost-complex structures.

Let M be an m-manifold and let $V=\sum_{i=1}^{m} V_i$ denote the WU class [29]. That is, if $u \in H^{m-i}(M; Z_2)$ then

$$\mathrm{Sq}^i(u) = u \cdot V_i,$$

where Sq^i denotes the mod 2 Steenrod operator of degree i, $i \geq 1$. The Theorem of Wu is that

$$w_k M = \sum_{i=0}^{k} \mathrm{Sq}^i V_{k-i}, \qquad k \geq 1.$$

Thus if m is even, say $m=2q$,

$$w_{2q} M = \mathrm{Sq}^q V_q = V_q^2.$$

But by definition, M is symplectic iff $V_q=0$, and so if M is symplectic then $w_{2q}M=0$, as asserted above. Also, by an easy extension of [16, Theorem III], one shows that if M is a spin m-manifold, then $V_{4k+2}=0$, $k \geq 0$ (since $\mathrm{Sq}^2 H^{m-2}(M; Z_2)=0$). Therefore if $m \equiv 4 \bmod 8$, M is symplectic as remarked above.

2. Proof of 1.1 and 1.3.

Throughout this section M will denote a closed oriented m-manifold, with $m \equiv 0$ or $1 \bmod 4$, $m>4$. We will show in § 7 that if (X_1, X_2) is a 2-field on M with isolated singularities, then the index is independent of the particular choice of 2-field. We define a mod 2 integer, $I_2 M$, by setting

$$I_2 M = \begin{cases} \mathrm{Index}_2(X_1, X_2), & \text{if} \quad m \equiv 0(4) \\ \mathrm{Index}\,(X_1, X_2), & \text{if} \quad m \equiv 1(4). \end{cases}$$

Let T denote the Thom complex of the tangent bundle of M and $U \in H^m(T; Z)$ the Thom class (see [25], [19]). $H^*(T)$ can be regarded as a module over $H^*(M)$ (integer or mod 2 coefficients). By THOM [25] the map $H^i(M) \to H^{m+i}(T)$, given by $x \to U \cdot x$, is an isomorphism for all $i>0$. Thus to determine the mod 2 integer $I_2 M$ it suffices to compute $U \cdot (I_2 M \mu)$, where $\mu \in H^m(M; Z_2)$ is the generator. For this we will need a secondary cohomology operation.

Recall that one has the following ADEM relation [2], when $m \equiv 0$, $1 \bmod 4$.

(*) $$\mathrm{Sq}^2 \mathrm{Sq}^{m-1} + \mathrm{Sq}^m \mathrm{Sq}^1 = \mathrm{Sq}^{m+1}.$$

If u is an integral cohomology class of $\dim < m+1$, then

$$\mathrm{Sq}^1 u = 0, \quad \mathrm{Sq}^{m+1} u = 0.$$

Also, if m is even we can write

$$\mathrm{Sq}^{m-1} = \mathrm{Sq}^1 \mathrm{Sq}^{m-2} = (\delta^* \mathrm{Sq}^{m-2}) \bmod 2.$$

Thus we have the following two non-stable relations:

$$\begin{aligned} m \equiv 0(4)&: \mathrm{Sq}^2(\delta^* \mathrm{Sq}^{m-2}) = 0, \\ m \equiv 1(4)&: \mathrm{Sq}^2 \mathrm{Sq}^{m-1} = 0, \end{aligned} \tag{2.1}$$

where in each case the relation obtains on integral classes of $\dim \leq m$.

Let Ω_m denote a (non-stable) secondary cohomology operation associated with each of the above relations, $m \equiv 0,1 \bmod 4$. (See [1] and [7].) Thus if X is a space and if $u \in H^j(X; Z)$, $j \leq m$, then Ω_m is defined on u, provided that

$$\delta^* \mathrm{Sq}^{m-2} u = 0 \quad \text{if} \quad m \equiv 0(4), \qquad \mathrm{Sq}^{m-1} u = 0 \quad \text{if} \quad m \equiv 1(4).$$

Furthermore

$$\Omega_m(u) \quad \text{is a coset in} \quad H^{m+j}(X; Z_2)$$

of the subgroup

$$\begin{aligned} &\mathrm{Sq}^2 H^{m+j-2}(X; Z), \quad &\text{if} \quad m \equiv 0(4), \\ &\mathrm{Sq}^2 H^{m+j-2}(X; Z_2), \quad &\text{if} \quad m \equiv 1(4). \end{aligned}$$

We will prove

THEOREM 2.2. *Let M be a closed spin manifold of* dim m, *where* $m \equiv 0$ *or* 1 mod 4 *and* $m > 4$. *If m is odd assume that* $w_{m-1} M = 0$, *while if m even assume that* $w_m M = 0$. *Then the operation Ω_m is defined on the Thom class U and the operation can be chosen so that*

$$\Omega_m(U) = U \cdot (I_2 M \mu).$$

with zero indeterminacy.

This will be proved in § 7, following the method of MAHOWALD-PETERSON [15]. (Theorem 2.2 is similar to Theorem 3.3.2 in [15], but the details of our proof will be somewhat different as we will use the point of view of § 5 in [27]).

To prove 1.1 and 1.3 we need to compute the operation Ω_m. This is done as follows. Assume that the tangent bundle of M has been given a Riemannian metric; let E denote the set of tangent vectors of length ≤ 1, and let E^1 denote the set of vectors of length 1. Then $T = E/E^1$ (= the space obtained from E by collapsing E^1 to a point). Moreover the collapsing map induces an isomorphism

$$H^*(E/E^1, *) \approx H^*(E, E^1),$$

and so we regard the Thom class U equally well as a class in $H^m(E, E^1; Z)$. MILNOR shows in [19] that there is an isomorphism

$$e: H^*(E, E^1) \approx H^*(M^2, M_2 - \text{diagonal}),$$

where $M^2 = M \times M$. Let $j: M^2 \subset (M^2, M^2 - \text{diagonal})$ denote the inclusion, and set

$$\underline{U} = j^* e(U) \in H^m(M^2; Z).$$

Now the isomorphism e is induced by maps and so commutes with all cohomology operations. Thus Ω_m is defined on $\underline{U}$. Assume that $w_2 M = 0$. Then

$$\mathrm{Sq}^2 H^{m-2}(M) = 0, \quad \mathrm{Sq}^2 H^{2m-2}(M^2) = 0,$$

and so Ω_m is defined with zero indeterminacy on U and $\underline{U}$. By naturality,

$$\Omega_m(\underline{U}) = j^* e \, \Omega_m(U).$$

But j^* is injective (as remarked in [3]) and so

$$\Omega_m(\underline{U}) = 0 \quad \text{if, and only if,} \quad \Omega_m(U) = 0.$$

Since a mod 2 integer is unchanged by squaring, we obtain from 2.2,

PROPOSITION 2.3. *Let M be a manifold as in 2.2. Then*

$$\Omega_m(\underline{U}) = I_2 M(\mu \oplus \mu) \in H^{2m}(M^2; Z_2).$$

To compute $\Omega_m(\underline{U})$ we reduce $\underline{U}$ mod 2. Consider the following non-stable relations (see (*)):

$$\begin{aligned} m \equiv 0(4)&: \mathrm{Sq}^2(\delta^* \mathrm{Sq}^{m-2}) + \mathrm{Sq}^m \mathrm{Sq}^1 = 0, \\ m \equiv 1(4)&: \mathrm{Sq}^2 \mathrm{Sq}^{m-1} + \mathrm{Sq}^1(\mathrm{Sq}^{m-1}\mathrm{Sq}^1) = 0, \end{aligned} \tag{2.4}$$

where in each case the relation obtains on mod 2 classes of dim $\leq m$. Let $\tilde{\Omega}_m$ denote a (non-stable) operation associated with each relation in 2.4.

Let M be a manifold as in 2.2. Regarding $\underline{U}$ as a class mod 2, $\tilde{\Omega}_m$ is defined on $\underline{U}$, and with zero indeterminacy when $m \equiv 1$. When $m \equiv 0$, $\tilde{\Omega}_m$ has $\mathrm{Sq}^m H^m(M^2)$ as indeterminacy subgroup. But if M is symplectic them $\mathrm{Sq}^m H^m(M^2) = 0$, and so $\tilde{\Omega}_m(\underline{U})$ will again be defined with zero indeterminacy. By considering the universal examples for Ω and $\tilde{\Omega}$ it is easily shown that, with all these hypotheses on M, $\tilde{\Omega}_m$ can be chosen so that

$$\tilde{\Omega}_m(\underline{U}) = \Omega_m(\underline{U}), \tag{2.5}$$

where Ω_m denotes the specific choice of operation given in 2.2.

Thus, as our final step, we compute $\tilde{\Omega}_m(\underline{U})$. Let $t: H^*(M^2) \to H^*(M^2)$ denote the isomorphism induced by interchanging the factors of M^2.

THEOREM 2.6. *Let M be an m-manifold as in 2.2. If m is even assume that M is symplectic. Then there is a* mod 2 *class* $A \in H^m(M^2)$ *such that*

a) $\underline{U}$ mod $2 = A + tA$,
b) $A \cup tA = \chi_2 M(\mu \otimes \mu)$,
c) $\tilde{\Omega}_m$ *is defined on* A.

The proof will be given in § 4.

Proof of 1.1 *and* 1.3. By 2.3 and 2.5,

$$\tilde{\Omega}_m(\underline{U}) = I_2 M(\mu \oplus \mu).$$

Now $\tilde{\Omega}_m$ is a non-stable operation of degree m. By 2.6 c) $\tilde{\Omega}$ is defined on A and thus also on tA. Therefore, by [7, cf. 2. 3],

$$\tilde{\Omega}(A + tA) = \tilde{\Omega}(A) + \tilde{\Omega}(tA) + A \cup tA.$$

Since t is the identity on $H^{2m}(M^2)$, we have by naturality,

$$\tilde{\Omega}_m(A) = t\tilde{\Omega}_m(A) = \tilde{\Omega}_m(tA).$$

Consequently, by 2.6 a) and b),

$$\tilde{\Omega}_m(\underline{U}) = \tilde{\Omega}_m(A + tA) = A \cup tA = \hat{\chi}_2 M(\mu \oplus \mu).$$

But $\tilde{\Omega}_m(\underline{U}) = I_2 M(\mu \otimes \mu)$, and so

$$I_2 M = \hat{\chi}_2 M,$$

which completes the proof of 1.1 and 1.3.

3. Mod 2 vector spaces

Most of the work in proving Theorem 2.6 will come in the case m even. This section develops some simple facts about mod 2 vector spaces needed for this case. The proof of 2.6 is then given in the next section.

Let V be a finite-dimensional mod 2 vector space. An endomorphism t of V is called an *involution* if $t^2=1$. An endomorphism d is called a *boundary* if $d^2=0$. Suppose that V has an involution t and a boundary d. We say that the pair (t, d) is *regular* if

$$td = dt, \tag{3.1}$$

and

there are subspaces A, B in V such that

$$dB = 0 \quad \text{and} \quad V = A \oplus tA \oplus dA \oplus tdA \oplus B \oplus tB. \tag{3.2}$$

Define

$$\Delta = t + 1 : V \to V.$$

LEMMA 3.3. *Let t be an involution on V and d a boundary such that the pair (t, d) is regular. Then*

$$(\mathrm{Ker}\, d) \cap (\mathrm{Ker}\, \Delta) = \Delta(\mathrm{Ker}\, d).$$

Proof. Because V is a Z_2-module, $\Delta^2=0$. Also by 3.1, $\Delta d = d\Delta$, and so

$$\Delta(\mathrm{Ker}\, d) \subset \mathrm{Ker}\, d \cap \mathrm{Ker}\, \Delta.$$

We prove 3.3 by showing that the opposite inclusion holds. Let $v \in V$ be an element such that

$$d v = 0, \quad \Delta v = 0.$$

By 3.2 we can write v as

$$v = a_1 + t a_2 + d a_3 + t d a_4 + b_1 + t b_2,$$

where the a's are in A and the b's in B. Since $dv=0$ and $dB=0$, we must have

$$d a_1 = d t a_2 = 0.$$

Furthermore

$$\Delta v = (a_1 + a_2) + (t a_1 + t a_2) + (d a_3 + d a_4) + (t d a_3 + t d a_4) + (b_1 + b_2) + (t b_1 + t b_2).$$

Since $\Delta v=0$ this means, by 3.2, that

$$a_1 = a_2, \quad d a_3 = d a_4, \quad b_1 = b_2.$$

Therefore

$$v = \Delta(a_1 + d a_3 + b_1), \quad \text{and} \quad d(a_1 + d a_3 + b_1) = 0,$$

which completes the proof.

Let X be a space whose total singular integral homology module is finitely generated. Let $H^*(X)$ denote the mod 2 cohomology algebra of X. By the Künneth theorem for cohomology,

$$H^*(X^2) \approx H^*(X) \otimes H^*(X),$$

where $X^2 = X \times X$.

Let $t: H^*(X^2) \to H^*(X^2)$ denote the involution induced by transposing the factors of X^2. We will call an element $v \in H^*(X^2)$ *symmetric* if $\Delta v=0$, where $\Delta = t+1$. Let $\alpha = (\alpha_1, \ldots, \alpha_q)$ be a basis for $H^*(X^2)$. An element $v \in H^*(X^2)$ will be called *symplectic* with respect to α if

$$v = \sum_{i,j} c_{ij} \alpha_i \otimes \alpha_j,$$

where all $c_{ii}=0$, $1 \le i \le q$.

LEMMA 3.4. *Let $v \in H^*(X^2)$ be a symmetric class. If v is symplectic with respect to one basis, then it is so with respect to any basis.*

Proof. With respect to a second basis for $H^*(X)$, the matrix $C=(c_{ij})$ becomes a matrix $C'=(c'_{ij})$, which is obtained from C by symmetric row and column operations [27, p. 188]. Thus C' is also symmetric. Moreover each such pair of row and column operations leaves unchanged the diagonal elements of C (since $c_{ii}=0$ and we are working over Z_2). Thus C' remains symplectic, i.e., $c'_{ii}=0$, $1 \le i \le q$. This completes the proof.

The main result of the section is the following.

PROPOSITION 3.5. *Let $v \in H^{2n}(X^2)$, $n>0$. Suppose that*

$$\Delta v = 0, \quad \mathrm{Sq}^1 v = 0$$

and that v is symplectic. Then there is a class u such that

$$\Delta u = v, \quad \mathrm{Sq}^1 u = 0.$$

Proof. Set $d = \mathrm{Sq}^1$. Then $d^2 = 0$ and $td = dt$. We choose a basis $\alpha_1, \dots, \alpha_q$ for $H^*(X)$ so that for some integer r,

$$\begin{aligned} d\alpha_i &= \alpha_{r+i}, & 1 \le i \le r, \\ d\alpha_j &= 0, & 2r+1 \le j \le q. \end{aligned}$$

Define $W \subset H^*(X^2)$ to be the subspace spanned by all basis elements $\alpha_i \otimes \alpha_j$, with $i \neq j$. Notice that the class v is in W because v is symplectic.

Now set $s = q - 2r$, and let $b_i = \alpha_{2r+i}$, $1 \le i \le s$, where r and q are given above. Define A, $B \subset W$ to be the subspaces spanned by the basis elements shown below:

$$\begin{aligned} A: \{&\alpha_i \otimes \alpha_j, \, d\alpha_i \otimes \alpha_j, \, 1 \le i < j \le r; \\ &\alpha_i \otimes d\alpha_j, \, 1 \le i \le j \le r; \\ &\alpha_i \otimes b_j, \, 1 \le i \le r, \, 1 \le j \le s.\}. \\ B: \{&d\alpha_i \otimes d\alpha_j, \, 1 \le i < j \le r; \\ &d\alpha_i \otimes b_j, \, 1 \le i \le r, \, 1 \le j \le s; \\ &b_i \otimes b_j, \, 1 \le i < j \le s.\}. \end{aligned}$$

Then, as is readily seen,

$$(*) \qquad W = A \oplus tA \oplus B \oplus tB, \quad dB = 0.$$

For any subspace $U \subset H^*(X^2)$, set $U^i = U \cap H^i(X^2)$, $i \ge 0$. Notice that the classes $d\alpha_i \otimes \alpha_i$, $\alpha_i \otimes d\alpha_i$ do not occur in A^{2p}, for any $i, p > 0$. Thus

$$dA^{2p} \cap dtA^{2p} = 0,$$

and so

$$(**) \qquad dW^{2p} = dA^{2p} \oplus dtA^{2p}, \quad p > 0.$$

Suppose now that the class v, given in 3.5, has degree $2n$, $n > 0$. We set

$$V = W^{2n} \oplus dW^{2n}.$$

By (*) and (**),

$$V = A^{2n} \oplus tA^{2n} \oplus B^{2n} \oplus tB^{2n} \oplus dA^{2n} \oplus dtA^{2n}.$$

Consequently the pair (t, d) is regular on V. By hypothesis $\Delta v = 0$, $dv = 0$, and so by 3.3 there is a class $u \in W^{2n}$ such that

$$\Delta u = v, \quad du = \mathrm{Sq}^1 u = 0.$$

This completes the proof.

4. Proof of Theorem 2.6

We retain the notation of §§ 2, 3. Let M be an m-manifold and let $\alpha_1, \dots, \alpha_q$ be a basis for $H^*(M)$ (mod 2 coefficients). Define y_{ij} to be the value of $\alpha_i \cup \alpha_j$ on the fundamental mod 2 homology class $[M]$. In particular $y_{ij}=0$ if $\deg \alpha_i + \deg \alpha_j \neq m$; and $y_{ij}=y_{ji}$, $1 \leq i, j \leq q$. Let Y be the $q \times q$ matrix (y_{ij}) and set $C=Y^{-1}$. Then by Milnor [19],

(*) $$\underline{U} = \sum_{i,j} c_{ij}\, \alpha_i \otimes \alpha_j,$$

where $C=(c_{ij})$. Since Y is symmetric so is C.

Notice that $q=\chi^+ M$. By the hypotheses of 2.6, q is even, say $q=2d$. We choose the basis $\{\alpha_i\}$ in a special way. Suppose first that m is odd, say $m=2k+1$. Let $\alpha_1, \dots, \alpha_d$ be an arbitrary basis for the graded vector space

$$\sum_{i=0}^{k} H^i(M).$$

By Poincaré duality, $H^i(M)$ and $H^{m-i}(M)$ are orthogonally paired by the cup-product. Consequently we can choose a basis $\beta_1, \dots, \beta_d$ for

$$\sum_{i=0}^{k} H^{m-i}(M)$$

such that if $\deg \alpha_i + \deg \beta_j = m$, then

$$\alpha_i \cup \beta_j = \delta_{ij}\, \mu .$$

Take as total basis for $H^*(M)$ the elements $\{\alpha_1, \dots, \alpha_d, \beta_d, \dots, \beta_1\}$. Then the matrix Y has the form shown below:

$$Y = \begin{bmatrix} & & & & 1 \\ & 0 & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & 0 & \\ 1 & & & & \end{bmatrix}.$$

Thus $C=Y$ and so by (*) we obtain

$$\underline{U} = \sum_{i=1}^{d} \alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i . \tag{4.1}$$

Suppose on the other hand that m is even, say $m=2k+2$. Let $\{\alpha_1, \dots, \alpha_r\}$, $\{\beta_1, \dots, \dots, \beta_r\}$ be bases for the respective vector spaces

$$\sum_{i=0}^{k} H^i(M), \quad \sum_{i=0}^{k} H^{m-i}(M),$$

chosen as above so that

$$\alpha_i \cup \beta_j = \delta_{ij}\,\mu\,,$$

if $\deg \alpha_i + \deg \beta_j = m$. Assume, as in 2.6, that M is symplectic. Then (see [28]) one can choose a basis $x_1, \dots, x_s, y_1, \dots, y_s$ for $H^{k+1}(M)$ such that

$$x_i \cup x_j = 0\,, \quad y_i \cup y_j = 0\,, \quad x_i \cup y_j = \delta_{ij}\,\mu\,.$$

Now by definition

$$2(r+s) = q = 2d\,.$$

Set

$$\alpha_{r+i} = x_i\,, \quad \beta_{r+i} = y_i\,, \quad 1 \le i \le s\,.$$

Then $\{\alpha_1, \dots, \alpha_d, \beta_d, \dots, \beta_1\}$ is a basis for $H^*(M)$ yielding as above

$$\underline{U} = \sum_{i=1}^{d} \alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i\,. \tag{4.2}$$

For m even or odd we set

$$A = \sum_{i=1}^{d} \alpha_i \otimes \beta_i\,.$$

Then by (4.1) and (4.2), $\underline{U} = A + tA$, which proves 2.6 i). Now

$$(\alpha_i \otimes \beta_i) \cup (\beta_j \otimes \alpha_j) = (\alpha_i \beta_j \otimes \beta_i \alpha_j) = 0$$

unless $i=j$. For if $\deg \alpha_i + \deg \beta_j = m$, then by definition $\alpha_i \cup \beta_j = \delta_{ij}\mu$, while if $\deg \alpha_i + \deg \beta_j \ne m$ then one of the pairs $\alpha_i \beta_j$, $\beta_i \alpha_j$ has degree greater than m and so is zero. Thus

$$A \cup tA = \sum_{i=1}^{d} \alpha_i \beta_i \otimes \alpha_i \beta_i = d(\mu \otimes \mu) = \chi_2 M(\mu \otimes \mu)\,,$$

since $2d = q = \chi^+ M$. Therefore the class A satisfies 2.6 ii).

To prove 2.6 iii) we need the following lemma.

LEMMA 4.3. *Let M be an orientable manifold of* $\dim m$, $m > 1$. *Let $u \in H^r(M)$, $v \in H^s(M)$, where $r+s=m$ and $0 < r \le s$.*

a) *Suppose that $m \equiv 0 \bmod 4$. If $r<s$, then*

$$\delta^* \operatorname{Sq}^{m-2}(u \otimes v) = 0\,.$$

If $r=s$, then

$$\delta^* \operatorname{Sq}^{m-2}(u \otimes v) = \delta^* \operatorname{Sq}^{r-2} u \otimes v^2 + u^2 \otimes \delta^* \operatorname{Sq}^{r-2} v\,.$$

b) *Suppose that m is odd. If $r < s-1$, then*

$$\operatorname{Sq}^{m-1}(u \otimes v) = 0\,.$$

If $r=s-1$, then

$$\operatorname{Sq}^{m-1}(u \otimes v) = u^2 \otimes \operatorname{Sq}^{s-1} v\,.$$

c) *Suppose that m is odd and that $w_2 M=0$. Then*

$$\mathrm{Sq}^{m-1}\,\mathrm{Sq}^1\,H^m(M^2)=0.$$

The proof of (a) and (b) follows at once by the Cartan formula, using the fact that $H^m(M;Z)\approx Z$. Thus

$$\delta^* H^{m-1}(M)=\mathrm{Sq}^1 H^{m-1}(M)=0.$$

We leave the details of the proof to the reader. For (c) suppose that $m=2k+1$. Then by Adem [2],

$$\begin{aligned}\mathrm{Sq}^{m-1}\,\mathrm{Sq}^1=\mathrm{Sq}^{2k}\,\mathrm{Sq}^1&=\mathrm{Sq}^2\,\mathrm{Sq}^{2k-1}+\varepsilon\,\mathrm{Sq}^{2k+1}\\&=\mathrm{Sq}^2\,\mathrm{Sq}^{2k-1}+\varepsilon\,\mathrm{Sq}^1\,\mathrm{Sq}^{2k}\end{aligned}$$

where $\varepsilon=0$ or 1. But

$$\mathrm{Sq}^2 H^{2m-2}(M^2)=0,\qquad \mathrm{Sq}^1 H^{2m-1}(M^2)=0,$$

since $w_1 M=w_2 M=0$. Therefore $\mathrm{Sq}^{m-1}\,\mathrm{Sq}^1 H^m(M^2)=0$, as claimed, which completes the proof of the lemma.

Proof of 2.6 iii). We must show that the operation Ω_m is defined on the class A.

Case I: $m\equiv 1 \bmod 4$. By 2.4 this means we must show that

$$\mathrm{Sq}^{m-1}\,\mathrm{Sq}^1 A=0,\qquad \mathrm{Sq}^{m-1} A=0.$$

The first assertion follows by 4.3 (c). To prove the second assertion, we assume that the basis $\alpha_1,\ldots,\alpha_d$ is ordered so that

$$\deg\alpha_i\le\deg\alpha_{i+1},\qquad 1\le i\le q-1.$$

Suppose that $\alpha_j,\ldots,\alpha_d$ are precisely those basis elements with degree $(m-1)/2$. Then by 4.3 (b),

$$\mathrm{Sq}^{m-1} A=\sum_{i=j}^{d}\alpha_i^2\otimes\mathrm{Sq}^{s-1}\beta_i,$$

where $s-1=(m-1)/2$. Consequently,

$$\mathrm{Sq}^{m-1}\,tA=t\,\mathrm{Sq}^{m-1}A=\sum_{i=j}^{d}\mathrm{Sq}^{s-1}\beta_i\otimes\alpha_i^2.$$

Now $\underline{U}=A+tA$, and by § 2 we know that $\mathrm{Sq}^{m-1}\,\underline{U}=0$, which means that

$$\mathrm{Sq}^{m-1}A+\mathrm{Sq}^{m-1}tA=0.$$

But, as is seen by the above calculation, $\mathrm{Sq}^{m-1}A$ and $\mathrm{Sq}^{m-1}tA$ occur in disjoint summands of the bi-graded vector space $H^*(M)\otimes H^*(M)$. Namely, $\mathrm{Sq}^{m-1}A$ has bi-degree $(m-1,m)$, while $\mathrm{Sq}^{m-1}tA$ has bi-degree $(m,m-1)$. Thus $\mathrm{Sq}^{m-1}A=0$, as claimed, which completes the proof of case I.

CASE II: $m \equiv 0 \bmod 4$. We will show that the class A can be replaced by a class B, which will continue to satisfy 2.6 i) and ii) and for which

$$\delta^* \mathrm{Sq}^{m-2} B = 0, \qquad \mathrm{Sq}^1 B = 0.$$

Thus the class B will satisfy 2.6 iii) (see 2.4) and so the proof of 2.6 will be completed.

By 4.1 (a) we see that $\delta^* \mathrm{Sq}^{m-2} H^m(M^2)=0$; for if the classes u and v in 4.1 (a) have degree $m/2$, then $u^2=v^2=0$, since M is symplectic by hypothesis.

In general it is not necessarily true that $\mathrm{Sq}^1 A=0$. Thus we must find a new class B, satisfying 2.6 i) and ii), such that $\mathrm{Sq}^1 B=0$.

As usual we set $\Delta=1+t$. Then $\Delta \underline{U}=0$, and so by 3.5 there is a class $B \in H^m(M^2)$ such that

$$\Delta B = \underline{U}, \qquad \mathrm{Sq}^1 B = 0.$$

Set $D=B-A$; since $\Delta A=\underline{U}$ it follows that $\Delta D=0$. Moreover,

$$B \cup tB = (A+D) \cup (tA+D) = A \cup tA + A \cup D + D \cup tA + D \cup D.$$

Since M is symplectic, an easy argument shows that M^2 is too; therefore $D \cup D=0$. In a moment we show that $A \cup D=D \cup tA$. This then implies that

$$B \cup tB = A \cup tA = \hat{\chi}_2 M(\mu \otimes \mu).$$

Thus the class B satisfies 2.6 (i)–(iii), and so the proof of 2.6 is complete.

We are left with showing that $A \cup D=D \cup tA$. By commutativity of the cup-product, $A \cup D=D \cup A$. Furthermore, since t is the identity on $H^{2m}(M^2)$, we have by naturality of the cup-product,

$$A \cup D = D \cup A = t(D \cup A) = tD \cup tA.$$

But $tD=D$ since $\Delta D=0$. Thus, $A \cup D=D \cup tA$ as claimed.

5. The relative Thom complex

Let ξ be an oriented n-plane bundle over a space B and suppose that ξ has a Riemannian metric [19, p. 21]. Denote by E, E^1 the respective subspaces of the total space of ξ consisting of those vectors of norm ≤ 1 and those of norm 1. (In order to avoid confusion we may sometimes write these spaces as $E(\xi)$, $E^1(\xi)$.) We define the Thom complex $T(\xi)$ to be E/E^1.

Let B' be a space and $f: B' \to B$ a map. Let $f^*\xi$ denote the bundle over B' induced from ξ by f. Give $f^*\xi$ the induced Riemannian metric. Then the natural bundle map $\bar{f}: f^*\xi \to \xi$ induces a map

$$T(f): T(f^*\xi) \to T(\xi).$$

Let B'' be a second space and $g:B''\to B'$ a map. Then, up to homeomorphism,

$$\begin{aligned} T(g^* f^* \xi) &= T((fg)^* \xi), \\ T(f)\cdot T(g) &= T(fg). \end{aligned} \tag{5.1}$$

Suppose that A is a subspace of B. Then the inclusion $A\subset B$ induces an inclusion $T(\xi_A)\subset T(\xi)$, where $\xi_A=\xi|A$. Thus, if $f:(B', A')\to(B, A)$ is a map of pairs, we obtain a map of pairs

$$T(f):(T(f^*\xi), T(f^*\xi_A))\to(T(\xi), T(\xi_A)).$$

Now let $U\in H^n(E, E^1)$ denote the Thom class of the bundle ξ and let $p:E\to B$ denote the projection. Thom shows that the homomorphism

$$H^i(B)\to H^{n+i}(E, E^1),$$

given by $x\to p^*x\cup U$, is an isomorphism ($i\geq 0$). Since the pair (E, E^1) enjoys the homotopy-extension property (e.g., we can regard E as the mapping cylinder of $p|E^1$), the collapsing map $(E, E^1)\to(T(\xi),*)$ induces an isomorphism in cohomology. Following THOM we define

$$\psi_B: H^i(B)\approx H^{n+i}(T(\xi), *)$$

to be the composite isomorphism. We prove[4])

LEMMA 5.2. *Let A be a closed subspace of B. Set $T_B=T(\xi)$, $T_A=T(\xi_A)$. Then there is a homomorphism*

$$\psi_{B,A}: H^q(B, A)\to H^{n+q}(T_B, T_A)$$

with the following properties.

a) *The following diagram is commutative:*

$$\begin{array}{ccccccccc} \cdots\to H^q(B, A) & \xrightarrow{j^*} & H^q(B) & \xrightarrow{i^*} & H^q(A) & \xrightarrow{\delta} & H^{q+1}(B, A) & \to\cdots \\ \downarrow{\scriptstyle \psi_{B,A}} & & \downarrow{\scriptstyle \psi_B} & & \downarrow{\scriptstyle \psi_A} & & \downarrow{\scriptstyle \psi_{B,A}} & \\ \cdots\to H^{q+n}(T_B, T_A) & \xrightarrow{j^*} & H^{q+n}(T_B) & \xrightarrow{i^*} & H^{q+n}(T_A) & \xrightarrow{\delta} & H^{q+n+1}(T_B, T_A) & \to\cdots \end{array}$$

Here i^*, j^* denote homomorphisms induced by inclusions and δ is the coboundary operator.

b) *$\psi_{B,A}$ is an isomorphism for all q.*

c) *Let $f:(B', A')\to(B, A)$ be a map of pairs. Then the following diagram commutes:*

$$\begin{array}{ccc} H^q(B, A) & \xrightarrow{f^*} & H^q(B', A') \\ \downarrow{\scriptstyle \psi_{B,A}} & & \downarrow{\scriptstyle \psi_{B',A'}} \\ H^{q+n}(T_B, T_A) & \xrightarrow[T(f)^*]{} & H^{q+n}(T_{B'}, T_{A'}). \end{array}$$

[4]) The result is well known, but I am unaware of a reference.

d) *Let* $x \in H^*(B, A)$, mod 2 *coefficients. Then,*

$$\mathrm{Sq}^k \psi_{B,A}(x) = \sum_{i+j=k} \psi_{B,A}(w_i \xi \cup \mathrm{Sq}^j x).$$

Proof: Following Spanier we define the *relative Thom pair* of the bundles (ξ, ξ_A) to be the pair $(E, E_A \cup E^1)$, where $E_A = E(\xi_A)$. Let $p' : (E, E_A) \to (B, A)$ denote the projection. Notice that if $x \in H^i(B, A)$, then

$$p'^* x \cup U \in H^{n+i}(E, E_A \cup E^1),$$

and so we obtain a homomorphism

$$\psi'_{B,A}: H^i(B, A) \to H^{n+i}(E, E_A \cup E^1), \qquad i \geq 0.$$

If A is empty then $\psi'_{B,A}$ is simply the isomorphism ψ_B given above.

Notice that if we collapse E^1 to a point in the pair $(E, E_A \cup E^1)$ we obtain the pair (T_B, T_A). Thus by the 5-lemma the collapsing map induces an isomorphism

$$H^*(E, E_A \cup E^1) \approx H^*(T_B, T_A).$$

We define $\psi_{B,A}: H^i(B, A) \to H^{n+i}(T_B, T_A)$ to be the composition of $\psi'_{B,A}$ with the isomorphism given above. The properties of $\psi_{B,A}$ will then follow from the analogous properties of $\psi'_{B,A}$. We proceed to develop the properties of $\psi'_{B,A}$.

By SPANIER [23, 5.4.9] we see that there is a coboundary operator

$$\Delta: H^j(E_A, E^1_A) \to H^{j+1}(E, E_A \cup E^1)$$

so that the following diagram commutes and has exact rows.

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{j^*} & H^q(B) & \xrightarrow{i^*} & H^q(A) & \xrightarrow{\delta} & H^{q+1}(B, A) & \xrightarrow{j^*} \cdots \\
& \downarrow \psi_B & & \downarrow \psi_A & & \downarrow \psi'_{B,A} & \\
\cdots \xrightarrow{j^*} & H^{q+n}(E, E^1) & \xrightarrow{i^*} & H^{q+n}(E_A, A^1_A) & \xrightarrow{\Delta} & H^{q+n+1}(E, E_A \cup E^1) & \xrightarrow{j^*} \cdots
\end{array}$$

(Because E and E_A are disk bundles the excision properties required in [23] are easily seen to be satisfied.) Since ψ_B and ψ_A are isomorphisms, it follows from the 5-lemma that $\psi'_{B,A}$ is an isomorphism.

Suppose that $f:(B', A') \to (B, A)$. Then one easily sees that f induces a map $f:(E_{B'}, E_{A'} \cup E^1_{B'}) \to (E, E_A \cup E^1)$, where $E_{B'} = E(f^* \xi)$, $E_{A'} = E(f^* \xi_A)$. Thus the following diagram commutes:

$$\begin{array}{ccc}
H^q(B, A) & \xrightarrow{f^*} & H^q(B', A') \\
\downarrow \psi'_{B,A} & & \downarrow \psi'_{B',A'} \\
H^{q+n}(E, E_A \cup E^1) & \xrightarrow[f^*]{} & H^{q+n}(E_{B'}, E_{A'} \cup E^1_{B'}).
\end{array}$$

Suppose finally that $x \in H^*(B, A)$. Then

$$\begin{aligned} \mathrm{Sq}^k(\psi'_{B,A}x) &= \mathrm{Sq}^k(p'^*x \cup U) = \sum_{i+j=k} p'^*\mathrm{Sq}^i x \cup \mathrm{Sq}^j U = \\ &= \sum_{i+j=k} p'^*\mathrm{Sq}^i x \cup (p^* w_j \xi \cup U) = \\ &= \sum_{i+j=k} p'^*(\mathrm{Sq}^i x \cup w_j \xi) \cup U = \sum_{i+j=k} \psi'_{B,A}(\mathrm{Sq}^i x \cup w_j \xi). \end{aligned}$$

(Here $w_j \xi$ denotes the j-th Stiefel-Whitney class of ξ, $j \geq 0$.) The proof of 5.2 now follows from these properties of $\psi'_{B\,A}$ and the definition of $\psi_{B,A}$.

Remark. As indicated in § 2, we sometimes will regard the Thom class U as an element of $H^n(T(\xi),*)$ – i.e., $U = \psi_B(1)$ – and then we write $\psi_B(x) = U \cdot x$, for $x \in H^i(B)$.

6. Lifting the Postnikov invariant

We suppose now that all spaces have basepoint (written *), and that all maps preserve basepoints.

Let B, B' be complexes, and $\pi: B' \to B$ a map. Let $w \in H^n(B; J)$, where $J = Z$ or Z_p, p a prime. Suppose that $w \neq 0$ but that $\pi^* w = 0$. We regard w as a map $B \to K(J, n)$ and let

$$\Omega K(J, n) \xrightarrow{i} E \xrightarrow{p} B$$

denote the principal fibration over B induced by w. (See [26]). Since $\pi^* w = 0$, there is a map $q: B' \to E$ such that $pq = \pi$. That is, we have the following commutative diagram, where $F = \Omega K(J, n)$:

$$(*) \qquad \begin{array}{ccc} & & F \\ & & \big\downarrow i \\ & & E \\ & \overset{q}{\nearrow} & \big\downarrow p \\ B' & \xrightarrow{\;\pi\;} & B. \end{array} \qquad \pi = pq$$

Let $k \in H^*(E, Z_p)$ be a class such that $q^* k = 0$. In our applications π will be a fiber map and k will be a Postinikov invariant for π. However in this section we consider k in the more general setting given above, and we study the problem of expressing such a class k in terms of cohomology invariants determined by B.

Suppose that k has degree t. We assume that the mod p cohomology morphism π^*

is surjective in degree t and that $t<2n-2$. Then there is an element α of the mod p Steenrod algebra such that

$$i^* k = \alpha\iota,$$

where ι denotes the fundamental class of $\Omega K(J, n)$.

For simplicity we now assume that $p=2$. We will say the class w is *realizable* if:

(6.1) *there is a vector bundle ξ over B (of* dim s, *say) such that*

$$w = w_n \xi.$$

Furthermore, if $J=Z$, we assume that $w \not\equiv 0$ mod 2.

Let T and U denote the Thom complex and class of the bundle ξ. If Y is any space and $g: Y \to B$ a map, we let T_Y, U_Y denote the Thom complex and class of $g^*\xi$.

Recall the cohomology operation α given above. We will say that the pair (w, α) is *admissible* if the following conditions are fulfilled.

(6.2) *There is a relation*

$$\alpha \mathrm{Sq}^n = 0,$$

which holds on integral cohomology classes of degree $\leq s$.

(6.3) *There is a secondary cohomology operation Ω associated with relation* 6.2 *such that*

$$\Omega(U_{B'}) = T(\pi)^* M,$$

where M is a coset in $H^{s+t}(T)$ of the indeterminacy subgroup of Ω.

Remark 1. If n is odd and $J=Z$, then in 6.1 we regard w_n as $\delta^* w_{n-1}$, while in 6.2, we regard Sq^n as $\delta^* \mathrm{Sq}^{n-1}$.

Remark 2. Recall that for any space X, Ω has indeterminacy subgroup $\alpha H^*(X; J)$.

Define $\kappa \subset H^t(E)$ to be the coset of k with respect to the subgroup

$$\text{Kernel } q^* \cap \text{Kernel } i^* \cap H^t(E).$$

We prove

THEOREM 6.4. *Let (w, α) be an admissible pair as defined above. Then there is a class $k' \in \kappa$ and a class $m \in H^t(B)$ such that*

$$U_B \cdot m \in M \quad \text{and} \quad U_E \cdot (k' + p^* m) \in \Omega(U_E).$$

Before giving the proof we note the following consequence.

Let X be a complex and $h: X \to B$ a map. Suppose that $h^* w = 0$. Then there is a map $l: X \to E$ such that $p \circ l = h$. By naturality we obtain from 6.4,

COROLLARY 6.5. *For any such map l, $U_X \cdot (l^* k' + h^* m) \in \Omega(U_X)$.*

We precede the proof of 6.4 with some remarks. Consider the following commutative diagram, with the notation defined below.

$$(**)\qquad \begin{array}{ccccc} & & T_F & & \Omega K(J, n+s) \\ & & \big\downarrow Ti & & \big\downarrow \hat{\imath} \\ & & T_E & \xrightarrow{f} & \hat{E} \\ & \overset{Tq}{\nearrow} & \big\downarrow Tp & & \big\downarrow \hat{p} \\ T_{B'} & \xrightarrow[T\pi]{} & T_B & = & T_B \xrightarrow{\psi_B(w)} K(J, n+s). \end{array}$$

The left hand portion of the diagram is obtained from diagram (*) by taking the Thom complex of the various bundles induced from ξ. Commutativity follows from 5.1. The map $\hat{p}$ in the above diagram is the principal fibration induced by the cohomology class $\psi_B(w)$. By 5.2 (c),

$$(T p)^* \psi_B(w) = \psi_E p^* w = 0,$$

and so the map Tp lifts to a map f as shown.

Let $\hat{\imath}$ denote the fundamental class of $\Omega K(J, n+s)$. At the end of the section we prove

LEMMA 6.6. *There is a class $\hat{k} \in H^{t+s}(\hat{E})$ such that*

$$\hat{\imath}^* \hat{k} = \alpha \hat{\imath}, \; T q^* f^* \hat{k} = 0.$$

Moreover, if $\hat{\mathscr{k}}$ denotes the coset in $H^{t+s}(\hat{E})$ of $\hat{k}$ with respect to the subgroup

$$\text{Kernel } \hat{\imath}^* \cap \text{Kernel } (f \circ T q)^* \cap H^{t+s}(\hat{E}),$$

then

$$f^* \hat{\mathscr{k}} \subset U_E \cdot \kappa.$$

We use 6.6 to prove 6.4.

Proof of Theorem 6.4. Since $w = w_n \xi$ it follows from THOM (see 5.2d) that

$$\psi_B(w) = \mathrm{Sq}^n\, U.$$

Thus we can regard the map $\psi_B(w)\colon T_B \to K(J, n+s)$ as the composite of the following maps:

$$T_B \xrightarrow{U} K(Z, s) \xrightarrow{\mathrm{Sq}^n \iota_s} K(J, n+s),$$

where ι_s denotes the fundamental class of $K(Z, s)$.

Let $f\colon T_E \to \hat{E}$ be the map given in diagram (**). Set $\bar{f} = Tq \circ f\colon T_{B'} \to \hat{E}$, and consider the following commutative diagram, where the notation is explained below:

$$\begin{array}{ccccccc}
 & & \Omega K(J, n+s) & = & \Omega K(J, n+s) & & \\
 & & \downarrow \hat{\imath} & & \downarrow j & & \\
 & & \hat{E} & \xrightarrow{v} & Y & & \\
 & \overset{\hat{f}}{\nearrow} & \downarrow \hat{p} & & \downarrow r & & \\
T_{B'} & \xrightarrow[T\pi]{} & T_B & \xrightarrow[U]{} & K(J, s) & \xrightarrow{\mathrm{Sq}^n \iota_s} & K(J, n+s).
\end{array}$$

The map r is the principal fibration with $\mathrm{Sq}^n \iota_s$ as classifying map, and j is the fiber inclusion. Since $\hat{p}$ is defined to be the fibration with $\psi_B(w)$ as classifying map and since $\psi_B(w) = \mathrm{Sq}^n U$, we may regard $\hat{p}$ as the fibration induced by U from r. Thus v is simply the natural map for the induced fibration.

Notice that Y is the universal space for the operation Ω. Let $\omega \in H^{t+s}(Y)$ denote a representative class for Ω, chosen according to the specific choice of Ω given in 6.3. Set $k_0 = v^* \omega \in H^{t+s}(\hat{E})$. Since $j^* \omega = \alpha \hat{\imath}$, we have $\hat{\imath}^* k_0 = \alpha \hat{\imath}$. Furthermore,

$$\hat{f}^* k_0 \in \Omega(T\pi^* U) = \Omega(U_{B'}).$$

But by 6.3 there is then a class $m \in H^t(B)$ such that

$$U \cdot m \in M \quad \text{and} \quad \hat{f}^* k_0 = T\pi^*(U \cdot m).$$

Set $k_0' = k_0 - \hat{p}^*(U \cdot m)$. Then,

$$\hat{\imath}^* k_0' = \hat{\imath}^* k_0 - \hat{\imath}^* \hat{p}^*(U \cdot m) = \hat{\imath}^* k_0 = \alpha \hat{\imath} = \hat{\imath}^* \hat{k},$$
$$\hat{f}^* k_0' = \hat{f}^* k_0 - \hat{f}^* \hat{p}^*(U \cdot m) = \hat{f}^* k_0 - T\pi^*(U \cdot m) = 0.$$

Consequently, by definition of the coset $\hat{\kappa}$, $k_0' \in \hat{\kappa}$. On the other hand $k_0 \in \Omega(\hat{p}^* U)$ and so

$$k_0' + \hat{p}^*(U \cdot m) \in \Omega(\hat{p}^* U).$$

By 6.6 there is a class $k' \in \kappa \subset H^t(E)$ such that

$$f^* k_0' = U_E \cdot k'.$$

Therefore, by naturality,

$$U_E \cdot (k' + p^* m) \in \Omega(U_E),$$

since

$$\hat{p} f = Tp, \; Tp^* U = U_E, \qquad Tp^*(U \cdot m) = U_E \cdot p^* m.$$

Thus k' is the desired class and the proof of 6.4 is complete.

We are left with proving 6.6. Before so doing we prove a preliminary result. Let ξ be the s-plane bundle over B given in 6.1. Now it is easily seen that the Thom complex of $\xi|*$ is simply an s-sphere S^s, which we may regard as embedded in T_B. Since the fiber map $p: E \to B$ maps F to $*$ in B, it follows that $T_p(T_F) = S^s \subset T_B$. Furthermore the map $\psi_B w: T_B \to K(J, n+s)$ can be chosen so that $\psi_B w(S^s) = *$ in $K(J, n+s)$. Since $\hat{E}$ is the fiber space induced by $\psi_B w$, it follows that S^s is embedded in $\hat{E}$ in a natural way. Set $K = K(J, n+s)$. Then, $\hat{p}^{-1}(S^s) = \Omega K \times S^s \subset \hat{E}$, and diagram (**) gives the commutative diagram shown below, where bold face letters denote maps of pairs.

$$\begin{array}{ccc} (T_E, T_F) & \xrightarrow{\mathbf{f}} & (\hat{E}, \Omega K \times S^s) \\ \Big\downarrow \mathbf{T}_P & & \Big\downarrow \hat{\mathbf{p}} \\ (T_B, S^s) & = & (T_B, S^s). \end{array}$$

Set $g = f \mid T_F: T_F \to \Omega K \times S^s$. We use the above diagram to prove

LEMMA 6.7. $g^*(\hat{\imath} \otimes 1) \bmod 2 = \psi_F(\imath) \bmod 2$, *where* $\hat{\imath}$ and $\imath$ *denote respectively the fundamental classes for* ΩK *and* F.

Proof. Let $\mathbf{p}: (E, F) \to (B, *)$ denote the map of pairs determined by p. Since p has w as classifying map, we have

$$\text{(a)} \qquad \delta \imath = -\mathbf{p}^* w \in H^n(E, F);$$

and similarly,

$$\text{(b)} \qquad \delta(\hat{\imath} \otimes 1) = -\mathbf{p}^* \psi_{B,*}(w) \in H^{n+s}(\hat{E}, \Omega K \times S^s).$$

Therefore by naturality and the commutative diagram above

$$\delta g^*(\hat{\imath} \otimes 1) = \mathbf{f}^* \delta(\hat{\imath} \otimes 1) = -\mathbf{f}^* \hat{\mathbf{p}}^* \psi_B w = -\mathbf{T}_p^* \psi_{B,*}(w).$$

By 5.2 (c) and by (a) above,

$$-\mathbf{T}_p^* \psi_{B,*}(w) = -\psi_{E,F} \mathbf{p}^* w = \psi_{E,F}(\delta \imath).$$

But by 5.2 (a), $\psi_{E,F}(\delta \imath) = \delta \psi_F(\imath)$. Thus, we obtain

$$\delta(g^*(\hat{\imath} \otimes 1)) = \delta \psi_F(\imath) \quad \text{in} \quad H^{n+s}(T_E, T_F; J).$$

By SERRE [21, p. 469], $\mathbf{p}^* w \not\equiv 0 \bmod 2$ since (by 6.1) $w \not\equiv 0 \bmod 2$. Thus by (a) above and 5.2 (a),

$$\delta: H^{n+s-1}(T_F; Z_2) \to H^{n+s}(T_E, T_F; Z_2)$$

is injective and so $g^*(\hat{\imath} \otimes 1) = \psi_F(\imath) \bmod 2$, as claimed.

Proof of 6.6. Since $w = w_n \xi$, it follows by THOM that

$$\psi_B w = \mathrm{Sq}^n \psi_B(1) = \mathrm{Sq}^n U.$$

Let α be the mod 2 Steenrod operation given at the neginning of the section. By 6.2, $\alpha \mathrm{Sq}^n = 0$ and so $\alpha \psi_B w = 0$. Applying the SERRE exact sequence [21, p. 468] to the fibration $\hat{p}$ (see diagram (**)), we see that by exactness there is a class $\hat{k} \in H^{t+s}(\hat{E})$ such that

$$\hat{\imath}^* \hat{k} = \alpha \hat{\imath}.$$

Furthermore, by using the exact sequence given in § 3 of [26] (with respect to the map $f \circ T_q : T_B' \to \hat{E}$) it is easily shown that $\hat{k}$ can be chosen so that, in addition, $T_q^* f^* \hat{k} = 0$.

Now the inclusion $\hat{\imath} : \Omega K \subset \hat{E}$ can be factored into the composite

$$\Omega K \xrightarrow{l} \Omega K \times S^s \xrightarrow{\hat{\jmath}} \hat{E},$$

where l is the natural injection and where $\hat{\jmath}$ is the inclusion. Since

$$\hat{\imath}^* \hat{k} = \alpha \hat{\iota},$$

it follows that

$$\hat{\jmath}^* \hat{k} = \alpha(\hat{\imath} \otimes 1).$$

Let $k_1 \in H^t(E)$ be the unique class such that

$$U_E \cdot k_1 = f^* \hat{k} \in H^{t+s}(T_E).$$

We will show that $k_1 \in \kappa$, which then will complete the proof of 6.6.

Using 5.2 we have:

$$U_{B'} \cdot q^* k_1 = T q^* (U_E \cdot k_1) = T q^* f^* \hat{k} = 0.$$

Therefore, $q^* k_1 = 0$. On the other hand,

$$U_F \cdot i^* k_1 = T i^* (U_E \cdot k_1) = T_i^* f^* \hat{k}.$$

But by definition of g and $\hat{\jmath}$, $f \cdot T_i = \hat{\jmath} \cdot g$. Thus

$$T_i^* f^* \hat{k} = g^* \hat{\jmath}^* \hat{k} = g^* (\alpha(\hat{\imath} \otimes 1)),$$

by the above computation. By 6.7,

$$g^* (\alpha(\hat{\imath} \otimes 1)) = \alpha g^* (\hat{\imath} \otimes 1) = \alpha \psi_F(\iota).$$

Now the bundle $i^* p^* \xi$ is trivial and so by 5.2 (d),

$$\alpha \psi_F(\iota) = \psi_F(\alpha \iota).$$

Also, by definition,

$$U_F \cdot i^* k_1 = \psi_F(i^* k_1).$$

Therefore

$$\psi_F(\alpha \iota - i^* k_1) = 0$$

and so $i^* k_1 = \alpha \iota$. Consequently, $k_1 \in \kappa$, which completes the proof of 6.6.

Remark 3. The theory leading up to 6.4 can be generalized in the following way. The single cohomology class w can be replaced by a vector of cohomology classes $w = (w_1, \ldots, w_a)$, with $\pi^* w_i = 0$. By making the appropriate changes in 6.1–6.3 one then can state a more general version of 6.4 so that it includes, for example, Theorem 3.3.2 of [15] as a special case.

Remark 4. Theorem 6.4 (as well as the generalization suggested above) is a special case of Theorem 5.9 in [27]. The Thom class U is a "generating class" for κ, in the language of § 5 of [27].

7. Proof of 2.2

Let n be an integer greater than three and set

$$B' = BS0(n-1),\, B = BS0(n+1).$$

For any group G we let BG denote the classifying space for G defined by MILNOR 21]. We denote the various rotation groups by $S0(q)$, $q\geq 2$.) The inclusion $S0(n-1) \subset S0(n+1)$ induces a map $\pi: B' \to B$. If we regard π as a fiber map, its fiber is the Stiefel manifold $V_{n+1,2}$.

Let X be a complex. Then a map $\xi: X \to B$ can be regarded as an oriented $(n+1)$-plane bundle over X. Moreover this bundle has two linearly independent cross-sections iff the map ξ can be factored through B' via π.

We construct a Postnikov resolution for the map π, through dimension $n+1$, as shown below.

$$\begin{array}{ccccc} K(J,n-1) & \xrightarrow{i} & E & & \\ & {}^{q}\nearrow & \downarrow p & & \\ B' & \xrightarrow{\pi} & B & \xrightarrow{w} & K(J,n). \end{array}$$

Here

$$\begin{array}{llll} J = Z_2, & w = w_n \gamma, & \text{if } n & \text{even} \\ J = Z, & w = \delta^* w_{n-1}\gamma, & \text{if } n & \text{odd}, \end{array}$$

where γ denotes the canonical $(n+1)$-plane bundle over B. The map p is the principal fibration with w as classifying map, and i is the inclusion of the fiber of p into E.

Let F denote the "fiber" of the map q (in the sense of [9]). By the choice of w, we see that F is $(n-1)$-connected and that

$$\pi_n F = Z_2 \text{ or } Z \oplus Z_2,$$

according to whether n is even or odd. Let $\gamma_n \in H^n(F; Z_2)$ denote the fundamental class if n is even; for n odd let it denote the cohomology class corresponding to the homomorphism $Z \oplus Z_2 \to Z_2$ given by projection on the right hand summand. Let $k \in H^{n+1}(E; Z_2)$ denote the transgression of the class γ_n. Then (see [10], [26]),

$$i^* k = \mathrm{Sq}^2 \iota,\, q^* k = 0,$$

where ι denotes the fundamental class of $K(J, n-1)$. Moreover, a simple argument using the transgression operator (e.g., see [18]) shows that

$$\text{Kernel } i^* \cap \text{Kernel } q^* \cap H^{n+1}(E) = \begin{cases} 0, & n \quad \text{even} \\ p^* w_{n+1}, & n \quad \text{odd}. \end{cases} \tag{7.1}$$

Let ξ be a bundle over a complex X as above, and suppose that $\xi^* w=0$. Then the map ξ lifts to the space E. We define

$$k(\xi) = \bigcup_\eta \eta^* k \subset H^{n+1}(X),$$

where the union is over all maps $\eta: X \to E$ such that $p\eta = \xi$. It is easily shown (see [14], [26]) that if n is even then $\xi|X^{n+1}$ lifts to B' iff $0 \in k(\xi)$, while if n is odd then $\xi|X^{n+1}$ lifts to B' iff $\chi(\xi)=0$ and $0 \in k(\xi)$. In particular, $\xi|X^n$ lifts to B'.

Furthermore, by a standard argument ([14], [26]), one sees that $k(\xi)$ is a coset in $H^{n+1}(X)$ of the subgroup $S^{n+1}(X, \xi)$ consisting of all classes of the form

$$\mathrm{Sq}^2(u) + u \cup w_2(\xi),$$

for all $u \in H^{n-1}(X; J)$. In particular if $\mathrm{Sq}^2 u = u \cup w_2 \xi$ for all such u, then $k(\xi)$ consists of a single class. We use the theory of § 6 to compute the coset $k(\xi)$.

At the end of the section we prove

LEMMA 7.2. *Let $n \equiv -1$, or 0 mod 4, $n \geq 4$. Then the operation Ω_{n+1} (see § 2) can be chosen so that*

$$U_{B'} \cdot (w_2 w_{n-1}) \in \Omega_{n+1}(U_{B'})$$

where $U_{B'}$ denotes the Thom class of $\pi^* \gamma$.

By definition, w is realizable as given in 6.1. Furthermore by relation 2.1 and by 7.2 it follows that the pair (w, Sq^2) is admissible, in the sense of 6.2 and 6.3. (To satisfy 6.3 we need only observe that $\pi^*: H^*(B) \to H^*(B')$ is surjective.)

Let T_X, U_X denote the Thom complex and Thom class for the bundle $\xi (= \xi^* \gamma)$. If $S^{n+1}(X, \xi)=0$, then one easily sees that $\mathrm{Sq}^2 H^{2n}(T_X; J)=0$. Therefore, if $\xi^* w=0$, then Ω_{n+1} is defined on U_X with zero indeterminacy.

Notice that by 7.1 κ is a coset of 0 if n is even, while if n is odd κ is a coset of the subgroup generated by $p^* w_{n+1}$. Thus by 6.5 and 7.2, we have

THEOREM 7.3. *Let ξ be an oriented $(n+1)$-plane bundle over X such that*

$$\xi^* w = 0, \qquad S^{n+1}(X, \xi) = 0.$$

Then

$$U_X \cdot (k(\xi) + w_2(\xi) w_{n-1}(\xi) + b_{n+1} w_{n+1}(\xi)) = \Omega_{n+1}(U_X),$$

with zero indeterminacy, where Ω_{n+1} is given in 7.2, where $b_{n+1} \in Z_2$, and where $b_{n+1}=0$ if n is even.

Proof of 2.2. Take X to be a spin manifold M of dim $m=n+1$, and take $\xi=\tau$, its tangent bundle. If $n=4s-1$, MASSEY shows that $\delta^* w_{4s-2}\tau=0$. If $n=4s$, we assume (as in 2.2) that $w_{4s}\tau=0$. Thus in either case $\tau^* w=0$ and so τ restricted to M^n has 2 independent cross-sections – i.e., there is a tangent 2-field on M with isolated singularities. By WU [29], $S^{n+1}(M,\tau)=0$. Thus the class $k(\tau)$ is independent of the particular choice of 2-field. In the language of § 2, $k(\tau)=(I_2 M)\mu$ and so 2.2 follows directly from 7.3 since $w_2 M=0$, and since we assume (in 2.2) that $w_m M=0$, when m is even.

Proof of 7.2. We recall the following facts about Thom complexes, due to ATIYAH [4].

(7.4) (ATIYAH). *Let X be a complex and let η be a vector bundle over X. Let ε denote (in general) the trivial line bundle. Then*

$$T(\eta\oplus\varepsilon)=\Sigma T(\eta),\quad U(\eta\oplus\varepsilon)=\Sigma U(\eta),$$

where *Σ denotes the reduced suspension operator and where T, U denote the appropriate Thom complex and Thom class. Furthermore, if $x\in H^*(X)$, then*

$$\Sigma(U(\eta)\cdot x)=(\Sigma U(\eta))\cdot x.$$

Let γ' denote the canonical $(n-1)$-plane bundle over the classifying space B'. Then

$$\pi^*\gamma=\gamma'\oplus 2\varepsilon,$$

and so by 7.4,

$$T_{B'}=\Sigma^2 T',\qquad U_{B'}=\Sigma^2 U',$$

where T', U' denote the Thom complex and class of γ'. Also by 7.4 we have

$$U_{B'}\cdot(w_2 w_{n-1})=\Sigma^2(U'\cdot w_2 w_{n-1}).$$

But

$$\Sigma^2(U'\cdot w_2 w_{n-1})=\Sigma^2(U'\cdot \mathrm{Sq}^2 U'),$$

since $\mathrm{Sq}^2 U'=U'\cdot w_2$, $U'\cdot w_{n-1}=\mathrm{Sq}^{n-1}U'=(U')^2$ mod 2. Thus

$$U_{B'}\cdot(w_2 w_{n-1})=\Sigma^2(U'\cdot \mathrm{Sq}^2 U'),$$

and so 7.2 is simply a special case of the following result.

LEMMA 7.5. *Let X be a complex and let $u\in H^{m-2}(X)$, $m\equiv 0$ or 1 mod 4. Then Ω_m is defined on $\Sigma^2 u$ and Ω_m can be chosen so that*

$$\Sigma^2(u\cdot \mathrm{Sq}^2 u)\in\Omega_m(\Sigma^2 u),$$

Proof. The proof is similar to that given by MAHOWALD-PETERSON for Theorem 2.2.1 in [15], and so is omitted.

REFERENCES

[1] J. F. ADAMS, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. *72* (1960), 20–104.

[2] J. ADEM, *The relations on Steenrod powers of cohomology classes*, in *Algebraic Geometry and Topology*, Princeton, 1957, 191–238.

[3] J. ADEM-S. GITLER, *Secondary characteristic classes and the immersion problem*, Bol. Soc. Mat. Mexicano *1963*, 53–78.

[4] M. ATIYAH, *Thom complexes*, Proc. London Math. Soc. [third series] *XI* (1961), 291–310.

[5] W. BARCUS-J. P. MEYER, *The suspension of a loop space*, Amer. J. Math. *80* (1958), 895–920.

[6] G. BREDON-A. KOSINSKI, *Vector fields on π-manifolds*, Ann. of Math., *84* (1966), 85–90.

[7] E. BROWN-F. PETERSON, *Whitehead products and cohomology operations*, Quart. J. Math. *15* (1964), 116–120.

[8] B. ECKMANN, *Espaces fibrés et homotopie*, Colloque de Topologie, Centre Belges de Recherches mathématiques, 1950, 83–99.

[9] B. ECKMANN-P. HILTON, *Operators and Cooperators in homotopy theory*, Math. Ann. *141* (1960), 1–21.

[10] R. HERMANN, *Secondary obstructions for fiber spaces*, Bull. Amer. Math. Soc. *65* (1959), 5–8.

[11] F. HIRZEBRUCH-H. HOPF, *Felder von Flachenelementen in 4-dimensionalen Mannigfaltigkeiten*, Math. Ann. *136* (1958), 156–172.

[12] H. HOPF, *Vectorfelder in n-dimensionalen Mannigfaltigkeiten*, Math. Ann. *96* (1927), 225–260.

[13] H. HOPF, *Zur topologie der komplexen Mannigfaltigkeiten*, in *Studies and Essays presented to R. Courant*, Interscience, 1941, 167–186.

[14] M. MAHOWALD, *On obstruction theory in orientable fiber bundles*, Trans. Amer. Math. Soc. *110* (1964), 315–349.

[15] M. MAHOWALD-F. PETERSON, *Secondary cohomology operations on the Thom class*, Topology *2* (1964), 367–377.

[16] W. MASSEY, *On the Stiefel-Whitney classes of a manifold*, Amer. J. Math. *82* (1960), 92–102.

[17] W. MASSEY, *On the Stiefel-Whitney classes of a manifold* II, Proc. Amer. Math. Soc. *13* (1962), 938–942.

[18] J. P. MEYER, *The characterization of Moore-Postnikov invariants*, Bol. Soc. Mat. Mexicana *1963*, 92–94.

[19] J. MILNOR, *Lectures on characteristic classes*, (mimeographed notes), Princeton Univ., 1957.

[20] J. MILNOR, *Construction of universal bundles* II, Ann. of Math. *63* (1956), 430–436.

[21] J. P. SERRE, *Homologie singulière des espaces fibrés*, Ann. of Math. *54* (1951), 425–505.

[22] J. P. SERRE, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comment. Math. Helv. *27* (1953), 198–231.

[23] E. SPANIER, *Algebraic Topology*, McGraw-Hill, 1966.

[24] N. STEENROD, *The topology of fiber bundles*, Princeton Univ., 1951.

[25] R. THOM, *Espaces fibrés en spheres et carrés de Steenrod*, Ann. Sci. Ecole Norm. Sup. *69* (1952), 109–182.

[26] E. THOMAS, *Seminar on fiber spaces*, Springer-Verlag, 1966.

[27] E. THOMAS, *Postnikov invariants and higher order cohomology operations*, Ann. of Math., to appear.

[28] O. VEBLEN, *Analysis Situs*, second edition, Amer. Math. Soc., 1931.

[29] W. T. WU, *Classes caractéristique et i-carrés d'une variété*, C. R. Acad. Sci. Paris *230* (1950), 918–920.

(Eingegangen, 23. Aug. 1966)