The Index of a Tangent 2-Field[1])

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Dedicated to Professor H. Hopf

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1. Introduction

Let M be a connected, smooth, Riemannian manifold, and let k be a positive integer. By a *k-field* on M we mean an ordered set of k orthonormal tangent vector fields. We say that M has a k-field with *finite singularities* if there is a k-field on the manifold obtained from M by removing a finite number of points. Let $(X_1, ..., X_k)$ be such a k-field. Choose a triangulation of M such that each singular point of the k-field lies in the interior of a distinct m-simplex $(m = \dim M)$. Let p be a singular point, say in the interior of the closed simplex σ. Suppose now that M is oriented. The tangent bundle of M restricted to σ is then isomorphic to the trivial bundle $\sigma \times R^m$, by an orientation preserving isomorphism, and this isomorphism can be chosen to be compatible with the standard Riemannian metric on $\sigma \times R^m$. Thus for each point q in $\sigma - \{p\}$ we can regard $(X_1(q), ..., X_k(q))$ as an orthonormal k-frame in R^m – that is, as a point in the Stiefel manifold $V_{m,k}$. Since M is oriented the boundary of σ, $\dot{\sigma}$, is then an oriented $(m-1)$-sphere. By the above remarks one sees that the k-field restricted to $\dot{\sigma}$ gives a map $\dot{\sigma} \to V_{m,k}$ and the homotopy class of this map is then an element of the homotopy group $\pi_{m-1}(V_{m,k})$. We define this homotopy class to be the *index* of the k-field at the singular point p (see Hopf [12], [13]), and write this Index $(X_1, ..., X_k)_p$. Now let $\{p_1, ..., p_r\}$ be the set of singular points of the k-field. We define

$$\text{Index}(X_1, ..., X_k) = \sum_i \text{Index}(X_1, ..., X_k)_{p_i} \in \pi_{m-1}(V_{m,k}).$$

One can show that this definition of the index agrees with the definition one obtains via obstruction theory. (See §§ 29–34 in [24].) This implies that the definition is independent of the choices made above; in particular it is independent of the orientation of M. Also, from obstruction theory it follows that Index$(X_1, ..., X_k) = 0$ iff there is a k-field without singularities on M which coincides with $(X_1, ..., X_k)$ on the $(m-2)$-skeleton of M. (See 34.2 of [24].)

A 1-field X on M is simply a field of unit tangent vectors. Since $V_{m,1} = (m-1)$-sphere and $\pi_{m-1}(V_{m,1}) = Z$, we may regard Index (X) as an integer. The celebrated theorem of H. Hopf [12] states that if X is a 1-field with finite singularities on a closed manifold[2]) M, then

$$\text{Index}(X) = \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M.

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[2]) By using local coefficients one can define the index on a non-orientable manifold (See [24, §39.5].)

Let (X_1, X_2) be a 2-field with finite singularities on a closed oriented manifold M of dim m, with $m > 4$. The index of (X_1, X_2) is then an element of the homotopy group $\pi_{m-1}(V_{m, 2})$. This group depends on the parity of m as is shown below (see [8]):

$$\pi_{m-1}(_{m, 2}) = \begin{cases} Z_2 & , \quad \text{if} \quad m \quad \text{odd} \\ Z \oplus Z_2, & \text{if} \quad m \quad \text{even}. \end{cases}$$

Thus if m is odd we can regard Index (X_1, X_2) as an integer mod 2. If m is even we write

$$\text{Index } (X_1, X_2) = (\text{Index}_0(X_1, X_2), \text{Index}_2(X_1, X_2)),$$

where $\text{Index}_0(X_1, X_2) \in Z$, $\text{Index}_2(X_1, X_2) \in Z_2$. It is easily shown (see § 7 below) that $\text{Index}_0(X_1, X_2) = \chi(M)$. In a previous paper [27] we have proved: *If* $m \equiv 2$ *or* 3 mod 4, *and if* (X_1, X_2) *is a 2-field with finite singularities, then*

$$\text{Index}_2(X_1, X_2) = 0, \quad \text{if} \quad m \equiv 2(4),$$
$$\text{Index } (X_1, X_2) = 0, \quad \text{if} \quad m \equiv 3(4).$$

The purpose of this paper is to consider 2-fields on m-manifolds where $m \equiv 0$, 1 mod 4.

The case of 4-manifolds has been completely solved by F. HIRZEBRUCH and H. HOPF [11]. For the rest of the section let M denote a closed oriented manifold of dim m, with $m > 4$. Let $w_i M \in H^i(M; Z_2)$ denote the ith Stiefel-Whitney class of M, $i \geq 1$. Recall (see § 39.1 in [24]) that if m is odd then M has a 2-field with finite singularities iff $w_{m-1} M = 0$, while if m is even then M has such a 2-field iff $\delta^* w_{m-2} M = 0$. (Here δ^* denotes the Bockstein coboundary from mod 2 coefficients to integer coefficients.) MASSEY [17] has shown that if m is even then one always has $\delta^* w_{m-2} M = 0$. Thus an orientable manifold of even dimension always has a 2-field with finite singularities.

Define

$$\chi^+ M = \sum_i \dim H_i(M; Z_2).$$

If $\chi^+ M$ is an even integer (as will be the case, for example, when m is odd), we define[3] an integer mod 2 by

$$\hat{\chi}_2 M = \tfrac{1}{2} \chi^+ M \bmod 2.$$

We will prove the following result. (Recall that M is called a *spin* manifold if $w_2 M = 0$.)

THEOREM 1.1. *Let* M *be a closed spin manifold of* dim $4k + 1$, $k > 0$, *such that* $w_{4k} M = 0$. *If* (X_1, X_2) *is any 2-field with finite singularities, then*

$$\text{Index } (X_1, X_2) = \hat{\chi}_2 M.$$

As an immediate consequence we have

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[3] See KERVAIRE, Math. Ann. *131* (1956) 220.

COROLLARY 1.2. *Let M be a closed spin manifold of* dim $4k+1$, $k>0$. *Then M has a 2-field without singularities if, and only if,*

$$w_{4k}M = 0, \qquad \hat{\chi}_2 M = 0.$$

In case M is a π-manifold, this is given as part of Theorem 2 in [6].

The case $m \equiv 0 \bmod 4$ requires an additional hypothesis. Let M be a manifold of even dimension, say $2q$. We call M *symplectic* if, for all classes $u \in H^q(M; Z_2)$, $u^2 = 0$. We show below that if M is a spin manifold of dim $8k+4$, $k \geq 0$, then M is symplectic. Also, we will show that if M is symplectic then $w_{2q}M = 0$, and so the Euler characteristic of M is an even integer. Therefore, by Poincaré duality, it follows that $\chi^+ M$ is also even and so $\hat{\chi}_2 M$ is defined. We will prove

THEOREM 1.3. *Let M be a closed spin manifold of* dim m, *where* $m \equiv 0 \bmod 4$ *and* $m > 4$. *If* $m \equiv 0 \bmod 8$ *assume that M is symplectic. Then for any 2-field (X_1, X_2) with finite singularities.*

$$\mathrm{Index}_2(X_1, X_2) = \hat{\chi}_2 M.$$

Suppose that dim $M = 4k$, $k > 0$; set $d_i = \dim H_i(M; Z_2)$. By Poincaré duality,

$$\chi(M) = \sum_{i=0}^{2k-1} (-1)^i 2d_i + d_{2k},$$

$$\chi^+ M = \sum_{i=0}^{2k-1} 2d_i + d_{2k}.$$

Therefore,

$$\chi^+ M = \left(\sum_{i=0}^{2k-1} 2(1 - (-1)^i) d_i \right) + \chi(M),$$

and so if $\chi(M)$ is even

$$\hat{\chi}_2 M = \left(\tfrac{1}{2}\chi(M) \right) \bmod 2.$$

In particular

$$\hat{\chi}_2 M = 0 \quad \text{if, and only if,} \quad \chi(M) \equiv 0 \bmod 4.$$

As a consequence we have

COROLLARY 1.4. *Let M be a closed spin manifold as in* 1.3. *Then M has a 2-field without singularities if, and only if, $\chi(M) = 0$.*

Recall that a manifold M of even dimension $2q$ is said to have an *almost-complex* structure if there is a complex q-plane bundle ω over M such that the tangent bundle of M is equivalent to the real bundle underlying ω. Now this complex bundle ω has a complex 1-field with finite singularities, and the index of this 1-field is simply $\chi(M)$ [19, pp. 61, 65]. Moreover the complex 1-field determines a (real) 2-field on M also with finite singularities and for this 2-field (X_1, X_2), $\mathrm{Index}_2(X_1, X_2) = bw_{2q}M$, $b \in Z_2$. Thus by 1.3 and the computation given above for $\hat{\chi}_2 M$, we obtain

COROLLARY 1.5. *Let M be a closed spin manifold as in* 1.3. *If M admits an almost-complex structure, then the Euler characteristic of M is divisible by* 4.

This argument was originally used by HOPF [13] to show that S^4 and S^8 do not admit almost-complex structures.

Let M be an m-manifold and let $V = \sum_{i=1}^{m} V_i$ denote the WU class [29]. That is, if $u \in H^{m-i}(M; Z_2)$ then

$$\mathrm{Sq}^i(u) = u \cdot V_i,$$

where Sq^i denotes the mod 2 Steenrod operator of degree i, $i \geq 1$. The Theorem of Wu is that

$$w_k M = \sum_{i=0}^{k} \mathrm{Sq}^i V_{k-i}, \qquad k \geq 1.$$

Thus if m is even, say $m = 2q$,

$$w_{2q} M = \mathrm{Sq}^q V_q = V_q^2.$$

But by definition, M is symplectic iff $V_q = 0$, and so if M is symplectic then $w_{2q} M = 0$, as asserted above. Also, by an easy extension of [16, Theorem III], one shows that if M is a spin m-manifold, then $V_{4k+2} = 0$, $k \geq 0$ (since $\mathrm{Sq}^2 H^{m-2}(M; Z_2) = 0$). Therefore if $m \equiv 4 \bmod 8$, M is symplectic as remarked above.

2. Proof of 1.1 and 1.3.

Throughout this section M will denote a closed oriented m-manifold, with $m \equiv 0$ or $1 \bmod 4$, $m > 4$. We will show in § 7 that if (X_1, X_2) is a 2-field on M with isolated singularities, then the index is independent of the particular choice of 2-field. We define a mod 2 integer, $I_2 M$, by setting

$$I_2 M = \begin{cases} \mathrm{Index}_2(X_1, X_2), & \text{if} \quad m \equiv 0\,(4) \\ \mathrm{Index}\,(X_1, X_2), & \text{if} \quad m \equiv 1\,(4). \end{cases}$$

Let T denote the Thom complex of the tangent bundle of M and $U \in H^m(T; Z)$ the Thom class (see [25], [19]). $H^*(T)$ can be regarded as a module over $H^*(M)$ (integer or mod 2 coefficients). By THOM [25] the map $H^i(M) \to H^{m+i}(T)$, given by $x \to U \cdot x$, is an isomorphism for all $i > 0$. Thus to determine the mod 2 integer $I_2 M$ it suffices to compute $U \cdot (I_2 M \mu)$, where $\mu \in H^m(M; Z_2)$ is the generator. For this we will need a secondary cohomology operation.

Recall that one has the following ADEM relation [2], when $m \equiv 0, 1 \bmod 4$.

$$(*) \qquad\qquad \mathrm{Sq}^2 \mathrm{Sq}^{m-1} + \mathrm{Sq}^m \mathrm{Sq}^1 = \mathrm{Sq}^{m+1}.$$

If u is an integral cohomology class of $\dim < m+1$, then

$$\mathrm{Sq}^1 u = 0, \qquad \mathrm{Sq}^{m+1} u = 0.$$

Also, if m is even we can write

$$\mathrm{Sq}^{m-1} = \mathrm{Sq}^1 \mathrm{Sq}^{m-2} = (\delta^* \mathrm{Sq}^{m-2}) \bmod 2.$$

Thus we have the following two non-stable relations:

$$
\begin{aligned}
m &\equiv 0(4): \mathrm{Sq}^2(\delta^* \mathrm{Sq}^{m-2}) = 0, \\
m &\equiv 1(4): \mathrm{Sq}^2 \mathrm{Sq}^{m-1} = 0,
\end{aligned}
\tag{2.1}
$$

where in each case the relation obtains on integral classes of $\dim \le m$.

Let Ω_m denote a (non-stable) secondary cohomology operation associated with each of the above relations, $m \equiv 0,1 \bmod 4$. (See [1] and [7].) Thus if X is a space and if $u \in H^j(X; Z), j \le m$, then Ω_m is defined on u, provided that

$$\delta^* \mathrm{Sq}^{m-2} u = 0 \quad \text{if} \quad m \equiv 0(4), \qquad \mathrm{Sq}^{m-1} u = 0 \quad \text{if} \quad m \equiv 1(4).$$

Furthermore

$$\Omega_m(u) \quad \text{is a coset in} \quad H^{m+j}(X; Z_2)$$

of the subgroup

$$
\begin{aligned}
\mathrm{Sq}^2 H^{m+j-2}(X; Z), &\quad \text{if} \quad m \equiv 0(4), \\
\mathrm{Sq}^2 H^{m+j-2}(X; Z_2), &\quad \text{if} \quad m \equiv 1(4).
\end{aligned}
$$

We will prove

THEOREM 2.2. *Let M be a closed spin manifold of* $\dim m$, *where* $m \equiv 0$ *or* 1 mod 4 *and* $m > 4$. *If m is odd assume that* $w_{m-1} M = 0$, *while if m even assume that* $w_m M = 0$. *Then the operation Ω_m is defined on the Thom class U and the operation can be chosen so that*

$$\Omega_m(U) = U \cdot (I_2 M \mu).$$

with zero indeterminacy.

This will be proved in § 7, following the method of MAHOWALD-PETERSON [15]. (Theorem 2.2 is similar to Theorem 3.3.2 in [15], but the details of our proof will be somewhat different as we will use the point of view of § 5 in [27]).

To prove 1.1 and 1.3 we need to compute the operation Ω_m. This is done as follows. Assume that the tangent bundle of M has been given a Riemannian metric; let E denote the set of tangent vectors of length ≤ 1, and let E^1 denote the set of vectors of length 1. Then $T = E/E^1$ ($=$ the space obtained from E by collapsing E^1 to a point). Moreover the collapsing map induces an isomorphism

$$H^*(E/E^1, *) \approx H^*(E, E^1),$$

and so we regard the Thom class U equally well as a class in $H^m(E, E^1; Z)$. MILNOR shows in [19] that there is an isomorphism

$$e: H^*(E, E^1) \approx H^*(M^2, M_2 - \text{diagonal}),$$

where $M^2 = M \times M$. Let $j: M^2 \subset (M^2, M^2 - \text{diagonal})$ denote the inclusion, and set

$$\underline{U} = j^* e(U) \in H^m(M^2; Z).$$

Now the isomorphism e is induced by maps and so commutes with all cohomology operations. Thus Ω_m is defined on $\underline{U}$. Assume that $w_2 M = 0$. Then

$$\text{Sq}^2 H^{m-2}(M) = 0, \quad \text{Sq}^2 H^{2m-2}(M^2) = 0,$$

and so Ω_m is defined with zero indeterminacy on U and $\underline{U}$. By naturality,

$$\Omega_m(\underline{U}) = j^* e \Omega_m(U).$$

But j^* is injective (as remarked in [3]) and so

$$\Omega_m(\underline{U}) = 0 \quad \text{if, and only if,} \quad \Omega_m(U) = 0.$$

Since a mod 2 integer is unchanged by squaring, we obtain from 2.2,

PROPOSITION 2.3. *Let M be a manifold as in 2.2. Then*

$$\Omega_m(\underline{U}) = I_2 M(\mu \oplus \mu) \in H^{2m}(M^2; Z_2).$$

To compute $\Omega_m(\underline{U})$ we reduce $\underline{U}$ mod 2. Consider the following non-stable relations (see (*)):

$$\begin{aligned} m &\equiv 0(4): \text{Sq}^2(\delta^* \text{Sq}^{m-2}) + \text{Sq}^m \text{Sq}^1 = 0, \\ m &\equiv 1(4): \text{Sq}^2 \text{Sq}^{m-1} + \text{Sq}^1(\text{Sq}^{m-1} \text{Sq}^1) = 0, \end{aligned} \tag{2.4}$$

where in each case the relation obtains on mod 2 classes of dim $\leq m$. Let $\tilde{\Omega}_m$ denote a (non-stable) operation associated with each relation in 2.4.

Let M be a manifold as in 2.2. Regarding $\underline{U}$ as a class mod 2, $\tilde{\Omega}_m$ is defined on $\underline{U}$, and with zero indeterminacy when $m \equiv 1$. When $m \equiv 0$, $\tilde{\Omega}_m$ has $\text{Sq}^m H^m(M^2)$ as indeterminacy subgroup. But if M is symplectic them $\text{Sq}^m H^m(M^2) = 0$, and so $\tilde{\Omega}_m(\underline{U})$ will again be defined with zero indeterminacy. By considering the universal examples for Ω and $\tilde{\Omega}$ it is easily shown that, with all these hypotheses on M, $\tilde{\Omega}_m$ can be chosen so that

$$\tilde{\Omega}_m(\underline{U}) = \Omega_m(\underline{U}), \tag{2.5}$$

where Ω_m denotes the specific choice of operation given in 2.2.

Thus, as our final step, we compute $\tilde{\Omega}_m(\underline{U})$. Let $t: H^*(M^2) \to H^*(M^2)$ denote the isomorphism induced by interchanging the factors of M^2.

THEOREM 2.6. *Let M be an m-manifold as in 2.2. If m is even assume that M is symplectic. Then there is a mod 2 class $A \in H^m(M^2)$ such that* ∗

a) $\underline{U} \bmod 2 = A + tA$,

b) $A \cup tA = \tilde{\chi}_2 M(\mu \otimes \mu)$,

c) $\tilde{\Omega}_m$ *is defined on A.*

The proof will be given in § 4.

Proof of 1.1 *and* 1.3. By 2.3 and 2.5,

$$\tilde{\Omega}_m(\underline{U}) = I_2 M(\mu \oplus \mu).$$

Now $\tilde{\Omega}_m$ is a non-stable operation of degree m. By 2.6 c) $\tilde{\Omega}$ is defined on A and thus also on tA. Therefore, by [7, cf. 2. 3],

$$\tilde{\Omega}(A + tA) = \tilde{\Omega}(A) + \tilde{\Omega}(tA) + A \cup tA.$$

Since t is the identity on $H^{2m}(M^2)$, we have by naturality,

$$\tilde{\Omega}_m(A) = t\,\tilde{\Omega}_m(A) = \tilde{\Omega}_m(tA).$$

Consequently, by 2.6 a) and b),

$$\tilde{\Omega}_m(\underline{U}) = \tilde{\Omega}_m(A + tA) = A \cup tA = \hat{\chi}_2 M(\mu \oplus \mu).$$

But $\tilde{\Omega}_m(\underline{U}) = I_2 M(\mu \otimes \mu)$, and so

$$I_2 M = \hat{\chi}_2 M,$$

which completes the proof of 1.1 and 1.3.

3. Mod 2 vector spaces

Most of the work in proving Theorem 2.6 will come in the case m even. This section develops some simple facts about mod 2 vector spaces needed for this case. The proof of 2.6 is then given in the next section.

Let V be a finite-dimensional mod 2 vector space. An endomorphism t of V is called an *involution* if $t^2 = 1$. An endomorphism d is called a *boundary* if $d^2 = 0$. Suppose that V has an involution t and a boundary d. We say that the pair (t, d) is *regular* if

$$t d = d t, \tag{3.1}$$

and

there are subspaces A, B in V such that

$$dB = 0 \quad \text{and} \quad V = A \oplus tA \oplus dA \oplus tdA \oplus B \oplus tB. \tag{3.2}$$

Define

$$\Delta = t + 1 : V \to V.$$

LEMMA 3.3. *Let t be an involution on V and d a boundary such that the pair (t, d) is regular. Then*

$$(\text{Ker } d) \cap (\text{Ker } \Delta) = \Delta(\text{Ker } d).$$

Proof. Because V is a Z_2-module, $\Delta^2 = 0$. Also by 3.1, $\Delta d = d\Delta$, and so

$$\Delta(\text{Ker } d) \subset \text{Ker } d \cap \text{Ker } \Delta.$$

We prove 3.3 by showing that the opposite inclusion holds. Let $v \in V$ be an element such that

$$dv = 0, \quad \Delta v = 0.$$

By 3.2 we can write v as

$$v = a_1 + t a_2 + d a_3 + t d a_4 + b_1 + t b_2,$$

where the a's are in A and the b's in B. Since $dv = 0$ and $dB = 0$, we must have

$$d a_1 = d t a_2 = 0.$$

Furthermore

$$\Delta v = (a_1 + a_2) + (t a_1 + t a_2) + (d a_3 + d a_4) + (t d a_3 + t d a_4)$$
$$+ (b_1 + b_2) + (t b_1 + t b_2).$$

Since $\Delta v = 0$ this means, by 3.2, that

$$a_1 = a_2, \quad d a_3 = d a_4, \quad b_1 = b_2.$$

Therefore

$$v = \Delta (a_1 + d a_3 + b_1), \quad \text{and} \quad d(a_1 + d a_3 + b_1) = 0,$$

which completes the proof.

Let X be a space whose total singular integral homology module is finitely generated. Let $H^*(X)$ denote the mod 2 cohomology algebra of X. By the Künneth theorem for cohomology,

$$H^*(X^2) \approx H^*(X) \otimes H^*(X),$$

where $X^2 = X \times X$.

Let $t: H^*(X^2) \rightarrow H^*(X^2)$ denote the involution induced by transposing the factors of X^2. We will call an element $v \in H^*(X^2)$ *symmetric* if $\Delta v = 0$, where $\Delta = t + 1$. Let $\alpha = (\alpha_1, \ldots, \alpha_q)$ be a basis for $H^*(X^2)$. An element $v \in H^*(X^2)$ will be called *symplectic* with respect to α if

$$v = \sum_{i,j} c_{ij} \alpha_i \otimes \alpha_j,$$

where all $c_{ii} = 0$, $1 \leq i \leq q$.

LEMMA 3.4. *Let $v \in H^*(X^2)$ be a symmetric class. If v is symplectic with respect to one basis, then it is so with respect to any basis.*

Proof. With respect to a second basis for $H^*(X)$, the matrix $C = (c_{ij})$ becomes a matrix $C' = (c'_{ij})$, which is obtained from C by symmetric row and column operations [27, p. 188]. Thus C' is also symmetric. Moreover each such pair of row and column operations leaves unchanged the diagonal elements of C (since $c_{ii} = 0$ and we are working over Z_2). Thus C' remains symplectic, i.e., $c'_{ii} = 0$, $1 \leq i \leq q$. This completes the proof.

The main result of the section is the following.

PROPOSITION 3.5. *Let* $v \in H^{2n}(X^2)$, $n > 0$. *Suppose that*

$$\Delta v = 0, \quad \mathrm{Sq}^1 v = 0$$

and that v *is symplectic. Then there is a class* u *such that*

$$\Delta u = v, \quad \mathrm{Sq}^1 u = 0.$$

Proof. Set $d = \mathrm{Sq}^1$. Then $d^2 = 0$ and $td = dt$. We choose a basis $\alpha_1, \ldots, \alpha_q$ for $H^*(X)$ so that for some integer r,

$$d\alpha_i = \alpha_{r+i}, \quad 1 \leq i \leq r,$$
$$d\alpha_j = 0, \quad 2r + 1 \leq j \leq q.$$

Define $W \subset H^*(X^2)$ to be the subspace spanned by all basis elements $\alpha_i \otimes \alpha_j$, with $i \neq j$. Notice that the class v is in W because v is symplectic.

Now set $s = q - 2r$, and let $b_i = \alpha_{2r+i}$, $1 \leq i \leq s$, where r and q are given above. Define $A, B \subset W$ to be the subspaces spanned by the basis elements shown below:

$$A : \{\alpha_i \otimes \alpha_j, d\alpha_i \otimes \alpha_j, 1 \leq i < j \leq r;$$
$$\alpha_i \otimes d\alpha_j, 1 \leq i \leq j \leq r;$$
$$\alpha_i \otimes b_j, 1 \leq i \leq r, 1 \leq j \leq s.\}.$$
$$B : \{d\alpha_i \otimes d\alpha_j, 1 \leq i < j \leq r;$$
$$d\alpha_i \otimes b_j, 1 \leq i \leq r, 1 \leq j \leq s;$$
$$b_i \otimes b_j, 1 \leq i < j \leq s.\}.$$

Then, as is readily seen,

(*) $$W = A \oplus tA \oplus B \oplus tB, \quad dB = 0.$$

For any subspace $U \subset H^*(X^2)$, set $U^i = U \cap H^i(X^2)$, $i \geq 0$. Notice that the classes $d\alpha_i \otimes \alpha_i$, $\alpha_i \otimes d\alpha_i$ do not occur in A^{2p}, for any $i, p > 0$. Thus

$$dA^{2p} \cap dt A^{2p} = 0,$$

and so

(**) $$dW^{2p} = dA^{2p} \oplus dt A^{2p}, \quad p > 0.$$

Suppose now that the class v, given in 3.5, has degree $2n$, $n > 0$. We set

$$V = W^{2n} \oplus dW^{2n}.$$

By (*) and (**),

$$V = A^{2n} \oplus t A^{2n} \oplus B^{2n} \oplus t B^{2n} \oplus d A^{2n} \oplus dt A^{2n}.$$

Consequently the pair (t, d) is regular on V. By hypothesis $\Delta v = 0$, $dv = 0$, and so by 3.3 there is a class $u \in W^{2n}$ such that

$$\Delta u = v, \quad du = \mathrm{Sq}^1 u = 0.$$

This completes the proof.

4. Proof of Theorem 2.6

We retain the notation of §§ 2, 3. Let M be an m-manifold and let $\alpha_1, \ldots, \alpha_q$ be a basis for $H^*(M)$ (mod 2 coefficients). Define y_{ij} to be the value of $\alpha_i \cup \alpha_j$ on the fundamental mod 2 homology class $[M]$. In particular $y_{ij} = 0$ if $\deg \alpha_i + \deg \alpha_j \neq m$; and $y_{ij} = y_{ji}$, $1 \leq i, j \leq q$. Let Y be the $q \times q$ matrix (y_{ij}) and set $C = Y^{-1}$. Then by MILNOR [19],

$$(*) \qquad \underline{U} = \sum_{i,j} c_{ij} \, \alpha_i \otimes \alpha_j,$$

where $C = (c_{ij})$. Since Y is symmetric so is C.

Notice that $q = \chi^+ M$. By the hypotheses of 2.6, q is even, say $q = 2d$. We choose the basis $\{\alpha_i\}$ in a special way. Suppose first that m is odd, say $m = 2k + 1$. Let $\alpha_1, \ldots, \alpha_d$ be an arbitrary basis for the graded vector space

$$\sum_{i=0}^{k} H^i(M).$$

By Poincaré duality, $H^i(M)$ and $H^{m-i}(M)$ are orthogonally paired by the cup-product. Consequently we can choose a basis $\beta_1, \ldots, \beta_d$ for

$$\sum_{i=0}^{k} H^{m-i}(M)$$

such that if $\deg \alpha_i + \deg \beta_j = m$, then

$$\alpha_i \cup \beta_j = \delta_{ij} \, \mu.$$

Take as total basis for $H^*(M)$ the elements $\{\alpha_1, \ldots, \alpha_d, \beta_d, \ldots, \beta_1\}$. Then the matrix Y has the form shown below:

$$Y = \begin{bmatrix} & & & & 1 \\ & & 0 & \cdot \\ & & 1 & \\ & \cdot & & 0 \\ 1 & & & \end{bmatrix}.$$

Thus $C = Y$ and so by $(*)$ we obtain

$$\underline{U} = \sum_{i=1}^{d} \alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i. \qquad (4.1)$$

Suppose on the other hand that m is even, say $m = 2k + 2$. Let $\{\alpha_1, \ldots, \alpha_r\}$, $\{\beta_1, \ldots, \beta_r\}$ be bases for the respective vector spaces

$$\sum_{i=0}^{k} H^i(M), \qquad \sum_{i=0}^{k} H^{m-i}(M),$$

chosen as above so that

$$\alpha_i \cup \beta_j = \delta_{ij}\,\mu,$$

if $\deg \alpha_i + \deg \beta_j = m$. Assume, as in 2.6, that M is symplectic. Then (see [28]) one can choose a basis $x_1, ..., x_s, y_1, ..., y_s$ for $H^{k+1}(M)$ such that

$$x_i \cup x_j = 0, \qquad y_i \cup y_j = 0, \qquad x_i \cup y_j = \delta_{ij}\,\mu.$$

Now by definition

$$2(r + s) = q = 2d.$$

Set

$$\alpha_{r+i} = x_i, \qquad \beta_{r+i} = y_i, \qquad 1 \le i \le s.$$

Then $\{\alpha_1, ..., \alpha_d, \beta_d, ..., \beta_1\}$ is a basis for $H^*(M)$ yielding as above

$$\underline{U} = \sum_{i=1}^{d} \alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i. \tag{4.2}$$

For m even or odd we set

$$A = \sum_{i=1}^{d} \alpha_i \otimes \beta_i.$$

Then by (4.1) and (4.2), $\underline{U} = A + tA$, which proves 2.6 i). Now

$$(\alpha_i \otimes \beta_i) \cup (\beta_j \otimes \alpha_j) = (\alpha_i \beta_j \otimes \beta_i \alpha_j) = 0$$

unless $i = j$. For if $\deg \alpha_i + \deg \beta_j = m$, then by definition $\alpha_i \cup \beta_j = \delta_{ij}\mu$, while if $\deg \alpha_i + \deg \beta_j \ne m$ then one of the pairs $\alpha_i \beta_j$, $\beta_i \alpha_j$ has degree greater than m and so is zero. Thus

$$A \cup tA = \sum_{i=1}^{d} \alpha_i \beta_i \otimes \alpha_i \beta_i = d(\mu \otimes \mu) = \chi_2 M(\mu \otimes \mu),$$

since $2d = q = \chi^+ M$. Therefore the class A satisfies 2.6 ii).

To prove 2.6 iii) we need the following lemma.

LEMMA 4.3. *Let M be an orientable manifold of* $\dim m, m > 1$. *Let $u \in H^r(M)$, $v \in H^s(M)$, where $r + s = m$ and $0 < r \le s$.*

a) *Suppose that $m \equiv 0 \bmod 4$. If $r < s$, then*

$$\delta^* \mathrm{Sq}^{m-2}(u \otimes v) = 0.$$

If $r = s$, then

$$\delta^* \mathrm{Sq}^{m-2}(u \otimes v) = \delta^* \mathrm{Sq}^{r-2} u \otimes v^2 + u^2 \otimes \delta^* \mathrm{Sq}^{r-2} v.$$

b) *Suppose that m is odd. If $r < s - 1$, then*

$$\mathrm{Sq}^{m-1}(u \otimes v) = 0.$$

If $r = s - 1$, then

$$\mathrm{Sq}^{m-1}(u \otimes v) = u^2 \otimes \mathrm{Sq}^{s-1} v.$$

c) *Suppose that m is odd and that* $w_2 M = 0$. *Then*

$$\text{Sq}^{m-1} \text{Sq}^1 H^m(M^2) = 0.$$

The proof of (a) and (b) follows at once by the Cartan formula, using the fact that $H^m(M;Z) \approx Z$. Thus

$$\delta^* H^{m-1}(M) = \text{Sq}^1 H^{m-1}(M) = 0.$$

We leave the details of the proof to the reader. For (c) suppose that $m = 2k + 1$. Then by ADEM [2],

$$\text{Sq}^{m-1} \text{Sq}^1 = \text{Sq}^{2k} \text{Sq}^1 = \text{Sq}^2 \text{Sq}^{2k-1} + \varepsilon \text{Sq}^{2k+1}$$
$$= \text{Sq}^2 \text{Sq}^{2k-1} + \varepsilon \text{Sq}^1 \text{Sq}^{2k}$$

where $\varepsilon = 0$ or 1. But

$$\text{Sq}^2 H^{2m-2}(M^2) = 0, \qquad \text{Sq}^1 H^{2m-1}(M^2) = 0,$$

since $w_1 M = w_2 M = 0$. Therefore $\text{Sq}^{m-1} \text{Sq}^1 H^m(M^2) = 0$, as claimed, which completes the proof of the lemma.

Proof of 2.6 iii). We must show that the operation Ω_m is defined on the class A.

CASE I: $m \equiv 1 \mod 4$. By 2.4 this means we must show that

$$\text{Sq}^{m-1} \text{Sq}^1 A = 0, \qquad \text{Sq}^{m-1} A = 0.$$

The first assertion follows by 4.3 (c). To prove the second assertion, we assume that the basis $\alpha_1, \ldots, \alpha_d$ is ordered so that

$$\deg \alpha_i \leq \deg \alpha_{i+1}, \qquad 1 \leq i \leq q - 1.$$

Suppose that $\alpha_j, \ldots, \alpha_d$ are precisely those basis elements with degree $(m-1)/2$. Then by 4.3 (b),

$$\text{Sq}^{m-1} A = \sum_{i=j}^{d} \alpha_i^2 \otimes \text{Sq}^{s-1} \beta_i,$$

where $s - 1 = (m-1)/2$. Consequently,

$$\text{Sq}^{m-1} t A = t \text{Sq}^{m-1} A = \sum_{i=j}^{d} \text{Sq}^{s-1} \beta_i \otimes \alpha_i^2.$$

Now $U = A + tA$, and by § 2 we know that $\text{Sq}^{m-1} U = 0$, which means that

$$\text{Sq}^{m-1} A + \text{Sq}^{m-1} t A = 0.$$

But, as is seen by the above calculation, $\text{Sq}^{m-1} A$ and $\text{Sq}^{m-1} t A$ occur in disjoint summands of the bi-graded vector space $H^*(M) \otimes H^*(M)$. Namely, $\text{Sq}^{m-1} A$ has bi-degree $(m-1, m)$, while $\text{Sq}^{m-1} t A$ has bi-degree $(m, m-1)$. Thus $\text{Sq}^{m-1} A = 0$, as claimed, which completes the proof of case I.

CASE II: $m \equiv 0 \bmod 4$. We will show that the class A can be replaced by a class B, which will continue to satisfy 2.6 i) and ii) and for which

$$\delta^* \mathrm{Sq}^{m-2} B = 0, \quad \mathrm{Sq}^1 B = 0.$$

Thus the class B will satisfy 2.6 iii) (see 2.4) and so the proof of 2.6 will be completed.

By 4.1 (a) we see that $\delta^* \mathrm{Sq}^{m-2} H^m(M^2) = 0$; for if the classes u and v in 4.1 (a) have degree $m/2$, then $u^2 = v^2 = 0$, since M is symplectic by hypothesis.

In general it is not necessarily true that $\mathrm{Sq}^1 A = 0$. Thus we must find a new class B, satisfying 2.6 i) and ii), such that $\mathrm{Sq}^1 B = 0$.

As usual we set $\Delta = 1 + t$. Then $\Delta \underline{U} = 0$, and so by 3.5 there is a class $B \in H^m(M^2)$ such that

$$\Delta B = \underline{U}, \quad \mathrm{Sq}^1 B = 0.$$

Set $D = B - A$; since $\Delta A = \underline{U}$ it follows that $\Delta D = 0$. Moreover,

$$B \cup tB = (A + D) \cup (tA + D) = A \cup tA + A \cup D + D \cup tA + D \cup D.$$

Since M is symplectic, an easy argument shows that M^2 is too; therefore $D \cup D = 0$. In a moment we show that $A \cup D = D \cup tA$. This then implies that

$$B \cup tB = A \cup tA = \hat{\chi}_2 M (\mu \otimes \mu).$$

Thus the class B satisfies 2.6 (i)–(iii), and so the proof of 2.6 is complete.

We are left with showing that $A \cup D = D \cup tA$. By commutativity of the cup-product, $A \cup D = D \cup A$. Furthermore, since t is the identity on $H^{2m}(M^2)$, we have by naturality of the cup-product,

$$A \cup D = D \cup A = t(D \cup A) = tD \cup tA.$$

But $tD = D$ since $\Delta D = 0$. Thus, $A \cup D = D \cup tA$ as claimed.

5. The relative Thom complex

Let ξ be an oriented n-plane bundle over a space B and suppose that ξ has a Riemannian metric [19, p. 21]. Denote by E, E^1 the respective subspaces of the total space of ξ consisting of those vectors of norm ≤ 1 and those of norm 1. (In order to avoid confusion we may sometimes write these spaces as $E(\xi)$, $E^1(\xi)$.) We define the Thom complex $T(\xi)$ to be E/E^1.

Let B' be a space and $f: B' \to B$ a map. Let $f^* \xi$ denote the bundle over B' induced from ξ by f. Give $f^* \xi$ the induced Riemannian metric. Then the natural bundle map $\tilde{f}: f^* \xi \to \xi$ induces a map

$$T(f): T(f^* \xi) \to T(\xi).$$

Let B'' be a second space and $g:B''\to B'$ a map. Then, up to homeomorphism,

$$T(g^* f^* \xi) = T((fg)^* \xi),$$
$$T(f)\cdot T(g) = T(fg). \tag{5.1}$$

Suppose that A is a subspace of B. Then the inclusion $A \subset B$ induces an inclusion $T(\xi_A) \subset T(\xi)$, where $\xi_A = \xi|A$. Thus, if $f:(B', A')\to(B, A)$ is a map of pairs, we obtain a map of pairs

$$T(f):(T(f^* \xi), T(f^* \xi_A)) \to (T(\xi), T(\xi_A)).$$

Now let $U \in H^n(E, E^1)$ denote the Thom class of the bundle ξ and let $p:E\to B$ denote the projection. Thom shows that the homomorphism

$$H^i(B) \to H^{n+i}(E, E^1),$$

given by $x \to p^* x \cup U$, is an isomorphism $(i \geq 0)$. Since the pair (E, E^1) enjoys the homotopy-extension property (e.g., we can regard E as the mapping cylinder of $p|E^1$), the collapsing map $(E, E^1)\to(T(\xi),*)$ induces an isomorphism in cohomology. Following Thom we define

$$\psi_B: H^i(B) \approx H^{n+i}(T(\xi), *)$$

to be the composite isomorphism. We prove[4]

LEMMA 5.2. *Let A be a closed subspace of B. Set $T_B = T(\xi)$, $T_A = T(\xi_A)$. Then there is a homomorphism*

$$\psi_{B, A}: H^q(B, A) \to H^{n+q}(T_B, T_A)$$

with the following properties.

a) *The following diagram is commutative:*

$$\cdots \to H^q(B, A) \xrightarrow{j^*} H^q(B) \xrightarrow{i^*} H^q(A) \xrightarrow{\delta} H^{q+1}(B, A) \to \cdots$$
$$\downarrow \psi_{B, A} \qquad \downarrow \psi_B \qquad \downarrow \psi_A \qquad \downarrow \psi_{B, A}$$
$$\cdots \to H^{q+n}(T_B, T_A) \xrightarrow{j^*} H^{q+n}(T_B) \xrightarrow{i^*} H^{q+n}(T_A) \xrightarrow{\delta} H^{q+n+1}(T_B, T_A) \to \cdots$$

Here i^*, j^* denote homomorphisms induced by inclusions and δ is the coboundary operator.

b) $\psi_{B, A}$ *is an isomorphism for all q.*

c) *Let $f:(B', A')\to(B, A)$ be a map of pairs. Then the following diagram commutes:*

$$H^q(B, A) \xrightarrow{f^*} H^q(B', A')$$
$$\downarrow \psi_{B, A} \qquad\qquad \downarrow \psi_{B', A'}$$
$$H^{q+n}(T_B, T_A) \xrightarrow{T(f)^*} H^{q+n}(T_{B'}, T_{A'}).$$

---

[4] The result is well known, but I am unaware of a reference.

d) *Let* $x \in H^*(B, A)$, mod 2 *coefficients. Then,*

$$\mathrm{Sq}^k \psi_{B,A}(x) = \sum_{i+j=k} \psi_{B,A}(w_i \xi \cup \mathrm{Sq}^j x).$$

Proof: Following Spanier we define the *relative Thom pair* of the bundles (ξ, ξ_A) to be the pair $(E, E_A \cup E^1)$, where $E_A = E(\xi_A)$. Let $p':(E, E_A) \rightarrow (B, A)$ denote the projection. Notice that if $x \in H^i(B, A)$, then

$$p'^* x \cup U \in H^{n+i}(E, E_A \cup E^1),$$

and so we obtain a homomorphism

$$\psi'_{B,A}: H^i(B, A) \rightarrow H^{n+i}(E, E_A \cup E^1), \quad i \geq 0.$$

If A is empty then $\psi'_{B,A}$ is simply the isomorphism ψ_B given above.

Notice that if we collapse E^1 to a point in the pair $(E, E_A \cup E^1)$ we obtain the pair (T_B, T_A). Thus by the 5-lemma the collapsing map induces an isomorphism

$$H^*(E, E_A \cup E^1) \approx H^*(T_B, T_A).$$

We define $\psi_{B,A}: H^i(B, A) \rightarrow H^{n+i}(T_B, T_A)$ to be the composition of $\psi'_{B,A}$ with the isomorphism given above. The properties of $\psi_{B,A}$ will then follow from the analogous properties of $\psi'_{B,A}$. We proceed to develop the properties of $\psi'_{B,A}$.

By SPANIER [23, 5.4.9] we see that there is a coboundary operator

$$\Delta: H^j(E_A, E_A^1) \rightarrow H^{j+1}(E, E_A \cup E^1)$$

so that the following diagram commutes and has exact rows.

$$\cdots \xrightarrow{\ j^*\ } H^q(B) \xrightarrow{\ i^*\ } H^q(A) \xrightarrow{\ \delta\ } H^{q+1}(B, A) \xrightarrow{\ j^*\ } \cdots$$
$$\downarrow \psi_B \qquad\qquad \downarrow \psi_A \qquad\qquad \downarrow \psi'_{B,A}$$
$$\cdots \xrightarrow{\ j^*\ } H^{q+n}(E, E^1) \xrightarrow{\ i^*\ } H^{q+n}(E_A, A_A^1) \xrightarrow{\ \Delta\ } H^{q+n+1}(E, E_A \cup E^1) \xrightarrow{\ j^*\ } \cdots$$

(Because E and E_A are disk bundles the excision properties required in [23] are easily seen to be satisfied.) Since ψ_B and ψ_A are isomorphisms, it follows from the 5-lemma that $\psi'_{B,A}$ is an isomorphism.

Suppose that $f:(B', A') \rightarrow (B, A)$. Then one easily sees that f induces a map $\bar{f}:(E_{B'}, E_{A'} \cup E_{B'}^1) \rightarrow (E, E_A \cup E^1)$, where $E_{B'} = E(f^* \xi)$, $E_{A'} = E(f^* \xi_A)$. Thus the following diagram commutes:

$$H^q(B, A) \xrightarrow{\ f^*\ } H^q(B', A')$$
$$\downarrow \psi'_{B,A} \qquad\qquad \downarrow \psi'_{B',A'}$$
$$H^{q+n}(E, E_A \cup E^1) \xrightarrow{\ \bar{f}^*\ } H^{q+n}(E_{B'}, E_{A'} \cup E_{B'}^1).$$

Suppose finally that $x \in H^*(B, A)$. Then

$$\mathrm{Sq}^k(\psi'_{B,A} x) = \mathrm{Sq}^k(p'^* x \cup U) = \sum_{i+j=k} p'^* \mathrm{Sq}^i x \cup \mathrm{Sq}^j U =$$

$$= \sum_{i+j=k} p'^* \mathrm{Sq}^i x \cup (p^* w_j \xi \cup U) =$$

$$= \sum_{i+j=k} p'^* (\mathrm{Sq}^i x \cup w_j \xi) \cup U = \sum_{i+j=k} \psi'_{B,A}(\mathrm{Sq}^i x \cup w_j \xi).$$

(Here $w_j \xi$ denotes the j-th Stiefel-Whitney class of ξ, $j \geq 0$.) The proof of 5.2 now follows from these properties of $\psi'_{B\ A}$ and the definition of $\psi_{B,A}$.

Remark. As indicated in § 2, we sometimes will regard the Thom class U as an element of $H^n(T(\xi),*)$ – i.e., $U = \psi_B(1)$ – and then we write $\psi_B(x) = U \cdot x$, for $x \in H^i(B)$.

6. Lifting the Postnikov invariant

We suppose now that all spaces have basepoint (written *), and that all maps preserve basepoints.

Let B, B' be complexes, and $\pi : B' \to B$ a map. Let $w \in H^n(B; J)$, where $J = Z$ or Z_p, p a prime. Suppose that $w \neq 0$ but that $\pi^* w = 0$. We regard w as a map $B \to K(J, n)$ and let

$$\Omega K(J, n) \overset{i}{\to} E \overset{p}{\to} B$$

denote the principal fibration over B induced by w. (See [26]). Since $\pi^* w = 0$, there is a map $q : B' \to E$ such that $pq = \pi$. That is, we have the following commutative diagram, where $F = \Omega K(J, n)$:

(*)

$$
\begin{array}{c}
F \\
\downarrow i \qquad \pi = pq \\
\to E \\
q \nearrow \quad \downarrow p \\
B' \longrightarrow B . \\
\quad \pi
\end{array}
$$

Let $k \in H^*(E, Z_p)$ be a class such that $q^* k = 0$. In our applications π will be a fiber map and k will be a Postinikov invariant for π. However in this section we consider k in the more general setting given above, and we study the problem of expressing such a class k in terms of cohomology invariants determined by B.

Suppose that k has degree t. We assume that the mod p cohomology morphism π^*

is surjective in degree t and that $t < 2n - 2$. Then there is an element α of the mod p Steenrod algebra such that

$$i^* k = \alpha \iota,$$

where ι denotes the fundamental class of $\Omega K(J, n)$.

For simplicity we now assume that $p = 2$. We will say the class w is *realizable* if:

(6.1) *there is a vector bundle ξ over B (of dim s, say) such that*

$$w = w_n \xi.$$

Furthermore, if $J = Z$, we assume that $w \not\equiv 0$ mod 2.

Let T and U denote the Thom complex and class of the bundle ξ. If Y is any space and $g: Y \to B$ a map, we let T_Y, U_Y denote the Thom complex and class of $g^* \xi$.

Recall the cohomology operation α given above. We will say that the pair (w, α) is *admissible* if the following conditions are fulfilled.

(6.2) *There is a relation*

$$\alpha \operatorname{Sq}^n = 0,$$

which holds on integral cohomology classes of degree $\leq s$.

(6.3) *There is a secondary cohomology operation Ω associated with relation 6.2 such that*

$$\Omega(U_{B'}) = T(\pi)^* M,$$

where M is a coset in $H^{s+t}(T)$ of the indeterminacy subgroup of Ω.

Remark 1. If n is odd and $J = Z$, then in 6.1 we regard w_n as $\delta^* w_{n-1}$, while in 6.2, we regard Sq^n as $\delta^* \operatorname{Sq}^{n-1}$.

Remark 2. Recall that for any space X, Ω has indeterminacy subgroup $\alpha H^*(X; J)$. Define $\kappa \subset H^t(E)$ to be the coset of k with respect to the subgroup

$$\text{Kernel } q^* \cap \text{Kernel } i^* \cap H^t(E).$$

We prove

THEOREM 6.4. *Let (w, α) be an admissible pair as defined above. Then there is a class $k' \in \kappa$ and a class $m \in H^t(B)$ such that*

$$U_B \cdot m \in M \quad \text{and} \quad U_E \cdot (k' + p^* m) \in \Omega(U_E).$$

Before giving the proof we note the following consequence.

Let X be a complex and $h: X \to B$ a map. Suppose that $h^* w = 0$. Then there is a map $l: X \to E$ such that $p \circ l = h$. By naturality we obtain from 6.4,

COROLLARY 6.5. *For any such map l, $U_X \cdot (l^* k' + h^* m) \in \Omega(U_X)$.*

We precede the proof of 6.4 with some remarks. Consider the following commutative diagram, with the notation defined below.

$$T_F \qquad \Omega K(J, n+s)$$

$$\Big\downarrow Ti \qquad \Big\downarrow \hat{\imath}$$

(**)
$$T_E \xrightarrow{\quad f \quad} \hat{E}$$

$$Tq \nearrow \quad \Big\downarrow Tp \qquad \Big\downarrow \hat{p}$$

$$T_{B'} \xrightarrow{\quad\quad} T_B \;=\; T_B \xrightarrow{\psi_B(w)} K(J, n+s).$$
$$T\pi$$

The left hand portion of the diagram is obtained from diagram (*) by taking the Thom complex of the various bundles induced from ξ. Commutativity follows from 5.1. The map $\hat{p}$ in the above diagram is the principal fibration induced by the cohomology class $\psi_B(w)$. By 5.2 (c),

$$(Tp)^* \psi_B(w) = \psi_E p^* w = 0,$$

and so the map Tp lifts to a map f as shown.

Let $\hat{\imath}$ denote the fundamental class of $\Omega K(J, n+s)$. At the end of the section we prove

LEMMA 6.6. *There is a class* $\hat{k} \in H^{t+s}(\hat{E})$ *such that*

$$\hat{\imath}^* \hat{k} = \alpha \hat{\imath}, \; T q^* f^* \hat{k} = 0.$$

Moreover, if $\hat{\kappa}$ *denotes the coset in* $H^{t+s}(\hat{E})$ *of* $\hat{k}$ *with respect to the subgroup*

$$\text{Kernel } \hat{\imath}^* \cap \text{Kernel } (f \circ T q)^* \cap H^{t+s}(\hat{E}),$$

then

$$f^* \hat{\kappa} \subset U_E \cdot \kappa.$$

We use 6.6 to prove 6.4.

Proof of Theorem 6.4. Since $w = w_n \xi$ it follows from THOM (see 5.2d) that

$$\psi_B(w) = \text{Sq}^n U.$$

Thus we can regard the map $\psi_B(w): T_B \to K(J, n+s)$ as the composite of the following maps:

$$T_B \xrightarrow{\quad U \quad} K(Z, s) \xrightarrow{\text{Sq}^n \iota_s} K(J, n+s),$$

where ι_s denotes the fundamental class of $K(Z, s)$.

Let $f: T_E \to \hat{E}$ be the map given in diagram (**). Set $\bar{f} = Tq \circ f: T_{B'} \to \hat{E}$, and consider the following commutative diagram, where the notation is explained below:

$$\Omega K(J, n+s) = \Omega K(J, n+s)$$

$$\downarrow \hat{\imath} \qquad \downarrow j$$

$$\hat{E} \xrightarrow{\quad v \quad} Y$$

$$f \nearrow \quad \downarrow \hat{p} \qquad \downarrow r$$

$$T_{B'} \xrightarrow{\quad\quad} T_B \xrightarrow{\quad\quad} K(J,s) \xrightarrow{\mathrm{Sq}^n \iota_s} K(J, n+s).$$
$$T\pi \qquad U$$

The map r is the principal fibration with $\mathrm{Sq}^n \iota_s$ as classifying map, and j is the fiber inclusion. Since $\hat{p}$ is defined to be the fibration with $\psi_B(w)$ as classifying map and since $\psi_B(w) = \mathrm{Sq}^n U$, we may regard $\hat{p}$ as the fibration induced by U from r. Thus v is simply the natural map for the induced fibration.

Notice that Y is the universal space for the operation Ω. Let $\omega \in H^{t+s}(Y)$ denote a representative class for Ω, chosen according to the specific choice of Ω given in 6.3. Set $k_0 = v^*\omega \in H^{t+s}(\hat{E})$. Since $j^*\omega = \alpha\iota$, we have $\hat{\imath}^*k_0 = \alpha\hat{\imath}$. Furthermore,

$$f^*k_0 \in \Omega(T\pi^* U) = \Omega(U_{B'}).$$

But by 6.3 there is then a class $m \in H^t(B)$ such that

$$U \cdot m \in M \quad \text{and} \quad f^*k_0 = T\pi^*(U \cdot m).$$

Set $k_0' = k_0 - \hat{p}^*(U \cdot m)$.   Then,

$$\hat{\imath}^* k_0' = \hat{\imath}^* k_0 - \hat{\imath}^* \hat{p}^*(U \cdot m) = \hat{\imath}^* k_0 = \alpha\hat{\imath} = \hat{\imath}^* \hat{k},$$
$$f^* k_0' = f^* k_0 - f^* \hat{p}^*(U \cdot m) = f^* k_0 - T\pi^*(U \cdot m) = 0.$$

Consequently, by definition of the coset $\hat{k}$, $k_0' \in \hat{k}$. On the other hand $k_0 \in \Omega(\hat{p}^* U)$ and so

$$k_0' + \hat{p}^*(U \cdot m) \in \Omega(\hat{p}^* U).$$

By 6.6 there is a class $k' \in \kappa \subset H^t(E)$ such that

$$f^* k_0' = U_E \cdot k'.$$

Therefore, by naturality,

$$U_E \cdot (k' + p^* m) \in \Omega(U_E),$$

since

$$\hat{p}f = Tp, \quad Tp^* U = U_E, \quad Tp^*(U \cdot m) = U_E \cdot p^* m.$$

Thus k' is the desired class and the proof of 6.4 is complete.

We are left with proving 6.6. Before so doing we prove a preliminary result. Let ξ be the s-plane bundle over B given in 6.1. Now it is easily seen that the Thom complex of $\xi|^*$ is simply an s-sphere S^s, which we may regard as embedded in T_B. Since the

fiber map $p: E \to B$ maps F to $*$ in B, it follows that $T_p(T_F) = S^s \subset T_B$. Furthermore the map $\psi_B w: T_B \to K(J, n+s)$ can be chosen so that $\psi_B w(S^s) = *$ in $K(J, n+s)$. Since $\hat{E}$ is the fiber space induced by $\psi_B w$, it follows that S^s is embedded in $\hat{E}$ in a natural way. Set $K = K(J, n+s)$. Then, $\hat{p}^{-1}(S^s) = \Omega K \times S^s \subset \hat{E}$, and diagram (**) gives the commutative diagram shown below, where bold face letters denote maps of pairs.

$$
\begin{array}{ccc}
(T_E, T_F) & \xrightarrow{\ \mathbf{f}\ } & (\hat{E}, \Omega K \times S^s) \\
\Big\downarrow \mathbf{T}_P & & \Big\downarrow \hat{\mathbf{p}} \\
(T_B, S^s) & = & (T_B, S^s).
\end{array}
$$

Set $g = f | T_F : T_F \to \Omega K \times S^s$. We use the above diagram to prove

LEMMA 6.7. $g^*(\hat{\imath} \otimes 1) \bmod 2 = \psi_F(\imath) \bmod 2$, where $\hat{\imath}$ and $\imath$ denote respectively the fundamental classes for ΩK and F.

Proof. Let $\mathbf{p}: (E, F) \to (B, *)$ denote the map of pairs determined by p. Since p has w as classifying map, we have

$$
\text{(a)} \qquad \delta \imath = - \mathbf{p}^* w \in H^n(E, F);
$$

and similarly,

$$
\text{(b)} \qquad \delta(\hat{\imath} \otimes 1) = - \mathbf{p}^* \psi_{B,*}(w) \in H^{n+s}(\hat{E}, \Omega K \times S^s).
$$

Therefore by naturality and the commutative diagram above

$$
\delta g^*(\hat{\imath} \otimes 1) = \mathbf{f}^* \delta(\hat{\imath} \otimes 1) = - \mathbf{f}^* \hat{\mathbf{p}}^* \psi_B w = - \mathbf{T}_p^* \psi_{B,*}(w).
$$

By 5.2 (c) and by (a) above,

$$
- \mathbf{T}_p^* \psi_{B,*}(w) = - \psi_{E,F} \mathbf{p}^* w = \psi_{E,F}(\delta \imath).
$$

But by 5.2 (a), $\psi_{E,F}(\delta \imath) = \delta \psi_F(\imath)$. Thus, we obtain

$$
\delta(g^*(\hat{\imath} \otimes 1)) = \delta \psi_F(\imath) \quad \text{in} \quad H^{n+s}(T_E, T_F; J).
$$

By SERRE [21, p. 469], $\mathbf{p}^* w \neq 0 \bmod 2$ since (by 6.1) $w \neq 0 \bmod 2$. Thus by (a) above and 5.2 (a),

$$
\delta: H^{n+s-1}(T_F; Z_2) \to H^{n+s}(T_E, T_F; Z_2)
$$

is injective and so $g^*(\hat{\imath} \otimes 1) = \psi_F(\imath) \bmod 2$, as claimed.

Proof of 6.6. Since $w = w_n \xi$, it follows by THOM that

$$
\psi_B w = \mathrm{Sq}^n \psi_B(1) = \mathrm{Sq}^n U.
$$

Let α be the mod 2 Steenrod operation given at the neginning of the section. By 6.2, $\alpha \mathrm{Sq}^n = 0$ and so $\alpha \psi_B w = 0$. Applying the SERRE exact sequence [21, p. 468] to the fibration $\hat{p}$ (see diagram (**)), we see that by exactness there is a class $\hat{k} \in H^{t+s}(\hat{E})$ such that

$$
\hat{\imath}^* \hat{k} = \alpha \hat{\imath}.
$$

Furthermore, by using the exact sequence given in § 3 of [26] (with respect to the map $f \circ T_q : T_B' \to \hat{E}$) it is easily shown that $\hat{k}$ can be chosen so that, in addition, $T_q^* f^* \hat{k} = 0$.

Now the inclusion $\hat{i} : \Omega K \subset \hat{E}$ can be factored into the composite

$$\Omega K \xrightarrow{\;l\;} \Omega K \times S^s \xrightarrow{\;\hat{j}\;} \hat{E},$$

where l is the natural injection and where $\hat{j}$ is the inclusion. Since

$$\hat{i}^* \hat{k} = \alpha \hat{i},$$

it follows that

$$\hat{j}^* \hat{k} = \alpha(\hat{i} \otimes 1).$$

Let $k_1 \in H^t(E)$ be the unique class such that

$$U_E \cdot k_1 = f^* \hat{k} \in H^{t+s}(T_E).$$

We will show that $k_1 \in \kappa$, which then will complete the proof of 6.6.

Using 5.2 we have:

$$U_{B'} \cdot q^* k_1 = T q^* (U_E \cdot k_1) = T q^* f^* \hat{k} = 0.$$

Therefore, $q^* k_1 = 0$. On the other hand,

$$U_F \cdot i^* k_1 = T i^* (U_E \cdot k_1) = T_i^* f^* \hat{k}.$$

But by definition of g and $\hat{j}$, $f \cdot T_i = \hat{j} \cdot g$. Thus

$$T_i^* f^* \hat{k} = g^* \hat{j}^* \hat{k} = g^* (\alpha(\hat{i} \otimes 1)),$$

by the above computation. By 6.7,

$$g^* (\alpha(\hat{i} \otimes 1)) = \alpha g^* (\hat{i} \otimes 1) = \alpha \psi_F(\iota).$$

Now the bundle $i^* p^* \xi$ is trivial and so by 5.2 (d),

$$\alpha \psi_F(\iota) = \psi_F(\alpha \iota).$$

Also, by definition,

$$U_F \cdot i^* k_1 = \psi_F(i^* k_1).$$

Therefore

$$\psi_F(\alpha \iota - i^* k_1) = 0$$

and so $i^* k_1 = \alpha \iota$. Consequently, $k_1 \in \kappa$, which completes the proof of 6.6.

Remark 3. The theory leading up to 6.4 can be generalized in the following way. The single cohomology class w can be replaced by a vector of cohomology classes $w = (w_1, \ldots, w_a)$, with $\pi^* w_i = 0$. By making the appropriate changes in 6.1–6.3 one then can state a more general version of 6.4 so that it includes, for example, Theorem 3.3.2 of [15] as a special case.

Remark 4. Theorem 6.4 (as well as the generalization suggested above) is a special case of Theorem 5.9 in [27]. The Thom class U is a "generating class" for κ, in the language of § 5 of [27].

7. Proof of 2.2

Let n be an integer greater than three and set

$$B' = BSO(n - 1), B = BSO(n + 1).$$

For any group G we let BG denote the classifying space for G defined by MILNOR 21]. We denote the various rotation groups by $SO(q)$, $q \geq 2$.) The inclusion $SO(n-1) \subset SO(n+1)$ induces a map $\pi: B' \to B$. If we regard π as a fiber map, its fiber is the Stiefel manifold $V_{n+1, 2}$.

Let X be a complex. Then a map $\xi: X \to B$ can be regarded as an oriented $(n+1)$-plane bundle over X. Moreover this bundle has two linearly independent cross-sections iff the map ξ can be factored through B' via π.

We construct a Postnikov resolution for the map π, through dimension $n+1$, as shown below.

Here

$$
\begin{array}{llll}
J = Z_2, & w = w_n \gamma, & \text{if} & n \quad \text{even} \\
J = Z, & w = \delta^* w_{n-1} \gamma, & \text{if} & n \quad \text{odd},
\end{array}
$$

where γ denotes the canonical $(n+1)$-plane bundle over B. The map p is the principal fibration with w as classifying map, and i is the inclusion of the fiber of p into E.

Let F denote the "fiber" of the map q (in the sense of [9]). By the choice of w, we see that F is $(n-1)$-connected and that

$$\pi_n F = Z_2 \text{ or } Z \oplus Z_2,$$

according to whether n is even or odd. Let $\gamma_n \in H^n(F; Z_2)$ denote the fundamental class if n is even; for n odd let it denote the cohomology class corresponding to the homomorphism $Z \oplus Z_2 \to Z_2$ given by projection on the right hand summand. Let $k \in H^{n+1}(E; Z_2)$ denote the transgression of the class γ_n. Then (see [10], [26]),

$$i^* k = \text{Sq}^2 \iota, q^* k = 0,$$

where ι denotes the fundamental class of $K(J, n-1)$. Moreover, a simple argument

using the transgression operator (e.g., see [18]) shows that

$$\text{Kernel } i^* \cap \text{Kernel } q^* \cap H^{n+1}(E) = \begin{cases} 0, & n \quad \text{even} \\ p^* w_{n+1}, & n \quad \text{odd}. \end{cases} \tag{7.1}$$

Let ξ be a bundle over a complex X as above, and suppose that $\xi^* w = 0$. Then the map ξ lifts to the space E. We define

$$k(\xi) = \bigcup_\eta \eta^* k \subset H^{n+1}(X),$$

where the union is over all maps $\eta: X \to E$ such that $p\eta = \xi$. It is easily shown (see [14], [26]) that if n is even then $\xi|X^{n+1}$ lifts to B' iff $0 \in k(\xi)$, while if n is odd then $\xi|X^{n+1}$ lifts to B' iff $\chi(\xi) = 0$ and $0 \in k(\xi)$. In particular, $\xi|X^n$ lifts to B'.

Furthermore, by a standard argument ([14], [26]), one sees that $k(\xi)$ is a coset in $H^{n+1}(X)$ of the subgroup $S^{n+1}(X, \xi)$ consisting of all classes of the form

$$\text{Sq}^2(u) + u \cup w_2(\xi),$$

for all $u \in H^{n-1}(X; J)$. In particular if $\text{Sq}^2 u = u \cup w_2 \xi$ for all such u, then $k(\xi)$ consists of a single class. We use the theory of § 6 to compute the coset $k(\xi)$.

At the end of the section we prove

LEMMA 7.2. *Let* $n \equiv -1$, *or* $0 \mod 4$, $n \geq 4$. *Then the operation* Ω_{n+1} (*see* § 2) *can be chosen so that*

$$U_{B'} \cdot (w_2 w_{n-1}) \in \Omega_{n+1}(U_{B'})$$

where $U_{B'}$ *denotes the Thom class of* $\pi^* \gamma$.

By definition, w is realizable as given in 6.1. Furthermore by relation 2.1 and by 7.2 it follows that the pair (w, Sq^2) is admissible, in the sense of 6.2 and 6.3. (To satisfy 6.3 we need only observe that $\pi^*: H^*(B) \to H^*(B')$ is surjective.)

Let T_X, U_X denote the Thom complex and Thom class for the bundle $\xi(=\xi^* \gamma)$. If $S^{n+1}(X, \xi) = 0$, then one easily sees that $\text{Sq}^2 H^{2n}(T_X; J) = 0$. Therefore, if $\xi^* w = 0$, then Ω_{n+1} is defined on U_X with zero indeterminacy.

Notice that by 7.1 κ is a coset of 0 if n is even, while if n is odd κ is a coset of the subgroup generated by $p^* w_{n+1}$. Thus by 6.5 and 7.2, we have

THEOREM 7.3. *Let* ξ *be an oriented* $(n+1)$-*plane bundle over* X *such that*

$$\xi^* w = 0, \qquad S^{n+1}(X, \xi) = 0.$$

Then

$$U_X \cdot (k(\xi) + w_2(\xi) w_{n-1}(\xi) + b_{n+1} w_{n+1}(\xi)) = \Omega_{n+1}(U_X),$$

with zero indeterminacy, where Ω_{n+1} *is given in* 7.2, *where* $b_{n+1} \in Z_2$, *and where* $b_{n+1} = 0$ *if* n *is even*.

Proof of 2.2. Take X to be a spin manifold M of dim $m = n+1$, and take $\xi = \tau$,

its tangent bundle. If $n=4s-1$, MASSEY shows that $\delta^* w_{4s-2} \tau = 0$. If $n=4s$, we assume (as in 2.2) that $w_{4s}\tau = 0$. Thus in either case $\tau^* w = 0$ and so τ restricted to M^n has 2 independent cross-sections – i.e., there is a tangent 2-field on M with isolated singularities. By WU [29], $S^{n+1}(M, \tau) = 0$. Thus the class $k(\tau)$ is independent of the particular choice of 2-field. In the language of § 2, $k(\tau) = (I_2 M)\mu$ and so 2.2 follows directly from 7.3 since $w_2 M = 0$, and since we assume (in 2.2) that $w_m M = 0$, when m is even.

Proof of 7.2. We recall the following facts about Thom complexes, due to ATIYAH [4].

(7.4) (ATIYAH). *Let X be a complex and let η be a vector bundle over X. Let ε denote (in general) the trivial line bundle. Then*

$$T(\eta \oplus \varepsilon) = \Sigma T(\eta), \quad U(\eta \oplus \varepsilon) = \Sigma U(\eta),$$

where Σ denotes the reduced suspension operator and where T, U denote the appropriate Thom complex and Thom class. Furthermore, if $x \in H^*(X)$, then

$$\Sigma(U(\eta) \cdot x) = (\Sigma U(\eta)) \cdot x.$$

Let γ' denote the canonical $(n-1)$-plane bundle over the classifying space B'. Then

$$\pi^* \gamma = \gamma' \oplus 2\varepsilon,$$

and so by 7.4,

$$T_{B'} = \Sigma^2 T', \qquad U_{B'} = \Sigma^2 U',$$

where T', U' denote the Thom complex and class of γ'. Also by 7.4 we have

$$U_{B'} \cdot (w_2 w_{n-1}) = \Sigma^2(U' \cdot w_2 w_{n-1}).$$

But

$$\Sigma^2(U' \cdot w_2 w_{n-1}) = \Sigma^2(U' \cdot Sq^2 U'),$$

since $Sq^2 U' = U' \cdot w_2$, $U' \cdot w_{n-1} = Sq^{n-1} U' = (U')^2$ mod 2. Thus

$$U_{B'} \cdot (w_2 w_{n-1}) = \Sigma^2(U' \cdot Sq^2 U'),$$

and so 7.2 is simply a special case of the following result.

LEMMA 7.5. *Let X be a complex and let $u \in H^{m-2}(X)$, $m \equiv 0$ or 1 mod 4. Then Ω_m is defined on $\Sigma^2 u$ and Ω_m can be chosen so that*

$$\Sigma^2(u \cdot Sq^2 u) \in \Omega_m(\Sigma^2 u),$$

Proof. The proof is similar to that given by MAHOWALD-PETERSON for Theorem 2.2.1 in [15], and so is omitted.

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