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The Index of a Tangent 2-Field¹)

by Emery Thomas (Berkeley)

Dedicated to Professor H. Hopf

1. Introduction

Let M be a connected, smooth, Riemannian manifold, and let k be a positive integer. By a k-field on M we mean an ordered set of k orthonormal tangent vector fields. We say that M has a k-field with finite singularities if there is a k-field on the manifold obtained from M by removing a finite number of points. Let $(X_1, ..., X_k)$ be such a k-field. Choose a triangulation of M such that each singular point of the k-field lies in the interior of a distinct m-simplex $(m = \dim M)$. Let p be a singular point, say in the interior of the closed simplex σ . Suppose now that M is oriented. The tangent bundle of M restricted to σ is then isomorphic to the trivial bundle $\sigma \times R^m$, by an orientation preserving isomorphism, and this isomorphism can be chosen to be compatible with the standard Riemannian metric on $\sigma \times R^m$. Thus for each point q in $\sigma - \{p\}$ we can regard $(X_1(q), ..., X_k(q))$ as an orthonormal k-frame in \mathbb{R}^m – that is, as a point in the Stiefel manifold $V_{m,k}$. Since M is oriented the boundary of σ , $\dot{\sigma}$, is then an oriented (m-1)-sphere. By the above remarks one sees that the k-field restricted to $\dot{\sigma}$ gives a map $\dot{\sigma} \rightarrow V_{m,k}$ and the homotopy class of this map is then an element of the homotopy group $\pi_{m-1}(V_{m,k})$. We define this homotopy class to be the index of the k-field at the singular point p (see HOPF [12], [13]), and write this Index $(X_1, ..., X_k)_p$. Now let $\{p_1, ..., p_r\}$ be the set of singular points of the k-field. We define

Index
$$(X_1, ..., X_k) = \sum_{i} \text{Index}(X_1, ..., X_k)_{p_i} \in \pi_{m-1}(V_{m,k}).$$

One can show that this definition of the index agrees with the definition one obtains via obstruction theory. (See §§ 29-34 in [24].) This implies that the definition is independent of the choices made above; in particular it is independent of the orientation of M. Also, from obstruction theory it follows that $Index(X_1, ..., X_k) = 0$ iff there is a k-field without singularities on M which coincides with $(X_1, ..., X_k)$ on the (m-2)-skeleton of M. (See 34.2 of [24].)

A 1-field X on M is simply a field of unit tangent vectors. Since $V_{m,1} = (m-1)$ sphere and $\pi_{m-1}(V_{m,1}) = Z$, we may regard Index (X) as an integer. The celebrated
theorem of H. HOPF [12] states that if X is a 1-field with finite singularities on a closed
manifold²) M, then
Index $(X) = \chi(M)$,

where $\chi(M)$ denotes the Euler characteristic of M.

¹⁾ Research supported by the National Science Foundation.

²⁾ By using local coefficients one can define the index on a non-orientable manifold (See [24, §39.5].)

Let (X_1, X_2) be a 2-field with finite singularities on a closed oriented manifold M of dim m, with m>4. The index of (X_1, X_2) is then an element of the homotopy group $\pi_{m-1}(V_{m,2})$. This group depends on the parity of m as is shown below (see [8]):

$$\pi_{m-1}(m, 2) = \begin{cases}
Z_2, & \text{if } m & \text{odd} \\
Z \oplus Z_2, & \text{if } m & \text{even}.
\end{cases}$$

Thus if m is odd we can regard Index (X_1, X_2) as an integer mod 2. If m is even we write

Index
$$(X_1, X_2) = (Index_0(X_1, X_2), Index_2(X_1, X_2)),$$

where $\operatorname{Index}_0(X_1, X_2) \in \mathbb{Z}$, $\operatorname{Index}_2(X_1, X_2) \in \mathbb{Z}_2$. It is easily shown (see § 7 below) that $\operatorname{Index}_0(X_1, X_2) = \chi(M)$. In a previous paper [27] we have proved: If $m \equiv 2$ or 3 mod 4, and if (X_1, X_2) is a 2-field with finite singularities, then

Index₂
$$(X_1, X_2) = 0$$
, if $m \equiv 2(4)$,
Index $(X_1, X_2) = 0$, if $m \equiv 3(4)$.

The purpose of this paper is to consider 2-fields on m-manifolds where $m \equiv 0$, 1 mod 4. The case of 4-manifolds has been completely solved by F. HIRZEBRUCH and H. HOPF [11]. For the rest of the section let M denote a closed oriented manifold of dim m, with m>4. Let $w_i M \in H^i(M; Z_2)$ denote the i^{th} Stiefel-Whitney class of M, $i \ge 1$. Recall (see § 39.1 in [24]) that if m is odd then M has a 2-field with finite singularities iff $w_{m-1} M = 0$, while if m is even then M has such a 2-field iff $\delta^* w_{m-2} M = 0$. (Here δ^* denotes the Bockstein coboundary from mod 2 coefficients to integer coefficients.) Massey [17] has shown that if m is even then one always has $\delta^* w_{m-2} M = 0$. Thus an orientable manifold of even dimension always has a 2-field with finite singularities.

Define

$$\chi^+ M = \sum_i \dim H_i(M; Z_2).$$

If $\chi^+ M$ is an even integer (as will be the case, for example, when m is odd), we define 3) an integer mod 2 by

$$\hat{\chi}_2 M = \frac{1}{2} \chi^+ M \mod 2.$$

We will prove the following result. (Recall that M is called a *spin* manifold if $w_2 M = 0$.)

THEOREM 1.1. Let M be a closed spin manifold of dim 4k+1, k>0, such that $w_{4k}M=0$. If (X_1, X_2) is any 2-field with finite singularities, then

$$\operatorname{Index}(X_1, X_2) = \hat{\chi}_2 M.$$

As an immediate consequence we have

³⁾ See Kervaire, Math. Ann. 131 (1956) 220.

COROLLARY 1.2. Let M be a closed spin manifold of dim 4k+1, k>0. Then M has a 2-field without singularities if, and only if,

$$w_{4k}M=0, \qquad \hat{\chi}_2M=0.$$

In case M is a π -manifold, this is given as part of Theorem 2 in [6].

The case $m \equiv 0 \mod 4$ requires an additional hypothesis. Let M be a manifold of even dimension, say 2q. We call M symplectic if, for all classes $u \in H^q(M; \mathbb{Z}_2)$, $u^2 = 0$. We show below that if M is a spin manifold of dim 8k + 4, $k \ge 0$, then M is symplectic. Also, we will show that if M is symplectic then $w_{2q}M = 0$, and so the Euler characteristic of M is an even integer. Therefore, by Poincaré duality, it follows that $\chi^+ M$ is also even and so $\hat{\chi}_2 M$ is defined. We will prove

THEOREM 1.3. Let M be a closed spin manifold of dim m, where $m \equiv 0 \mod 4$ and m>4. If $m \equiv 0 \mod 8$ assume that M is symplectic. Then for any 2-field (X_1, X_2) with finite singularities. Index₂ $(X_1, X_2) = \hat{\chi}_2 M$.

Suppose that dim M=4k, k>0; set $d_i=\dim H_i(M; Z_2)$. By Poincaré duality,

$$\chi(M) = \sum_{i=0}^{2k-1} (-1)^{i} 2d_{i} + d_{2k},$$

$$\chi^{+} M = \sum_{i=0}^{2k-1} 2d_{i} + d_{2k}.$$

Therefore,

$$\chi^{+} M = \left(\sum_{i=0}^{2k-1} 2(1-(-1)^{i}) d_{i}\right) + \chi(M),$$

and so if $\chi(M)$ is even

$$\hat{\chi}_2 M = \left(\frac{1}{2}\chi(M)\right) \bmod 2.$$

In particular

$$\hat{\chi}_2 M = 0$$
 if, and only if, $\chi(M) \equiv 0 \mod 4$.

As a consequence we have

COROLLARY 1.4. Let M be a closed spin manifold as in 1.3. Then M has a 2-field without singularities if, and only if, $\chi(M)=0$.

Recall that a manifold M of even dimension 2q is said to have an almost-complex structure if there is a complex q-plane bundle ω over M such that the tangent bundle of M is equivalent to the real bundle underlying ω . Now this complex bundle ω has a complex 1-field with finite singularities, and the index of this 1-field is simply $\chi(M)$ [19, pp. 61, 65]. Moreover the complex 1-field determines a (real) 2-field on M also with finite singularities and for this 2-field (X_1, X_2) , Index₂ $(X_1, X_2) = bw_{2q}M$, $b \in \mathbb{Z}_2$. Thus by 1.3 and the computation given above for $\hat{\chi}_2 M$, we obtain

COROLLARY 1.5. Let M be a closed spin manifold as in 1.3. If M admits an almost-complex structure, then the Euler characteristic of M is divisible by 4.

This argument was originally used by Hopf [13] to show that S^4 and S^8 do not admit almost-complex structures.

Let M be an m-manifold and let $V = \sum_{i=1}^{m} V_i$ denote the WU class [29]. That is, if $u \in H^{m-i}(M; \mathbb{Z}_2)$ then

$$\mathrm{Sq}^i(u)=u\cdot V_i,$$

where Sq^i denotes the mod 2 Steenrod operator of degree $i, i \ge 1$. The Theorem of Wu is that

$$w_k M = \sum_{i=0}^k \operatorname{Sq}^i V_{k-i}, \quad k \ge 1.$$

Thus if m is even, say m=2q,

$$w_{2q}M = \operatorname{Sq}^q V_q = V_q^2$$
.

But by definition, M is symplectic iff $V_q = 0$, and so if M is symplectic then $w_{2q}M = 0$, as asserted above. Also, by an easy extension of [16, Theorem III], one shows that if M is a spin m-manifold, then $V_{4k+2} = 0$, $k \ge 0$ (since $\operatorname{Sq}^2 H^{m-2}(M; \mathbb{Z}_2) = 0$). Therefore if $m = 4 \mod 8$, M is symplectic as remarked above.

2. Proof of 1.1 and 1.3.

Throughout this section M will denote a closed oriented m-manifold, with $m \equiv 0$ or $1 \mod 4$, m > 4. We will show in § 7 that if (X_1, X_2) is a 2-field on M with isolated singularities, then the index is independent of the particular choice of 2-field. We define a mod 2 integer, $I_2 M$, by setting

$$I_2 M = \begin{cases} \operatorname{Index}_2(X_1, X_2), & \text{if } m \equiv 0(4) \\ \operatorname{Index}(X_1, X_2), & \text{if } m \equiv 1(4). \end{cases}$$

Let T denote the Thom complex of the tangent bundle of M and $U \in H^m(T; Z)$ the Thom class (see [25], [19]). $H^*(T)$ can be regarded as a module over $H^*(M)$ (integer or mod 2 coefficients). By Thom [25] the map $H^i(M) \to H^{m+i}(T)$, given by $x \to U \cdot x$, is an isomorphism for all i > 0. Thus to determine the mod 2 integer I_2M it suffices to compute $U \cdot (I_2M\mu)$, where $\mu \in H^m(M; Z_2)$ is the generator. For this we will need a secondary cohomology operation.

Recall that one has the following ADEM relation [2], when $m \equiv 0$, 1 mod 4.

(*)
$$Sq^2 Sq^{m-1} + Sq^m Sq^1 = Sq^{m+1}$$
.

If u is an integral cohomology class of $\dim < m+1$, then

$$\operatorname{Sq}^{1} u = 0, \quad \operatorname{Sq}^{m+1} u = 0.$$

Also, if m is even we can write

$$Sq^{m-1} = Sq^{1}Sq^{m-2} = (\delta^{*}Sq^{m-2}) \mod 2$$
.

Thus we have the following two non-stable relations:

$$m \equiv 0(4) : \operatorname{Sq}^{2}(\delta^{*} \operatorname{Sq}^{m-2}) = 0,$$

 $m \equiv 1(4) : \operatorname{Sq}^{2} \operatorname{Sq}^{m-1} = 0,$ (2.1)

where in each case the relation obtains on integral classes of dim $\leq m$.

Let Ω_m denote a (non-stable) secondary cohomology operation associated with each of the above relations, $m \equiv 0.1 \mod 4$. (See [1] and [7].) Thus if X is a space and if $u \in H^j(X; \mathbb{Z})$, $j \leq m$, then Ω_m is defined on u, provided that

$$\delta^* \operatorname{Sq}^{m-2} u = 0$$
 if $m \equiv 0(4)$, $\operatorname{Sq}^{m-1} u = 0$ if $m \equiv 1(4)$.

Furthermore

$$\Omega_m(u)$$
 is a coset in $H^{m+j}(X; \mathbb{Z}_2)$

of the subgroup

$$\operatorname{Sq}^{2} H^{m+j-2}(X; Z), \quad \text{if} \quad m \equiv 0(4),$$

 $\operatorname{Sq}^{2} H^{m+j-2}(X; Z_{2}), \quad \text{if} \quad m \equiv 1(4).$

We will prove

THEOREM 2.2. Let M be a closed spin manifold of $\dim m$, where $m \equiv 0$ or $1 \mod 4$ and m>4. If m is odd assume that w_{m-1} M=0, while if m even assume that $w_m M=0$. Then the operation Ω_m is defined on the Thom class U and the operation can be chosen so that

$$\Omega_m(U) = U \cdot (I_2 M \mu).$$

with zero indeterminacy.

This will be proved in § 7, following the method of Mahowald-Peterson [15]. (Theorem 2.2 is similar to Theorem 3.3.2 in [15], but the details of our proof will be somewhat different as we will use the point of view of § 5 in [27]).

To prove 1.1 and 1.3 we need to compute the operation Ω_m . This is done as follows. Assume that the tangent bundle of M has been given a Riemannian metric; let E denote the set of tangent vectors of length ≤ 1 , and let E^1 denote the set of vectors of length 1. Then $T = E/E^1$ (= the space obtained from E by collapsing E^1 to a point). Moreover the collapsing map induces an isomorphism

$$H^*(E/E^1,*) \approx H^*(E,E^1),$$

and so we regard the Thom class U equally well as a class in $H^m(E, E^1; Z)$. MILNOR shows in [19] that there is an isomorphism

$$e: H^*(E, E^1) \approx H^*(M^2, M_2 - \text{diagonal}),$$

where $M^2 = M \times M$. Let $j: M^2 \subset (M^2, M^2 - \text{diagonal})$ denote the inclusion, and set

$$\underline{U}=j^*e(U)\in H^m(M^2;Z).$$

Now the isomorphism e is induced by maps and so commutes with all cohomology operations. Thus Ω_m is defined on \underline{U} . Assume that $w_2 M = 0$. Then

$$\operatorname{Sq}^{2} H^{m-2}(M) = 0$$
, $\operatorname{Sq}^{2} H^{2m-2}(M^{2}) = 0$,

and so Ω_m is defined with zero indeterminacy on U and \underline{U} . By naturality,

$$\Omega_m(\underline{U}) = j^* e \Omega_m(\underline{U}).$$

But j^* is injective (as remarked in [3]) and so

$$\Omega_m(\underline{U}) = 0$$
 if, and only if, $\Omega_m(U) = 0$.

Since a mod 2 integer is unchanged by squaring, we obtain from 2.2,

Proposition 2.3. Let M be a manifold as in 2.2. Then

$$\Omega_m(\underline{U}) = I_2 M(\mu \oplus \mu) \in H^{2m}(M^2; Z_2).$$

To compute $\Omega_m(\underline{U})$ we reduce \underline{U} mod 2. Consider the following non-stable relations (see (*)):

$$m \equiv 0(4) : \operatorname{Sq}^{2}(\delta^{*} \operatorname{Sq}^{m-2}) + \operatorname{Sq}^{m} \operatorname{Sq}^{1} = 0, m \equiv 1(4) : \operatorname{Sq}^{2} \operatorname{Sq}^{m-1} + \operatorname{Sq}^{1}(\operatorname{Sq}^{m-1} \operatorname{Sq}^{1}) = 0,$$
 (2.4)

where in each case the relation obtains on mod 2 classes of dim $\leq m$. Let $\tilde{\Omega}_m$ denote a (non-stable) operation associated with each relation in 2.4.

Let M be a manifold as in 2.2. Regarding \underline{U} as a class mod 2, $\widetilde{\Omega}_m$ is defined on \underline{U} , and with zero indeterminacy when $m \equiv 1$. When $m \equiv 0$, $\widetilde{\Omega}_m$ has $\operatorname{Sq}^m H^m(M^2)$ as indeterminacy subgroup. But if M is symplectic them $\operatorname{Sq}^m H^m(M^2) = 0$, and so $\widetilde{\Omega}_m(\underline{U})$ will again be defined with zero indeterminacy. By considering the universal examples for Ω and $\widetilde{\Omega}$ it is easily shown that, with all these hypotheses on M, $\widetilde{\Omega}_m$ can be chosen so that

$$\tilde{\Omega}_m(\underline{U}) = \Omega_m(\underline{U}), \tag{2.5}$$

where Ω_m denotes the specific choice of operation given in 2.2.

Thus, as our final step, we compute $\widetilde{\Omega}_m(\underline{U})$. Let $t:H^*(M^2)\to H^*(M^2)$ denote the isomorphism induced by interchanging the factors of M^2 .

THEOREM 2.6. Let M be an m-manifold as in 2.2. If m is even assume that M is symplectic. Then there is a mod 2 class $A \in H^m(M^2)$ such that

- a) $\underline{U} \mod 2 = A + tA$,
- b) $A \cup tA = \hat{\chi}_2 M(\mu \otimes \mu)$,
- c) $\tilde{\Omega}_m$ is defined on A.

The proof will be given in § 4.

Proof of 1.1 and 1.3. By 2.3 and 2.5,

$$\widetilde{\Omega}_m(\underline{U}) = I_2 M(\mu \oplus \mu).$$

Now $\tilde{\Omega}_m$ is a non-stable operation of degree m. By 2.6 c) $\tilde{\Omega}$ is defined on A and thus also on tA. Therefore, by [7, cf. 2. 3],

$$\widetilde{\Omega}(A+tA)=\widetilde{\Omega}(A)+\widetilde{\Omega}(tA)+A\cup tA.$$

Since t is the identity on $H^{2m}(M^2)$, we have by naturality,

$$\widetilde{\Omega}_m(A) = t \, \widetilde{\Omega}_m(A) = \widetilde{\Omega}_m(t \, A).$$

Consequently, by 2.6 a) and b),

$$\widetilde{\Omega}_{m}(\underline{U}) = \widetilde{\Omega}_{m}(A + tA) = A \cup tA = \widehat{\chi}_{2} M(\mu \oplus \mu).$$

But $\widetilde{\Omega}_m(\underline{U}) = I_2 M(\mu \otimes \mu)$, and so

$$I_2 M = \hat{\chi}_2 M$$
,

which completes the proof of 1.1 and 1.3.

3. Mod 2 vector spaces

Most of the work in proving Theorem 2.6 will come in the case m even. This section develops some simple facts about mod 2 vector spaces needed for this case. The proof of 2.6 is then given in the next section.

Let V be a finite-dimensional mod 2 vector space. An endomorphism t of V is called an *involution* if $t^2=1$. An endomorphism d is called a *boundary* if $d^2=0$. Suppose that V has an involution t and a boundary d. We say that the pair (t, d) is regular if

$$t d = d t, (3.1)$$

and

there are subspaces A, B in V such that

$$dB = 0$$
 and $V = A \oplus tA \oplus dA \oplus tdA \oplus B \oplus tB$. (3.2)

Define

$$\Delta = t + 1: V \rightarrow V$$
.

LEMMA 3.3. Let t be an involution on V and d a boundary such that the pair (t, d) is regular. Then

$$(\operatorname{Ker} d) \cap (\operatorname{Ker} \Delta) = \Delta (\operatorname{Ker} d).$$

Proof. Because V is a \mathbb{Z}_2 -module, $\Delta^2 = 0$. Also by 3.1, $\Delta d = d\Delta$, and so

$$\Delta(\operatorname{Ker} d) \subset \operatorname{Ker} d \cap \operatorname{Ker} \Delta$$
.

We prove 3.3 by showing that the opposite inclusion holds. Let $v \in V$ be an element such that

$$dv=0$$
, $\Delta v=0$.

By 3.2 we can write v as

$$v = a_1 + t a_2 + d a_3 + t d a_4 + b_1 + t b_2$$

where the a's are in A and the b's in B. Since dv = 0 and dB = 0, we must have

$$d a_1 = d t a_2 = 0$$
.

Furthermore

$$\Delta v = (a_1 + a_2) + (t a_1 + t a_2) + (d a_3 + d a_4) + (t d a_3 + t d a_4) + (b_1 + b_2) + (t b_1 + t b_2).$$

Since $\Delta v = 0$ this means, by 3.2, that

$$a_1 = a_2$$
, $da_3 = da_4$, $b_1 = b_2$.

Therefore

$$v = \Delta(a_1 + da_3 + b_1)$$
, and $d(a_1 + da_3 + b_1) = 0$,

which completes the proof.

Let X be a space whose total singular integral homology module is finitely generated. Let $H^*(X)$ denote the mod 2 cohomology algebra of X. By the Künneth theorem for cohomology,

$$H^*(X^2) \approx H^*(X) \otimes H^*(X)$$

where $X^2 = X \times X$.

Let $t: H^*(X^2) \to H^*(X^2)$ denote the involution induced by transposing the factors of X^2 . We will call an element $v \in H^*(X^2)$ symmetric if $\Delta v = 0$, where $\Delta = t + 1$. Let $\alpha = (\alpha_1, ..., \alpha_q)$ be a basis for $H^*(X^2)$. An element $v \in H^*(X^2)$ will be called symplectic with respect to α if

$$v = \sum_{i,j} c_{ij} \alpha_i \otimes \alpha_j,$$

where all $c_{ii} = 0$, $1 \le i \le q$.

LEMMA 3.4. Let $v \in H^*(X^2)$ be a symmetric class. If v is symplectic with respect to one basis, then it is so with respect to any basis.

Proof. With respect to a second basis for $H^*(X)$, the matrix $C = (c_{ij})$ becomes a matrix $C' = (c'_{ij})$, which is obtained from C by symmetric row and column operations [27, p. 188]. Thus C' is also symmetric. Moreover each such pair of row and column operations leaves unchanged the diagonal elements of C (since $c_{ii} = 0$ and we are working over Z_2). Thus C' remains symplectic, i.e., $c'_{ii} = 0$, $1 \le i \le q$. This completes the proof.

The main result of the section is the following.

PROPOSITION 3.5. Let $v \in H^{2n}(X^2)$, n > 0. Suppose that

$$\Delta v = 0, \quad \operatorname{Sq}^1 v = 0$$

and that v is symplectic. Then there is a class u such that

$$\Delta u = v$$
, $\operatorname{Sq}^1 u = 0$.

Proof. Set $d=\operatorname{Sq}^1$. Then $d^2=0$ and td=dt. We choose a basis $\alpha_1,...,\alpha_q$ for $H^*(X)$ so that for some integer r,

$$d\alpha_i = \alpha_{r+i}, \quad 1 \le i \le r,$$

$$d\alpha_j = 0, \quad 2r + 1 \le j \le q.$$

Define $W \subset H^*(X^2)$ to be the subspace spanned by all basis elements $\alpha_i \otimes \alpha_j$, with $i \neq j$. Notice that the class v is in W because v is symplectic.

Now set s=q-2r, and let $b_i=\alpha_{2r+i}$, $1 \le i \le s$, where r and q are given above. Define A, $B \subset W$ to be the subspaces spanned by the basis elements shown below:

$$A: \{\alpha_{i} \otimes \alpha_{j}, d \alpha_{i} \otimes \alpha_{j}, 1 \leq i < j \leq r; \\ \alpha_{i} \otimes d \alpha_{j}, 1 \leq i \leq j \leq r; \\ \alpha_{i} \otimes b_{j}, 1 \leq i \leq r, 1 \leq j \leq s. \}.$$

$$B: \{d \alpha_{i} \otimes d \alpha_{j}, 1 \leq i < j \leq r; \\ d \alpha_{i} \otimes b_{j}, 1 \leq i \leq r, 1 \leq j \leq s; \\ b_{i} \otimes b_{i}, 1 \leq i < j \leq s. \}.$$

Then, as is readily seen,

(*)
$$W = A \oplus t A \oplus B \oplus t B, \quad dB = 0.$$

For any subspace $U \subset H^*(X^2)$, set $U^i = U \cap H^i(X^2)$, $i \ge 0$. Notice that the classes $d\alpha_i \otimes \alpha_i$, $\alpha_i \otimes d\alpha_i$ do not occur in A^{2p} , for any i, p > 0. Thus

$$dA^{2p} \cap dtA^{2p} = 0,$$

and so

(**)
$$dW^{2p} = dA^{2p} \oplus dtA^{2p}, \quad p > 0.$$

Suppose now that the class v, given in 3.5, has degree 2n, n>0. We set

$$V = W^{2n} \oplus dW^{2n}.$$

By (*) and (**),

$$V = A^{2n} \oplus t A^{2n} \oplus B^{2n} \oplus t B^{2n} \oplus d A^{2n} \oplus d t A^{2n}.$$

Consequently the pair (t, d) is regular on V. By hypothesis $\Delta v = 0$, dv = 0, and so by 3.3 there is a class $u \in W^{2n}$ such that

$$\Delta u = v$$
, $du = \operatorname{Sq}^1 u = 0$.

This completes the proof.

4. Proof of Theorem 2.6

We retain the notation of §§ 2, 3. Let M be an m-manifold and let $\alpha_1, \ldots, \alpha_q$ be a basis for $H^*(M)$ (mod 2 coefficients). Define y_{ij} to be the value of $\alpha_i \cup \alpha_j$ on the fundamental mod 2 homology class [M]. In particular $y_{ij} = 0$ if deg $\alpha_i + \deg \alpha_j \neq m$; and $y_{ij} = y_{ji}$, $1 \le i, j \le q$. Let Y be the $q \times q$ matrix (y_{ij}) and set $C = Y^{-1}$. Then by MILNOR [19],

$$\underline{U} = \sum_{i,j} c_{ij} \, \alpha_i \otimes \alpha_j,$$

where $C = (c_{ij})$. Since Y is symmetric so is C.

Notice that $q = \chi^+ M$. By the hypotheses of 2.6, q is even, say q = 2d. We choose the basis $\{\alpha_i\}$ in a special way. Suppose first that m is odd, say m = 2k + 1. Let $\alpha_1, \ldots, \alpha_d$ be an arbitrary basis for the graded vector space

$$\sum_{i=0}^k H^i(M).$$

By Poincaré duality, $H^i(M)$ and $H^{m-i}(M)$ are orthogonally paired by the cup-product. Consequently we can choose a basis $\beta_1, ..., \beta_d$ for

$$\sum_{i=0}^k H^{m-i}(M)$$

such that if deg $\alpha_i + \text{deg } \beta_i = m$, then

$$\alpha_i \cup \beta_j = \delta_{ij} \, \mu \, .$$

Take as total basis for $H^*(M)$ the elements $\{\alpha_1, ..., \alpha_d, \beta_d, ..., \beta_1\}$. Then the matrix Y has the form shown below:

$$Y = \left(\begin{array}{c} & & 1 \\ 0 & \cdot \\ & 1 \\ & \cdot & 0 \\ 1 & & \end{array}\right).$$

Thus C = Y and so by (*) we obtain

$$\underline{U} = \sum_{i=1}^{d} \alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i. \tag{4.1}$$

Suppose on the other hand that m is even, say m=2k+2. Let $\{\alpha_1, ..., \alpha_r\}$, $\{\beta_1, ..., ..., \beta_r\}$ be bases for the respective vector spaces

$$\sum_{i=0}^{k} H^{i}(M), \quad \sum_{i=0}^{k} H^{m-i}(M),$$

chosen as above so that

$$\alpha_i \cup \beta_j = \delta_{ij} \mu$$
,

if $\deg \alpha_i + \deg \beta_j = m$. Assume, as in 2.6, that M is symplectic. Then (see [28]) one can choose a basis $x_1, ..., x_s, y_1, ..., y_s$ for $H^{k+1}(M)$ such that

$$x_i \cup x_j = 0$$
, $y_i \cup y_j = 0$, $x_i \cup y_j = \delta_{ij} \mu$.

Now by definition

$$2(r+s)=q=2d.$$

Set

$$\alpha_{r+i} = x_i, \quad \beta_{r+i} = y_i, \quad 1 \le i \le s.$$

Then $\{\alpha_1, ..., \alpha_d, \beta_d, ..., \beta_1\}$ is a basis for $H^*(M)$ yielding as above

$$\underline{U} = \sum_{i=1}^{d} \alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i. \tag{4.2}$$

For m even or odd we set

$$A=\sum_{i=1}^d\alpha_i\otimes\beta_i.$$

Then by (4.1) and (4.2), U = A + tA, which proves 2.6 i). Now

$$(\alpha_i \otimes \beta_i) \cup (\beta_j \otimes \alpha_j) = (\alpha_i \beta_j \otimes \beta_i \alpha_j) = 0$$

unless i=j. For if $\deg \alpha_i + \deg \beta_j = m$, then by definition $\alpha_i \cup \beta_j = \delta_{ij}\mu$, while if $\deg \alpha_i + \deg \beta_j \neq m$ then one of the pairs $\alpha_i \beta_j$, $\beta_i \alpha_j$ has degree greater than m and so is zero. Thus

$$A \cup t A = \sum_{i=1}^{d} \alpha_i \beta_i \otimes \alpha_i \beta_i = d(\mu \otimes \mu) = \hat{\chi}_2 M(\mu \otimes \mu),$$

since $2d = q = \chi^+ M$. Therefore the class A satisfies 2.6 ii).

To prove 2.6 iii) we need the following lemma.

LEMMA 4.3. Let M be an orientable manifold of dim m, m > 1. Let $u \in H^r(M)$, $v \in H^s(M)$, where r+s=m and $0 < r \le s$.

a) Suppose that $m \equiv 0 \mod 4$. If r < s, then

$$\delta^* \operatorname{Sq}^{m-2}(u \otimes v) = 0.$$

If r = s, then

$$\delta^* \operatorname{Sq}^{m-2}(u \otimes v) = \delta^* \operatorname{Sq}^{r-2} u \otimes v^2 + u^2 \otimes \delta^* \operatorname{Sq}^{r-2} v.$$

b) Suppose that m is odd. If r < s-1, then

$$\operatorname{Sq}^{m-1}(u\otimes v)=0.$$

If r = s - 1, then

$$\operatorname{Sq}^{m-1}(u \otimes v) = u^2 \otimes \operatorname{Sq}^{s-1} v$$
.

c) Suppose that m is odd and that $w_2 M = 0$. Then

$$\operatorname{Sq}^{m-1} \operatorname{Sq}^1 H^m(M^2) = 0.$$

The proof of (a) and (b) follows at once by the Cartan formula, using the fact that $H^m(M; \mathbb{Z}) \approx \mathbb{Z}$. Thus

$$\delta^* H^{m-1}(M) = \operatorname{Sq}^1 H^{m-1}(M) = 0.$$

We leave the details of the proof to the reader. For (c) suppose that m=2k+1. Then by ADEM [2],

$$Sq^{m-1} Sq^1 = Sq^{2k} Sq^1 = Sq^2 Sq^{2k-1} + \varepsilon Sq^{2k+1}$$

= $Sq^2 Sq^{2k-1} + \varepsilon Sq^1 Sq^{2k}$

where $\varepsilon = 0$ or 1. But

$$\operatorname{Sq}^{2} H^{2m-2}(M^{2}) = 0, \quad \operatorname{Sq}^{1} H^{2m-1}(M^{2}) = 0,$$

since $w_1 M = w_2 M = 0$. Therefore $\operatorname{Sq}^{m-1} \operatorname{Sq}^1 H^m(M^2) = 0$, as claimed, which completes the proof of the lemma.

Proof of 2.6 iii). We must show that the operation Ω_m is defined on the class A.

Case I: $m \equiv 1 \mod 4$. By 2.4 this means we must show that

$$Sq^{m-1} Sq^1 A = 0$$
, $Sq^{m-1} A = 0$.

The first assertion follows by 4.3 (c). To prove the second assertion, we assume that the basis $\alpha_1, ..., \alpha_d$ is ordered so that

$$\deg \alpha_i \leq \deg \alpha_{i+1}$$
, $1 \leq i \leq q-1$.

Suppose that $\alpha_j, ..., \alpha_d$ are precisely those basis elements with degree (m-1)/2. Then by 4.3 (b),

$$\operatorname{Sq}^{m-1} A = \sum_{i=j}^{d} \alpha_i^2 \otimes \operatorname{Sq}^{s-1} \beta_i,$$

where s-1=(m-1)/2. Consequently,

$$\operatorname{Sq}^{m-1} t A = t \operatorname{Sq}^{m-1} A = \sum_{i=1}^{d} \operatorname{Sq}^{s-1} \beta_i \otimes \alpha_i^2.$$

Now $\underline{U} = A + tA$, and by § 2 we know that $\operatorname{Sq}^{m-1} \underline{U} = 0$, which means that

$$Sq^{m-1}A + Sq^{m-1}tA = 0.$$

But, as is seen by the above calculation, $\operatorname{Sq}^{m-1}A$ and $\operatorname{Sq}^{m-1}tA$ occur in disjoint summands of the bi-graded vector space $H^*(M) \otimes H^*(M)$. Namely, $\operatorname{Sq}^{m-1}A$ has bi-degree (m-1,m), while $\operatorname{Sq}^{m-1}tA$ has bi-degree (m,m-1). Thus $\operatorname{Sq}^{m-1}A=0$, as claimed, which completes the proof of case I.

CASE II: $m \equiv 0 \mod 4$. We will show that the class A can be replaced by a class B, which will continue to satisfy 2.6 i) and ii) and for which

$$\delta^* \operatorname{Sq}^{m-2} B = 0$$
, $\operatorname{Sq}^1 B = 0$.

Thus the class B will satisfy 2.6 iii) (see 2.4) and so the proof of 2.6 will be completed.

By 4.1 (a) we see that $\delta * \operatorname{Sq}^{m-2} H^m(M^2) = 0$; for if the classes u and v in 4.1 (a) have degree m/2, then $u^2 = v^2 = 0$, since M is symplectic by hypothesis.

In general it is not necessarily true that $Sq^1A = 0$. Thus we must find a new class B, satisfying 2.6 i) and ii), such that $Sq^1B = 0$.

As usual we set $\Delta = 1 + t$. Then $\Delta \underline{U} = 0$, and so by 3.5 there is a class $B \in H^m(M^2)$ such that

$$\Delta B = \underline{U}, \quad \operatorname{Sq}^1 B = 0.$$

Set D=B-A; since $\Delta A=U$ it follows that $\Delta D=0$. Moreover,

$$B \cup tB = (A+D) \cup (tA+D) = A \cup tA + A \cup D + D \cup tA + D \cup D.$$

Since M is symplectic, an easy argument shows that M^2 is too; therefore $D \cup D = 0$. In a moment we show that $A \cup D = D \cup tA$. This then implies that

$$B \cup t B = A \cup t A = \hat{\chi}_2 M(\mu \otimes \mu).$$

Thus the class B satisfies 2.6 (i)-(iii), and so the proof of 2.6 is complete.

We are left with showing that $A \cup D = D \cup tA$. By commutativity of the cup-product, $A \cup D = D \cup A$. Furthermore, since t is the identity on $H^{2m}(M^2)$, we have by naturality of the cup-product,

$$A \cup D = D \cup A = t(D \cup A) = tD \cup tA$$
.

But tD=D since $\Delta D=0$. Thus, $A \cup D=D \cup tA$ as claimed.

5. The relative Thom complex

Let ξ be an oriented *n*-plane bundle over a space B and suppose that ξ has a Riemannian metric [19, p. 21]. Denote by E, E^1 the respective subspaces of the total space of ξ consisting of those vectors of norm ≤ 1 and those of norm 1. (In order to avoid confusion we may sometimes write these spaces as $E(\xi)$, $E^1(\xi)$.) We define the Thom complex $T(\xi)$ to be E/E^1 .

Let B' be a space and $f: B' \to B$ a map. Let $f^*\xi$ denote the bundle over B' induced from ξ by f. Give $f^*\xi$ the induced Riemannian metric. Then the natural bundle map $f: f^*\xi \to \xi$ induces a map

$$T(f): T(f^*\xi) \to T(\xi).$$

Let B" be a second space and $g:B'' \rightarrow B'$ a map. Then, up to homeomorphism,

$$T(g^*f^*\xi) = T((fg)^*\xi),$$

$$T(f) \cdot T(g) = T(fg).$$
(5.1)

Suppose that A is a subspace of B. Then the inclusion $A \subset B$ induces an inclusion $T(\xi_A) \subset T(\xi)$, where $\xi_A = \xi | A$. Thus, if $f:(B', A') \to (B, A)$ is a map of pairs, we obtain a map of pairs

$$T(f):(T(f^*\xi), T(f^*\xi_A)) \to (T(\xi), T(\xi_A)).$$

Now let $U \in H^n(E, E^1)$ denote the Thom class of the bundle ξ and let $p: E \to B$ denote the projection. Thom shows that the homomorphism

$$H^i(B) \rightarrow H^{n+i}(E, E^1),$$

given by $x \to p^* x \cup U$, is an isomorphism $(i \ge 0)$. Since the pair (E, E^1) enjoys the homotopy-extension property (e.g., we can regard E as the mapping cylinder of $p|E^1$), the collapsing map $(E, E^1) \to (T(\xi), *)$ induces an isomorphism in cohomology. Following Thom we define $\psi_R: H^i(B) \approx H^{n+i}(T(\xi), *)$

to be the composite isomorphism. We prove⁴)

LEMMA 5.2. Let A be a closed subspace of B. Set $T_B = T(\xi)$, $T_A = T(\xi_A)$. Then there is a homomorphism $\psi_{B-A}: H^q(B,A) \to H^{n+q}(T_B,T_A)$

with the following properties.

a) The following diagram is commutative:

$$\cdots \to H^{q}(B, A) \xrightarrow{j^{*}} H^{q}(B) \xrightarrow{i^{*}} H^{q}(A) \xrightarrow{\delta} H^{q+1}(B, A) \to \cdots$$

$$\downarrow \psi_{B, A} \qquad \downarrow \psi_{B} \qquad \downarrow \psi_{A} \qquad \downarrow \psi_{B, A}$$

$$\cdots \to H^{q+n}(T_{B}, T_{A}) \xrightarrow{j^{*}} H^{q+n}(T_{B}) \xrightarrow{i^{*}} H^{q+n}(T_{A}) \xrightarrow{\delta} H^{q+n+1}(T_{B}, T_{A}) \to \cdots$$

Here i^* , j^* denote homomorphisms induced by inclusions and δ is the coboundary operator.

- b) $\psi_{B,A}$ is an isomorphism for all q.
- c) Let $f:(B', A') \rightarrow (B, A)$ be a map of pairs. Then the following diagram commutes:

$$H^{q}(B, A) \xrightarrow{f^{*}} H^{q}(B', A')$$

$$\downarrow \psi_{B, A} \qquad \qquad \downarrow \psi_{B', A'}$$

$$H^{q+n}(T_{B}, T_{A}) \xrightarrow{T(f)^{*}} H^{q+n}(T_{B'}, T_{A'}).$$

⁴⁾ The result is well known, but I am unaware of a reference.

d) Let $x \in H^*(B, A)$, mod 2 coefficients. Then,

$$\operatorname{Sq}^{k} \psi_{B,A}(x) = \sum_{i+j=k} \psi_{B,A}(w_{i} \xi \cup \operatorname{Sq}^{j} x).$$

Proof: Following Spanier we define the *relative Thom pair* of the bundles (ξ, ξ_A) to be the pair $(E, E_A \cup E^1)$, where $E_A = E(\xi_A)$. Let $p': (E, E_A) \to (B, A)$ denote the projection. Notice that if $x \in H^i(B, A)$, then

$$p'^*x \cup U \in H^{n+i}(E, E_A \cup E^1),$$

and so we obtain a homomorphism

$$\psi'_{B,A}: H^{i}(B,A) \to H^{n+i}(E,E_{A} \cup E^{1}), \quad i \geq 0.$$

If A is empty then $\psi'_{B,A}$ is simply the isomorphism ψ_B given above.

Notice that if we collapse E^1 to a point in the pair $(E, E_A \cup E^1)$ we obtain the pair (T_B, T_A) . Thus by the 5-lemma the collapsing map induces an isomorphism

$$H^*(E, E_A \cup E^1) \approx H^*(T_B, T_A).$$

We define $\psi_{B,A}: H^i(B,A) \to H^{n+i}(T_B,T_A)$ to be the composition of $\psi'_{B,A}$ with the isomorphism given above. The properties of $\psi_{B,A}$ will then follow from the analogous properties of $\psi'_{B,A}$. We proceed to develop the properties of $\psi'_{B,A}$.

By Spanier [23, 5.4.9] we see that there is a coboundary operator

$$\Delta: H^{j}(E_{A}, E_{A}^{1}) \to H^{j+1}(E, E_{A} \cup E^{1})$$

so that the following diagram commutes and has exact rows.

$$\cdots \xrightarrow{j^*} H^q(B) \xrightarrow{i^*} H^q(A) \xrightarrow{\delta} H^{q+1}(B, A) \xrightarrow{j^*} \cdots$$

$$\downarrow \psi_B \qquad \qquad \downarrow \psi_A \qquad \qquad \downarrow \psi'_{B, A}$$

$$\cdots \xrightarrow{j^*} H^{q+n}(E, E^1) \xrightarrow{i^*} H^{q+n}(E_A, A_A^1) \xrightarrow{\Delta} H^{q+n+1}(E, E_A \cup E^1) \xrightarrow{j^*} \cdots$$

(Because E and E_A are disk bundles the excision properties required in [23] are easily seen to be satisfied.) Since ψ_B and ψ_A are isomorphisms, it follows from the 5-lemma that $\psi'_{B,A}$ is an isomorphism.

Suppose that $f:(B',A')\to(B,A)$. Then one easily sees that f induces a map $\bar{f}:(E_{B'},E_{A'}\cup E_{B'}^1)\to(E,E_A\cup E^1)$, where $E_{B'}=E(f^*\xi)$, $E_{A'}=E(f^*\xi_A)$. Thus the following diagram commutes:

$$H^{q}(B, A) \xrightarrow{f^{*}} H^{q}(B', A')$$

$$\downarrow \psi'_{B, A} \qquad \qquad \downarrow \psi'_{B', A'}$$

$$H^{q+n}(E, E_{A} \cup E^{1}) \xrightarrow{} H^{q+n}(E_{B'}, E_{A'} \cup E^{1}_{B'}).$$

$$f^{*}$$

Suppose finally that $x \in H^*(B, A)$. Then

$$Sq^{k}(\psi'_{B,A}x) = Sq^{k}(p'^{*}x \cup U) = \sum_{i+j=k} p'^{*}Sq^{i}x \cup Sq^{j}U =$$

$$= \sum_{i+j=k} p'^{*}Sq^{i}x \cup (p^{*}w_{j}\xi \cup U) =$$

$$= \sum_{i+j=k} p'^{*}(Sq^{i}x \cup w_{j}\xi) \cup U = \sum_{i+j=k} \psi'_{B,A}(Sq^{i}x \cup w_{j}\xi).$$

(Here $w_j \xi$ denotes the j-th Stiefel-Whitney class of ξ , $j \ge 0$.) The proof of 5.2 now follows from these properties of $\psi'_{B,A}$ and the definition of $\psi_{B,A}$.

Remark. As indicated in § 2, we sometimes will regard the Thom class U as an element of $H^n(T(\xi),*)$ -i.e., $U=\psi_B(1)$ - and then we write $\psi_B(x)=U\cdot x$, for $x\in H^i(B)$.

6. Lifting the Postnikov invariant

We suppose now that all spaces have basepoint (written *), and that all maps preserve basepoints.

Let B, B' be complexes, and $\pi: B' \to B$ a map. Let $w \in H^n(B; J)$, where J = Z or Z_p , p a prime. Suppose that $w \neq 0$ but that $\pi^* w = 0$. We regard w as a map $B \to K(J, n)$ and let

$$\Omega K(J,n) \xrightarrow{i} E \xrightarrow{p} B$$

denote the principal fibration over B induced by w. (See [26]). Since $\pi^* w = 0$, there is a map $q: B' \to E$ such that $pq = \pi$. That is, we have the following commutative diagram, where $F = \Omega K(J, n)$:

$$\begin{array}{ccc}
F \\
\downarrow i \\
 & \pi = pq \\
 & \downarrow p \\
 & \downarrow p \\
 & B' \xrightarrow{\pi} B.
\end{array}$$

Let $k \in H^*(E, \mathbb{Z}_p)$ be a class such that $q^*k = 0$. In our applications π will be a fiber map and k will be a Postinikov invariant for π . However in this section we consider k in the more general setting given above, and we study the problem of expressing such a class k in terms of cohomology invariants determined by B.

Suppose that k has degree t. We assume that the mod p cohomology morphism π^*

is surjective in degree t and that t < 2n-2. Then there is an element α of the mod p Steenrod algebra such that

$$i^*k = \alpha i$$
,

where ι denotes the fundamental class of $\Omega K(J, n)$.

For simplicity we now assume that p=2. We will say the class w is realizable if: (6.1) there is a vector bundle ξ over B (of dim s, say) such that

$$w = w_n \xi$$
.

Furthermore, if J=Z, we assume that $w \not\equiv 0 \mod 2$.

Let T and U denote the Thom complex and class of the bundle ξ . If Y is any space and $g: Y \rightarrow B$ a map, we let T_Y , U_Y denote the Thom complex and class of $g^*\xi$.

Recall the cohomology operation α given above. We will say that the pair (w, α) is *admissible* if the following conditions are fulfilled.

(6.2) There is a relation

$$\alpha Sq^n = 0$$
,

which holds on integral cohomology classes of degree $\leq s$.

(6.3) There is a secondary cohomology operation Ω associated with relation 6.2 such that

$$\Omega(U_{B'})=T(\pi)^* M,$$

where M is a coset in $H^{s+t}(T)$ of the indeterminacy subgroup of Ω .

Remark 1. If n is odd and J=Z, then in 6.1 we regard w_n as $\delta^* w_{n-1}$, while in 6.2, we regard Sq^n as $\delta^* Sq^{n-1}$.

Remark 2. Recall that for any space X, Ω has indeterminacy subgroup $\alpha H^*(X; J)$. Define $\kappa \subset H^t(E)$ to be the coset of k with respect to the subgroup

Kernel
$$q^* \cap \text{Kernel } i^* \cap H^t(E)$$
.

We prove

THEOREM 6.4. Let (w, α) be an admissible pair as defined above. Then there is a class $k' \in \kappa$ and a class $m \in H^t(B)$ such that

$$U_B \cdot m \in M$$
 and $U_E \cdot (k' + p^* m) \in \Omega(U_E)$.

Before giving the proof we note the following consequence.

Let X be a complex and $h: X \to B$ a map. Suppose that $h^*w = 0$. Then there is a map $l: X \to E$ such that $p \circ l = h$. By naturality we obtain from 6.4,

COROLLARY 6.5. For any such map l, $U_X \cdot (l^*k' + h^*m) \in \Omega(U_X)$.

We precede the proof of 6.4 with some remarks. Consider the following commutative diagram, with the notation defined below.

$$T_{F} \qquad \Omega K(J, n + s)$$

$$\downarrow T i \qquad \downarrow \hat{i}$$

$$T_{E} \xrightarrow{f} \hat{E}$$

$$Tq \qquad \downarrow T p \qquad \downarrow \hat{p}$$

$$T_{B'} \xrightarrow{T_{\pi}} T_{B} = T_{B} \xrightarrow{\psi_{B}(w)} K(J, n + s).$$

The left hand portion of the diagram is obtained from diagram (*) by taking the Thom complex of the various bundles induced from ξ . Commutativity follows from 5.1. The map \hat{p} in the above diagram is the principal fibration induced by the cohomology class $\psi_B(w)$. By 5.2 (c),

$$(T p)^* \psi_B(w) = \psi_E p^* w = 0,$$

and so the map Tp lifts to a map f as shown.

Let $\hat{\imath}$ denote the fundamental class of $\Omega K(J, n+s)$. At the end of the section we prove

LEMMA 6.6. There is a class $\hat{k} \in H^{t+s}(\hat{E})$ such that

$$\hat{i}^* \hat{k} = \alpha \hat{i}, T q^* f^* \hat{k} = 0.$$

Moreover, if \hat{k} denotes the coset in $H^{t+s}(\hat{E})$ of \hat{k} with respect to the subgroup

Kernel
$$\hat{i}^* \cap \text{Kernel } (f \circ T q)^* \cap H^{t+s}(\hat{E}),$$

then

$$f^*\hat{\kappa} \subset U_E \cdot \kappa$$
.

We use 6.6 to prove 6.4.

Proof of Theorem 6.4. Since $w = w_n \xi$ it follows from Thom (see 5.2d) that

$$\psi_B(w) = \operatorname{Sq}^n U.$$

Thus we can regard the map $\psi_B(w)$: $T_B \to K(J, n+s)$ as the composite of the following maps:

$$T_B \xrightarrow{U} K(Z, s) \xrightarrow{\operatorname{Sq}^n \iota_s} K(J, n + s),$$

where ι_s denotes the fundamental class of K(Z, s).

Let $f: T_E \to \hat{E}$ be the map given in diagram (**). Set $\hat{f} = Tq \circ f: T_{B'} \to \hat{E}$, and consider the following commutative diagram, where the notation is explained below:

$$\Omega K(J, n + s) = \Omega K(J, n + s)$$

$$\downarrow \hat{i} \qquad \qquad \downarrow j$$

$$\hat{E} \xrightarrow{v} Y$$

$$\hat{f} \qquad \qquad \downarrow \hat{p} \qquad \qquad \downarrow r$$

$$T_{B'} \xrightarrow{T_B} T_B \xrightarrow{V} K(J, s) \xrightarrow{Sq^n l_s} K(J, n + s).$$

The map r is the principal fibration with $\operatorname{Sq}^n \iota_s$ as classifying map, and j is the fiber inclusion. Since \hat{p} is defined to be the fibration with $\psi_B(w)$ as classifying map and since $\psi_B(w) = \operatorname{Sq}^n U$, we may regard \hat{p} as the fibration induced by U from r. Thus v is simply the natural map for the induced fibration.

Notice that Y is the universal space for the operation Ω . Let $\omega \in H^{t+s}(Y)$ denote a representative class for Ω , chosen according to the specific choice of Ω given in 6.3. Set $k_0 = v^* \omega \in H^{t+s}(\hat{E})$. Since $j^* \omega = \alpha \hat{\imath}$, we have $\hat{\imath}^* k_0 = \alpha \hat{\imath}$. Furthermore,

$$\hat{f}^* k_0 \in \Omega(T \pi^* U) = \Omega(U_{B'}).$$

But by 6.3 there is then a class $m \in H^t(B)$ such that

$$U \cdot m \in M$$
 and $\hat{f}^* k_0 = T \pi^* (U \cdot m)$.

Set $k_0' = k_0 - \hat{p}^*(U \cdot m)$. Then,

$$\hat{i}^* k_0' = \hat{i}^* k_0 - \hat{i}^* \hat{p}^* (U \cdot m) = \hat{i}^* k_0 = \alpha \hat{i} = \hat{i}^* \hat{k},$$

$$\hat{f}^* k_0' = \hat{f}^* k_0 - \hat{f}^* \hat{p}^* (U \cdot m) = \hat{f}^* k_0 - T \pi^* (U \cdot m) = 0.$$

Consequently, by definition of the coset $\hat{\kappa}$, $k'_0 \in \hat{\kappa}$. On the other hand $k_0 \in \Omega(\hat{p}^*U)$ and so

$$k_0' + \hat{p}^*(U \cdot m) \in \Omega(\hat{p}^*U).$$

By 6.6 there is a class $k' \in \kappa \subset H^t(E)$ such that

$$f^*k'_0 = U_E \cdot k'.$$

Therefore, by naturality,

$$U_E \cdot (k' + p^* m) \in \Omega(U_E),$$

since

$$\hat{p}f = T p$$
, $T p^* U = U_E$, $T p^* (U \cdot m) = U_E \cdot p^* m$.

Thus k' is the desired class and the proof of 6.4 is complete.

We are left with proving 6.6. Before so doing we prove a preliminary result. Let ξ be the s-plane bundle over B given in 6.1. Now it is easily seen that the Thom complex of ξ |* is simply an s-sphere S^s , which we may regard as embedded in T_B . Since the

fiber map $p: E \to B$ maps F to * in B, it follows that $T_p(T_F) = S^s \subset T_B$. Furthermore the map $\psi_B w: T_B \to K(J, n+s)$ can be chosen so that $\psi_B w(S^s) = *$ in K(J, n+s). Since \hat{E} is the fiber space induced by $\psi_B w$, it follows that S^s is embedded in \hat{E} in a natural way. Set K = K(J, n+s). Then, $\hat{p}^{-1}(S^s) = \Omega K \times S^s \subset \hat{E}$, and diagram (**) gives the commutative diagram shown below, where bold face letters denote maps of pairs.

$$(T_E, T_F) \xrightarrow{\mathbf{f}} (\hat{E}, \Omega K \times S^s)$$

$$\downarrow \mathbf{T}_P \qquad \qquad \downarrow \hat{\mathbf{p}}$$

$$(T_B, S^s) = (T_B, S^s).$$

Set $g = f \mid T_F : T_F \to \Omega K \times S^s$. We use the above diagram to prove

LEMMA 6.7. $g^*(\hat{\imath} \otimes 1) \mod 2 = \psi_F(\imath) \mod 2$, where $\hat{\imath}$ and \imath denote respectively the fundamental classes for ΩK and F.

Proof. Let $\mathbf{p}:(E,F)\to(B,*)$ denote the map of pairs determined by p. Since p has w as classifying map, we have

(a)
$$\delta \iota = -\mathbf{p}^* w \in H^n(E, F);$$

and similarly,

(b)
$$\delta(\hat{\iota} \otimes 1) = -\mathbf{p}^* \psi_{B,*}(w) \in H^{n+s}(\hat{E}, \Omega K \times S^s).$$

Therefore by naturality and the commutative diagram above

$$\delta g^*(\hat{\imath}\otimes 1) = \mathbf{f}^*\delta(\hat{\imath}\otimes 1) = -\mathbf{f}^*\hat{\mathbf{p}}^*\psi_B w = -\mathbf{T}_p^*\psi_{B,*}(w).$$

By 5.2 (c) and by (a) above,

$$-\mathbf{T}_{p}^{*}\psi_{B,*}(w) = -\psi_{E,F}\mathbf{p}^{*}w = \psi_{E,F}(\delta \iota).$$

But by 5.2 (a), $\psi_{E,F}(\delta \iota) = \delta \psi_{F}(\iota)$. Thus, we obtain

$$\delta(g^*(\hat{\imath}\otimes 1)) = \delta\psi_F(\imath)$$
 in $H^{n+s}(T_E, T_F; J)$.

By Serre [21, p. 469], $p^*w \neq 0 \mod 2$ since (by 6.1) $w \neq 0 \mod 2$. Thus by (a) above and 5.2 (a),

$$\delta: H^{n+s-1}(T_F; Z_2) \to H^{n+s}(T_E, T_F; Z_2)$$

is injective and so $g^*(\hat{\imath} \otimes 1) = \psi_F(\imath) \mod 2$, as claimed.

Proof of 6.6. Since $w = w_n \xi$, it follows by Thom that

$$\psi_B w = \operatorname{Sq}^n \psi_B(1) = \operatorname{Sq}^n U.$$

Let α be the mod 2 Steenrod operation given at the neginning of the section. By 6.2, $\alpha \operatorname{Sq}^n = 0$ and so $\alpha \psi_B w = 0$. Applying the SERRE exact sequence [21, p. 468] to the fibration \hat{p} (see diagram (**)), we see that by exactness there is a class $\hat{k} \in H^{t+s}(\hat{E})$ such that $\hat{i}^* \hat{k} = \alpha \hat{i}$.

Furthermore, by using the exact sequence given in § 3 of [26] (with respect to the map $f \circ T_q : T_{B'} \to \hat{E}$) it is easily shown that \hat{k} can be chosen so that, in addition, $T_q^* f^* \hat{k} = 0$. Now the inclusion $\hat{i} : \Omega K \subset \hat{E}$ can be factored into the composite

$$\Omega K \xrightarrow{l} \Omega K \times S^s \xrightarrow{\hat{\jmath}} \hat{E}$$
,

where l is the natural injection and where \hat{j} is the inclusion. Since

$$\hat{i}^* \hat{k} = \alpha \hat{i}$$

it follows that

$$\hat{\jmath}^*\hat{k} = \alpha(\hat{\imath} \otimes 1).$$

Let $k_1 \in H^t(E)$ be the unique class such that

$$U_E \cdot k_1 = f^* \hat{k} \in H^{t+s}(T_E).$$

We will show that $k_1 \in \kappa$, which then will complete the proof of 6.6. Using 5.2 we have:

$$U_{B'} \cdot q^* k_1 = T q^* (U_E \cdot k_1) = T q^* f^* \hat{k} = 0.$$

Therefore, $q*k_1=0$. On the other hand,

$$U_F \cdot i^* k_1 = T i^* (U_E \cdot k_1) = T_i^* f^* \hat{k}.$$

But by definition of g and \hat{j} , $f \cdot T_i = \hat{j} \cdot g$. Thus

$$T_i^* f^* \hat{k} = g^* \hat{\jmath}^* \hat{k} = g^* (\alpha(\hat{\imath} \otimes 1)),$$

by the above computation. By 6.7,

$$g^*(\alpha(\hat{\imath}\otimes 1)) = \alpha g^*(\hat{\imath}\otimes 1) = \alpha \psi_F(\imath).$$

Now the bundle $i * p * \xi$ is trivial and so by 5.2 (d),

$$\alpha \psi_F(\iota) = \psi_F(\alpha \iota).$$

Also, by definition,

$$U_F \cdot i^* k_1 = \psi_F(i^* k_1).$$

Therefore

$$\psi_F(\alpha \,\iota - i^* k_1) = 0$$

and so $i * k_1 = \alpha \iota$. Consequently, $k_1 \in \kappa$, which completes the proof of 6.6.

Remark 3. The theory leading up to 6.4 can be generalized in the following way. The single cohomology class w can be replaced by a vector of cohomology classes $w = (w_1, ..., w_a)$, with $\pi^* w_i = 0$. By making the appropriate changes in 6.1-6.3 one then can state a more general version of 6.4 so that it includes, for example, Theorem 3.3.2 of [15] as a special case.

Remark 4. Theorem 6.4 (as well as the generalization suggested above) is a special case of Theorem 5.9 in [27]. The Thom class U is a "generating class" for κ , in the language of § 5 of [27].

7. Proof of 2.2

Let *n* be an integer greater than three and set

$$B' = BSO(n-1), B = BSO(n+1).$$

For any group G we let BG denote the classifying space for G defined by MILNOR 21]. We denote the various rotation groups by SO(q), $q \ge 2$.) The inclusion $SO(n-1) \subset SO(n+1)$ induces a map $\pi: B' \to B$. If we regard π as a fiber map, its fiber is the Stiefel manifold $V_{n+1,2}$.

Let X be a complex. Then a map $\xi: X \to B$ can be regarded as an oriented (n+1)plane bundle over X. Moreover this bundle has two linearly independent cross-sections iff the map ξ can be factored through B' via π .

We construct a Postnikov resolution for the map π , through dimension n+1, as shown below.

$$K(J, n-1) \xrightarrow{q} E$$

$$\downarrow p$$

$$B' \xrightarrow{\pi} B \xrightarrow{w} K(J, n).$$

Here

$$J = Z_2$$
, $w = w_n \gamma$, if n even $J = Z$, $w = \delta^* w_{n-1} \gamma$, if n odd,

where γ denotes the canonical (n+1)-plane bundle over B. The map p is the principal fibration with w as classifying map, and i is the inclusion of the fiber of p into E.

Let F denote the "fiber" of the map q (in the sense of [9]). By the choice of w, we see that F is (n-1)-connected and that

$$\pi_n F = Z_2 \text{ or } Z \oplus Z_2$$

according to whether n is even or odd. Let $\gamma_n \in H^n(F; \mathbb{Z}_2)$ denote the fundamental class if n is even; for n odd let it denote the cohomology class corresponding to the homomorphism $Z \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$ given by projection on the right hand summand. Let $k \in H^{n+1}(E; \mathbb{Z}_2)$ denote the transgression of the class γ_n . Then (see [10], [26]),

$$i^* k = \operatorname{Sq}^2 \iota, q^* k = 0,$$

where i denotes the fundamental class of K(J, n-1). Moreover, a simple argument

using the transgression operator (e.g., see [18]) shows that

Kernel
$$i^* \cap \text{Kernel } q^* \cap H^{n+1}(E) = \begin{cases} 0, & n \text{ even} \\ p^* w_{n+1}, & n \text{ odd.} \end{cases}$$
 (7.1)

Let ξ be a bundle over a complex X as above, and suppose that $\xi^* w = 0$. Then the map ξ lifts to the space E. We define

$$k(\xi) = \bigcup_{n} \eta^* k \subset H^{n+1}(X),$$

where the union is over all maps $\eta: X \to E$ such that $p\eta = \xi$. It is easily shown (see [14], [26]) that if n is even then $\xi | X^{n+1}$ lifts to B' iff $0 \in k(\xi)$, while if n is odd then $\xi | X^{n+1}$ lifts to B' iff $\chi(\xi) = 0$ and $0 \in k(\xi)$. In particular, $\xi | X^n$ lifts to B'.

Furthermore, by a standard argument ([14], [26]), one sees that $k(\xi)$ is a coset in $H^{n+1}(X)$ of the subgroup $S^{n+1}(X, \xi)$ consisting of all classes of the form

$$\mathrm{Sq}^2(u) + u \cup w_2(\xi),$$

for all $u \in H^{n-1}(X; J)$. In particular if $\operatorname{Sq}^2 u = u \cup w_2 \xi$ for all such u, then $k(\xi)$ consists of a single class. We use the theory of § 6 to compute the coset $k(\xi)$.

At the end of the section we prove

LEMMA 7.2. Let $n \equiv -1$, or $0 \mod 4$, $n \geq 4$. Then the operation Ω_{n+1} (see § 2) can be chosen so that

$$U_{B'} \cdot (w_2 w_{n-1}) \in \Omega_{n+1} (U_{B'})$$

where $U_{B'}$ denotes the Thom class of $\pi^* \gamma$.

By definition, w is realizable as given in 6.1. Furthermore by relation 2.1 and by 7.2 it follows that the pair (w, Sq^2) is admissible, in the sense of 6.2 and 6.3. (To satisfy 6.3 we need only observe that $\pi^*: H^*(B) \to H^*(B')$ is surjective.)

Let T_X , U_X denote the Thom complex and Thom class for the bundle $\xi(=\xi^*\gamma)$. If $S^{n+1}(X,\xi)=0$, then one easily sees that $\operatorname{Sq}^2H^{2n}(T_X;J)=0$. Therefore, if $\xi^*w=0$, then Ω_{n+1} is defined on U_X with zero indeterminacy.

Notice that by 7.1 κ is a coset of 0 if n is even, while if n is odd κ is a coset of the subgroup generated by p^*w_{n+1} . Thus by 6.5 and 7.2, we have

THEOREM 7.3. Let ξ be an oriented (n+1)-plane bundle over X such that

$$\xi^* w = 0$$
, $S^{n+1}(X, \xi) = 0$.

Then

$$U_X \cdot (k(\xi) + w_2(\xi) w_{n-1}(\xi) + b_{n+1} w_{n+1}(\xi)) = \Omega_{n+1}(U_X),$$

with zero indeterminacy, where Ω_{n+1} is given in 7.2, where $b_{n+1} \in \mathbb{Z}_2$, and where $b_{n+1} = 0$ if n is even.

Proof of 2.2. Take X to be a spin manifold M of dim m=n+1, and take $\xi=\tau$,

its tangent bundle. If n=4s-1, Massey shows that $\delta^* w_{4s-2}\tau=0$. If n=4s, we assume (as in 2.2) that $w_{4s}\tau=0$. Thus in either case $\tau^* w=0$ and so τ restricted to M^n has 2 independent cross-sections – i.e., there is a tangent 2-field on M with isolated singularities. By Wu [29], $S^{n+1}(M,\tau)=0$. Thus the class $k(\tau)$ is independent of the particular choice of 2-field. In the language of § 2, $k(\tau)=(I_2M)\mu$ and so 2.2 follows directly from 7.3 since $w_2M=0$, and since we assume (in 2.2) that $w_mM=0$, when m is even.

Proof of 7.2. We recall the following facts about Thom complexes, due to ATIYAH [4].

(7.4) (ATIYAH). Let X be a complex and let η be a vector bundle over X. Let ε denote (in general) the trivial line bundle. Then

$$T(\eta \oplus \varepsilon) = \Sigma T(\eta), \ U(\eta \oplus \varepsilon) = \Sigma U(\eta),$$

where Σ denotes the reduced suspension operator and where T, U denote the appropriate Thom complex and Thom class. Furthermore, if $x \in H^*(X)$, then

$$\Sigma(U(\eta)\cdot x) = (\Sigma U(\eta))\cdot x.$$

Let γ' denote the canonical (n-1)-plane bundle over the classifying space B'. Then

$$\pi^* \gamma = \gamma' \oplus 2\varepsilon$$
,

and so by 7.4,

$$T_{B'}=\Sigma^2 T', \qquad U_{B'}=\Sigma^2 U',$$

where T', U' denote the Thom complex and class of γ' . Also by 7.4 we have

$$U_{B'} \cdot (w_2 w_{n-1}) = \Sigma^2 (U' \cdot w_2 w_{n-1}).$$

But

$$\Sigma^{2}(U' \cdot w_{2} w_{n-1}) = \Sigma^{2}(U' \cdot \operatorname{Sq}^{2} U'),$$

since $\operatorname{Sq}^2 U' = U' \cdot w_2$, $U' \cdot w_{n-1} = \operatorname{Sq}^{n-1} U' = (U')^2 \mod 2$. Thus

$$U_{B'}\cdot(w_2\,w_{n-1})=\Sigma^2(\,U'\cdot\operatorname{Sq}^2\,U')\,,$$

and so 7.2 is simply a special case of the following result.

LEMMA 7.5. Let X be a complex and let $u \in H^{m-2}(X)$, $m \equiv 0$ or $1 \mod 4$. Then Ω_m is defined on $\Sigma^2 u$ and Ω_m can be chosen so that

$$\Sigma^2(u\cdot \operatorname{Sq}^2 u)\in \Omega_m(\Sigma^2 u),$$

Proof. The proof is similar to that given by Mahowald-Peterson for Theorem 2.2.1 in [15], and so is omitted.

REFERENCES

- [1] J. F. Adams, On the non-existence of elements of Hopf invariant one, Ann. of Math. 72 (1960), 20-104.
- [2] J. ADEM, The relations on Steenrod powers of cohomology classes, in Algebraic Geometry and Topology, Princeton, 1957, 191-238.
- [3] J. ADEM-S. GITLER, Secondary characteristic classes and the immersion problem, Bol. Soc. Mat. Mexicano 1963, 53-78.
- [4] M. ATIYAH, Thom complexes, Proc. London Math. Soc. [third series] XI (1961), 291-310.
- [5] W. BARCUS-J. P. MEYER, The suspension of a loop space, Amer. J. Math. 80 (1958), 895-920.
- [6] G. Bredon-A. Kosinski, Vector fields on π-manifolds, Ann. of Math., 84 (1966), 85-90.
- [7] E. Brown-F. Peterson, Whitehead products and cohomology operations, Quart. J. Math. 15 (1964), 116-120.
- [8] B. ECKMANN, Espaces fibrés et homotopie, Colloque de Topologie, Centre Belges de Recherches mathématiques, 1950, 83-99.
- [9] B. ECKMANN-P. HILTON, Operators and Cooperators in homotopy theory, Math. Ann. 141 (1960), 1-21.
- [10] R. HERMANN, Secondary obstructions for fiber spaces, Bull. Amer. Math. Soc. 65 (1959), 5-8.
- [11] F. Hirzebruch-H. Hopf, Felder von Flachenelementen in 4-dimensionalen Mannigfaltigkeiten, Math. Ann. 136 (1958), 156-172.
- [12] H. HOPF, Vectorfelder in n-dimensionalen Mannigfaltigkeiten, Math. Ann. 96 (1927), 225-260.
- [13] H. HOPF, Zur topologie der komplexen Mannigfaltigkeiten, in Studies and Essays presented to R. Courant, Interscience, 1941, 167–186.
- [14] M. MAHOWALD, On obstruction theory in orientable fiber bundles, Trans. Amer. Math. Soc. 110 (1964), 315-349.
- [15] M. MAHOWALD-F. Peterson, Secondary cohomology operations on the Thom class, Topology 2 (1964), 367-377.
- [16] W. Massey, On the Stiefel-Whitney classes of a manifold, Amer. J. Math. 82 (1960), 92-102.
- [17] W. Massey, On the Stiefel-Whitney classes of a manifold II, Proc. Amer. Math. Soc. 13 (1962), 938-942.
- [18] J. P. MEYER, The characterization of Moore-Postnikov invariants, Bol. Soc. Mat. Mexicana 1963, 92-94.
- [19] J. MILNOR, Lectures on characteristic classes, (mimeographed notes), Princeton Univ., 1957.
- [20] J. MILNOR, Construction of universal bundles II, Ann. of Math. 63 (1956), 430-436.
- [21] J. P. SERRE, Homologie singulière des espaces fibrés, Ann. of Math. 54 (1951), 425-505.
- [22] J. P. Serre, Cohomologie modulo 2 des complexes d'Eilenberg-MacLane, Comment. Math. Helv. 27 (1953), 198-231.
- [23] E. Spanier, Algebraic Topology, McGraw-Hill, 1966.
- [24] N. Steenrod, The topology of fiber bundles, Princeton Univ., 1951.
- [25] R. Thom, Espaces fibrés en spheres et carrés de Steenrod, Ann. Sci. Ecole Norm. Sup. 69 (1952), 109-182.
- [26] E. THOMAS, Seminar on fiber spaces, Springer-Verlag, 1966.
- [27] E. Thomas, Postnikov invariants and higher order cohomology operations, Ann. of Math., to appear.
- [28] O. Veblen, Analysis Situs, second edition, Amer. Math. Soc., 1931.
- [29] W. T. Wu, Classes caractéristique et i-carrés d'une variété, C. R. Acad. Sci. Paris 230 (1950), 918-920.

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