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# On the Isoperimetric Problem in a Riemann Space

*To Professor Heinz Hopf on his 70th birthday*

By YOSHIE KATSURADA, Sapporo

## Introduction

As well-known, the isoperimetric problem in an Euclidean space of two dimensions is to find the shortest simple closed curve enclosing a fixed area. The solution is a circle. The analogous problem in an Euclidean space of three dimensions is to find the simple closed surface with minimum area enclosing a fixed volume. Here again the classical answer is the sphere.

One knows (see, for instance, [1, 2]) that the closed surfaces with constant mean curvature are closely related to the isoperimetric problem, because of the following.

**THEOREM.** *Let  $S$  be a simple closed surface, then  $S$  has constant mean curvature  $H$  if and only if  $S$  is stationary with respect to the isoperimetric problem ([1], p. 75).*

In previous papers ([3, 4]), the author has investigated some properties of a closed orientable hypersurface with the first mean curvature  $H_1 = \text{constant}$  in an  $(m+1)$ -dimensional Riemann space  $R^{m+1}$ .

It is the aim of the present paper to generalize the above Theorem to hypersurfaces in  $R^{m+1}$  and to investigate the connection with the isoperimetric problem in  $R^{m+1}$ . In §1 some integral formulas for a closed orientable hypersurface which is the boundary of a domain in  $R^{m+1}$  are derived; §2 gives a variational interpretation for these formulas and for a formula (I) of Minkowski type in  $R^{m+1}$  ([3], p. 288). In §3 the main theorem is proved.

## § 1. Some integral formulas

We consider a Riemann space  $R^{m+1}$  ( $m+1 \geq 3$ ) of class  $C^v$  ( $v \geq 3$ ) which admits a one-parameter continuous group  $G$  of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x) \delta\tau \quad (1.1)$$

(where  $x^i$  are local coordinates in  $R^{m+1}$  and  $\xi^i$  are the components of a contravariant vector  $\xi$ ). We suppose that the paths of these transformations cover  $R^{m+1}$  simply and that  $\xi$  is everywhere continuous and  $\neq 0$ . If  $\xi$  is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. ([5], p. 32), then the group  $G$  is called isometric, homothetic, conformal, etc. respectively.

We now consider a domain  $D$  in  $R^{m+1}$  such that its boundary is a closed hyper-

surface  $V^m$  of class  $C^3$  imbedded in  $R^{m+1}$ , locally given by

$$x^i = x^i(u^\alpha); \quad (1.2)$$

here and henceforth, Latin indices run from 1 to  $m+1$  and Greek indices from 1 to  $m$ .

Let us consider a differential form of degree  $m$  at a point  $P$  of the domain  $D$ , defined by

$$((\xi, \underbrace{dx, \dots, dx}_m)) = \sqrt{g}(\xi, dx, \dots, dx) \quad (1.3)$$

where  $dx^k$  is a displacement in the domain  $D$  and  $g$  denotes the determinant of the metric tensor  $g_{ij}$  of  $R^{m+1}$ . Then the exterior differential of the differential form (1.3) divided by  $m!$  becomes as follows

$$\frac{1}{m!} d((\xi, dx, \dots, dx)) = -\frac{1}{2} g^{ij} \mathcal{L}_\xi g_{ij} dV \quad (1.4)$$

where  $\mathcal{L}_\xi g_{ij}$  is the Lie derivative of the tensor  $g_{ij}$  with respect to the infinitesimal point transformation (1.1), and  $dV$  means the volume element of  $D$ .

Integrating both members of (1.4) over the whole domain  $D$ , and applying Stokes' theorem, we have

$$-\frac{1}{2} \int_D \dots \int_D g^{ij} \mathcal{L}_\xi g_{ij} dV = \frac{1}{m!} \int_D \dots \int_D d((\xi, dx, \dots, dx)) = \frac{1}{m!} \int_{V^m} \dots \int_{V^m} ((\xi, dx, \dots, dx)) \quad (1.5)$$

$V^m$  being the boundary of  $D$ . On the other hand, we can easily see the following relation  $((\xi, dx, \dots, dx)) = \xi^i n_i m! dA$ , where  $dx^k$  means a displacement along the hypersurface  $V^m$ , i.e.,  $dx^k = (\partial x^k / \partial u^\alpha) du^\alpha$ , and  $n_i$  is a unit normal covariant vector at a point  $P$  of the hypersurface  $V^m$  and  $dA$  denotes the area element of  $V^m$ . Thus we obtain the integral formula

$$-\frac{1}{2} \int_D \dots \int_D g^{ij} \mathcal{L}_\xi g_{ij} dV = \int_{V^m} \dots \int_{V^m} \xi^i n_i dA \quad (\alpha).$$

Let the group  $G$  be conformal, that is,  $\xi^i$  satisfy the equation

$$\mathcal{L}_\xi g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\phi(x) g_{ij}$$

(cf. [5], p. 32), where the symbol "; " always means the covariant derivative, then  $(\alpha)$  becomes

$$-(m+1) \int_D \dots \int_D \phi dV = \int_{V^m} \dots \int_{V^m} \xi^i n_i dA \quad (\alpha)_c.$$

Let  $G$  be homothetic, that is,  $\phi \equiv C = \text{constant}$ , then

$$-(m+1)CV = \int \cdots \int_{V^m} \xi^i n_i dA \quad (\alpha)_h$$

$V$  being the total volume of  $D$ . Especially, if our space  $R^{m+1}$  is an Euclidean space  $E^{m+1}$  and if we take a point of  $D$  as origin of the euclidean coordinates  $x^i$  and attach to each point  $x$  the vector  $\xi^i$  with the components  $\xi^i = x^i$  (i.e., the position vector of  $x$ ), then the transformations (1.1) are homothetic, that is,  $C=1$ , thus the formula  $(\alpha)_h$  becomes the following well-known formula

$$(m+1)V = - \int \cdots \int_{V^m} x^i n_i dA.$$

In the case  $m+1=3$ , we have  $3V = - \int \cdots \int_{V^2} x^i n_i dA$  (cf. [2], p. 18).

Furthermore in the Riemann space  $R^{m+1}$ , let  $G$  be isometric, that is,  $C=0$ , then we have

$$\int \cdots \int_{V^m} \xi^i n_i dA = 0 \quad (\alpha)_i.$$

By making use of the formula  $(\alpha)_c$  and the formula  $(I)_c$  of the previous paper ([4], p. 3), we have the following

**THEOREM 1.1.** *If  $D$  is a domain in  $R^{m+1}$  admitting a conformal Killing vector  $\xi$  (i.e.,  $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$ ) and if its boundary  $V^m$  is a closed hypersurface with  $H_1 = \text{constant}$ , then it follows that*

$$(m+1)H_1 \int_D \cdots \int \phi dV = \int_{V^m} \cdots \int \phi dA \quad (1.6)$$

where  $H_1$  means the first mean curvature of  $V^m$ .

*Proof.* Multiplying the formula  $(\alpha)_c$  by  $H_1 (= \text{const.})$ , we obtain

$$-(m+1)H_1 \int_D \cdots \int \phi dV = H_1 \int \cdots \int_{V^m} \xi^i n_i dA.$$

By making use of the formula  $(I)_c$  of the previous paper  $H_1 \int \cdots \int_{V^m} \xi^i n_i dA = - \int \cdots \int_{V^m} \phi dA$  (cf. [4], p. 3), we see that  $(m+1)H_1 \int_D \cdots \int \phi dV = \int \cdots \int_{V^m} \phi dA$ .

**COROLLARY.** *If  $D$  is a domain in  $R^{m+1}$  admitting a homothetic Killing vector  $\xi$  (i.e.,  $\xi_{i;j} + \xi_{j;i} = 2C g_{ij}$ ) and if its boundary  $V^m$  is a closed hypersurface with  $H_1 = \text{const.}$ , then we have*

$$V = \frac{1}{m+1} \cdot \frac{A}{H_1} \quad (1.7)$$

where  $A$  is the total area of  $V^m$ .

*Proof.* Substituting  $\phi = C$  (=const.) into both members of (1.6), we obtain easily (1.7).

Especially, if our space  $R^{m+1}$  is an Euclidean space  $E^{m+1}$  and if  $V^m$  is a hypersphere with radius  $\gamma$ , then the formula (1.7) becomes  $V = \gamma \cdot A / m + 1$ .

## § 2. On variational problems of integral formulas

In this section, we shall discuss the preceding integral formulas and the integral formulas of the previous paper ([4], p. 3) from the point of view of the calculus of variations.

We now consider a variation of a geometrical object in  $R^{m+1}$ , defined by

$$\bar{x}^i = x^i + \xi^i(x) \varepsilon \quad (2.1)$$

where  $\varepsilon$  is a parameter near  $\varepsilon=0$ ; then substituting (1.2) into (2.1), we get a family  $\bar{x}^i = \bar{x}^i(u^\alpha, \varepsilon)$  of admissible hypersurfaces of the form

$$\bar{x}^i = x^i(u^\alpha) + \xi^i(x^j(u^\alpha)) \varepsilon. \quad (2.2)$$

For each value of  $\varepsilon$  near  $\varepsilon=0$ , we thus obtain a domain  $D(\varepsilon)$  with a boundary  $V^m(\varepsilon)$ , where  $D(0)=D$ ,  $V^m(0)=V^m$ ; let  $V(\varepsilon)$  be the total volume of  $D(\varepsilon)$ . Now we have the following

**THEOREM 2.1.** *If  $(\delta V / \partial \varepsilon)_{\varepsilon=0}$  is the first variation of the total volume of  $D(\varepsilon)$  along  $D$  with respect to a direction  $\xi^i$ , then*

$$\left( \frac{\delta V}{\partial \varepsilon} \right)_{\varepsilon=0} = \frac{1}{2} \int_D \dots \int_D g^{ij} \mathcal{L}_\xi g_{ij} dV. \quad (2.3)$$

*Proof.* Let  $\bar{V}$  be the total volume of  $D(\varepsilon)$ , which is given by the integral form

$$\bar{V} = \int_{D(\varepsilon)} \dots \int \sqrt{\bar{g}}(d\bar{x}, \dots, d\bar{x})$$

where  $\bar{g} = g(x, \varepsilon)$  and  $d\bar{x}^i = dx^i + (\partial \xi^i / \partial x^l) dx^l \varepsilon$ . For the first variation of  $\bar{V}$  along  $D$  we have

$$\begin{aligned} \frac{\delta V}{\partial \varepsilon} &= \int_{D(\varepsilon)} \dots \int \frac{\partial}{\partial \varepsilon} \sqrt{\bar{g}}(d\bar{x}, \dots, d\bar{x}) + \sqrt{\bar{g}} \frac{\partial}{\partial \varepsilon} (d\bar{x}, \dots, d\bar{x}), \\ \left( \frac{\delta V}{\partial \varepsilon} \right)_{\varepsilon=0} &= \frac{1}{2} \int_D \dots \int \sqrt{g} g^{ij} \left( \frac{\partial g_{ij}}{\partial x^l} \xi^l + g_{lj} \frac{\partial \xi^l}{\partial x^i} + g_{li} \frac{\partial \xi^l}{\partial x^j} \right) (dx, \dots, dx) \\ &= \frac{1}{2} \int_D \dots \int g^{ij} \mathcal{L}_\xi g_{ij} dV \end{aligned}$$

because of  $dV = \sqrt{g}(dx, \dots, dx)$  and  $\mathcal{L}_\xi g_{ij} = (\partial g_{ij}/\partial x^l)\xi^l + g_{lj}(\partial \xi^l/\partial x^i) + g_{li}(\partial \xi^l/\partial x^j)$  (cf. [5], p. 4).

Therefore we evidently have the following

**COROLLARY 2.1.** The first variation of the total volume of  $D(\varepsilon)$  along  $D$ , with respect to a direction  $\xi^i$  becomes as follows

$$\left(\frac{\delta V}{\partial \varepsilon}\right)_{\varepsilon=0} = (m+1) \int_D \dots \int \phi dV, \quad \left(\frac{\delta V}{\partial \varepsilon}\right)_{\varepsilon=0} = (m+1)CV, \quad \text{or} \quad \left(\frac{\delta V}{\partial \varepsilon}\right)_{\varepsilon=0} = 0$$

according to  $\xi^i$  being a conformal Killing vector ( $\mathcal{L}_\xi g_{ij} = 2\phi g_{ij}$ ), a homothetic Killing vector, or a Killing vector.

**COROLLARY 2.2.** The first variation of the total volume of  $D(\varepsilon)$  along  $D$ , with respect to a direction  $\xi^i$ , is given by

$$\left(\frac{\delta V}{\partial \varepsilon}\right)_{\varepsilon=0} = - \int_{V^m} \dots \int \xi^i n_i dA. \quad (2.4)$$

The proof easily follows from the integral formula ( $\alpha$ ) and (2.3).

We consider next a closed orientable hypersurface  $V^m$  of class  $C^3$  imbedded in  $R^{m+1}$ , locally given by (1.2). then we obtain a family  $\bar{x}^i = x^i(u^\alpha, \varepsilon)$  of admissible hypersurfaces of the form (2.2). For each value of  $\varepsilon$  near  $\varepsilon=0$ , we have a hypersurface  $V^m(\varepsilon)$ , where  $V^m(0) = V^m$ , and we have a value  $A(\varepsilon)$  of the total area of  $V^m(\varepsilon)$ . Then we shall prove the following theorem.

**THEOREM 2.2.** Let  $(\delta A/\partial \varepsilon)_{\varepsilon=0}$  be the first variation of the total area of  $V^m(\varepsilon)$  along  $V^m$ , with respect to a direction  $\xi^i$ , then

$$\left(\frac{\delta A}{\partial \varepsilon}\right)_{\varepsilon=0} = \frac{1}{2} \int_{V^m} \dots \int \mathcal{L}_\xi g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA. \quad (2.5)$$

*Proof.* As well-known, the total area of  $V^m(\varepsilon)$  is given by the form

$$A(\varepsilon) = \int_{V^m(\varepsilon)} \dots \int \sqrt{\tilde{g}(\varepsilon)}(du, \dots, du)$$

where  $\tilde{g}(\varepsilon)$  means the determinant of the metric tensor  $g_{\alpha\beta}(\varepsilon)$  of the hypersurface  $V^m(\varepsilon)$  (i.e.,  $g_{\alpha\beta}(\varepsilon) = g_{ij}(\bar{x})(\partial \bar{x}^i/\partial u^\alpha)(\partial \bar{x}^j/\partial u^\beta)$ ).

Differentiating the above integral form with respect to  $\varepsilon$ , we have

$$\frac{\delta A}{\partial \varepsilon} = \int_{V^m(\varepsilon)} \dots \int \frac{\partial}{\partial \varepsilon} \sqrt{\tilde{g}(\varepsilon)}(du, \dots, du)$$

where  $u^\alpha$  and  $\varepsilon$  are independent parameters.

On making use of the following results

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} \sqrt{\tilde{g}} &= \frac{1}{2\sqrt{\tilde{g}}} \left\{ \frac{\partial \tilde{g}}{\partial \bar{x}^k} \xi^k + \frac{\partial \tilde{g}}{\partial (\partial \bar{x}^k / \partial u^\alpha)} \frac{\partial \xi^k}{\partial u^\alpha} \right\}, \\ \frac{\partial \tilde{g}}{\partial \bar{x}^k} &= \frac{\partial \tilde{g}_{ij}}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial u^\alpha} \frac{\partial \bar{x}^j}{\partial u^\beta} g^{\alpha\beta}(\varepsilon) \tilde{g}, \quad \frac{\partial \tilde{g}}{\partial (\partial \bar{x}^k / \partial u^\alpha)} = 2 g_{kj}(\bar{x}) \frac{\partial \bar{x}^j}{\partial u^\beta} g^{\alpha\beta}(\varepsilon) \tilde{g}, \\ \frac{\partial \tilde{g}}{\partial (\partial \bar{x}^k / \partial u^\alpha)} \frac{\partial \xi^k}{\partial u^\alpha} &= \left( g_{kj}(\bar{x}) \frac{\partial \xi^k}{\partial x^i} + g_{ki}(\bar{x}) \frac{\partial \xi^k}{\partial x^j} \right) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta}(\varepsilon) \tilde{g}, \end{aligned}$$

we obtain

$$\left( \frac{\partial \sqrt{\tilde{g}}}{\partial \varepsilon} \right)_{\varepsilon=0} = \frac{1}{2} \mathcal{L}_\xi g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} d\sqrt{\tilde{g}} A.$$

Consequently for the first variation of the total area of  $V^m(\varepsilon)$  along  $V^m$ , we can see

$$\left( \frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = \frac{1}{2} \int \dots \int_{V^m} \mathcal{L}_\xi g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA.$$

**COROLLARY 2.3.** The first variation of the total area of  $V^m(\varepsilon)$  along  $V^m$ , with respect to a direction  $\xi^i$ , becomes as follows

$$\left( \frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = m \int \dots \int_{V^m} \phi dA, \quad \left( \frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = m C A \quad \text{or} \quad \left( \frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = 0$$

according to  $\xi^i$  being a conformal Killing vector, a homothetic Killing vector or a Killing vector.

From Theorem 2.2 and the formula (I) of the previous paper (cf. [4], p. 3), we can see easily the following

**COROLLARY 2.4.** The first variation of the total area  $V^m(\varepsilon)$  along  $V^m$  with respect to a direction  $\xi^i$ , has the form

$$\frac{1}{m} \left( \frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int \dots \int_{V^m} H_1 \xi^i n_i dA. \tag{2.6}$$

If our space  $R^{m+1}$  is an Euclidean space  $E^{m+1}$  and if we take to each point  $x$  the vector  $\xi^i(x)$  with the components  $\xi^i = x^i$  (i.e., the position vector of  $x$ ), then the vector  $\xi^i$  is a homothetic Killing vector with  $C=1$ , and  $\xi^i n_i$  is the support function  $p$  for  $x \in V^m$ . In this case, the formula (2.6) becomes

$$\int \dots \int_{V^m} H_1 p dA + A = 0,$$

this being nothing but the formula of Minkowski type of  $V^m$  in  $E^{m+1}$  given by C. C. HSIUNG (cf. [6], p. 286). Therefore we can see the formula (2.6):

$$\int_{V^m} \cdots \int H_1 \xi^i n_i dA + \frac{1}{m} \left( \frac{\delta A}{\partial \varepsilon} \right)_{\varepsilon=0} = 0$$

as a generalization of the formula of Minkowski type.

*Remark 1.* Although the vector field  $\xi^i(x)$  is not defined on the whole Riemann space but defined on a certain domain including both  $D$  and  $V^m$ , all the preceding theorems are valid.

*Remark 2.* In case an arbitrary vector  $\eta^i$  is defined on the hypersurface  $V^m$  given by (1.2), we can find also the following formulas

$$\left( \frac{\delta V}{\partial \varepsilon} \right)_{\varepsilon=0} = - \int_{V^m} \cdots \int \eta^i(u^\alpha) n_i dA \quad (2.7)$$

and

$$\left( \frac{\delta A}{\partial \varepsilon} \right)_{\varepsilon=0} = - m \int_{V^m} \cdots \int H_1 \eta^i(u^\alpha) n_i dA \quad (2.8)$$

for the first variation of the total volume of  $D(\varepsilon)$  along  $D$  and the first variation of the total area of  $V^m(\varepsilon)$  along  $V^m$ , by means of a family  $\bar{x}^i = x^i(u^\alpha, \varepsilon)$  of the hypersurfaces of the form

$$\bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \eta^i(u^\alpha) \varepsilon.$$

### § 3. The isoperimetric problems

In this section, we shall prove the following theorems closely related to what may be called an isoperimetric problem in  $R^{m+1}$ .

If  $(\delta A / \partial \varepsilon)_{\varepsilon=0} = 0$  for all variations with respect to a direction such that  $(\delta V / \partial \varepsilon)_{\varepsilon=0} = 0$ , then the hypersurface  $V^m$  is called a pseudo-stationary hypersurface.

**THEOREM 3.1.** *Let  $V^m$  be a closed orientable hypersurface in  $R^{m+1}$ . Then the first mean curvature of  $V^m$  is constant if and only if  $V^m$  is a pseudo-stationary hypersurface.*

*Proof.* Suppose  $H_1$  is constant; if  $(\delta V / \partial \varepsilon)_{\varepsilon=0} = 0$ , then we get from (2.7)

$$\left( \frac{\delta V}{\partial \varepsilon} \right)_{\varepsilon=0} = - \int_{V^m} \cdots \int \eta^i n_i dA = 0$$

and hence from (2.8)

$$\left( \frac{\delta A}{\partial \varepsilon} \right)_{\varepsilon=0} = - m \int_{V^m} \cdots \int H_1 \eta^i n_i dA = - m H_1 \int_{V^m} \cdots \int \eta^i n_i dA = 0.$$

Thus  $V^m$  is a pseudo-stationary hypersurface.

Conversely suppose  $(\delta A/\delta \varepsilon)_{\varepsilon=0} = 0$  for every variation with respect to a direction  $\eta^i$  such that  $(\delta V/\delta \varepsilon)_{\varepsilon=0} = 0$ ; we must prove that  $H_1$  is constant. Let  $\varphi$  be an arbitrary function defined on  $V^m$  such that  $\int \dots \int_{V^m} \varphi dA = 0$ . We wish to show first that  $\varphi$  is in fact the normal component of a variation vector  $\eta^i$  such that  $(\delta V/\delta \varepsilon)_{\varepsilon=0} = 0$ . Let us consider the family of hypersurfaces  $\bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \varphi n^i \varepsilon$ , then from (2.4) we see

$$\left(\frac{\delta V}{\delta \varepsilon}\right)_{\varepsilon=0} = - \int \dots \int_{V^m} \varphi n^i n_i dA = - \int \dots \int_{V^m} \varphi dA = 0.$$

Thus  $\varphi$  is the normal component of a variation vector such that  $(\delta V/\delta \varepsilon)_{\varepsilon=0} = 0$ .

By hypothesis,  $V^m$  is pseudo-stationary, therefore it follows that

$$\left(\frac{\delta A}{\delta \varepsilon}\right)_{\varepsilon=0} = - m \int \dots \int_{V^m} H_1 \varphi dA = 0.$$

Thus we have  $\int \dots \int_{V^m} H_1 \varphi dA = 0$ . Also if  $h$  is an arbitrary constant, we have  $\int \dots \int_{V^m} h \varphi dA = 0$ , and hence for any function  $\varphi$  such that  $\int \dots \int_{V^m} \varphi dA = 0$  and for any constant  $h$ , we obtain

$$\int \dots \int_{V^m} (H_1 - h) \varphi dA = 0.$$

Now let  $h$  be the mean value of  $H_1$ :

$$h = \frac{1}{A} \int \dots \int_{V^m} H_1 dA,$$

then we have

$$\begin{aligned} \int \dots \int_{V^m} (H_1 - h) dA &= \int \dots \int_{V^m} H_1 dA - h \int \dots \int_{V^m} dA \\ &= \int \dots \int_{V^m} H_1 dA - h \cdot A = \int \dots \int_{V^m} H_1 dA - \int \dots \int_{V^m} H_1 dA = 0. \end{aligned}$$

Consequently taking  $H_1 - h$  for  $\varphi$ , we obtain

$$\int \dots \int_{V^m} (H_1 - h)^2 dA = 0.$$

Therefore  $H_1 \equiv h$ , which concludes the proof.

This theorem is nothing but a generalization of the same theorem in an Euclidean space given already in [2], p. 19, and this proof follows the same argument as in [2].

A. D. Alexandrov has already proved the following result in his paper ([7], p. 304), where in the case of positive curvature,  $R^{m+1}$  shall be a sphere and  $V^m$  contained in a hemisphere of  $R^{m+1}$ :

**THEOREM A.** *If  $R^{m+1}$  has constant curvature and if  $V^m$  is a simple closed hypersurface with  $H_1 = \text{constant}$ , then  $V^m$  is a hypersphere.*

From this result, we have (under the same assumptions as above):

**COROLLARY 3.1.** *If  $V^m$  is a simple closed hypersurface in  $R^{m+1}$  with constant curvature, then  $V^m$  is a hypersphere if and only if  $V^m$  is a pseudo-stationary hypersurface.*

Now in  $R^{m+1}$ , let  $S$  be the collection of all closed orientable hypersurfaces  $V^m$  enclosing a fixed volume. Then the total area  $A$  of  $V^m$  is a function on  $S$ . Let  $V^m$  be a fixed hypersurface and consider a one parameter family of continuous and differentiable variations of  $V^m$ , indexed by a parameter  $\varepsilon$ . Let  $V^m(\varepsilon)$  denote the varied hypersurface. Then we require that  $V^m(0) = V^m$  and that for each  $\varepsilon$ ,  $V^m(\varepsilon) \in S$  (i.e. these variations are volume preserving).

The total area  $A(\varepsilon)$  of  $V^m(\varepsilon)$  is a differentiable function of  $\varepsilon$ . If  $(\delta A / \partial \varepsilon)_{\varepsilon=0} = 0$  for all volume preserving variations, then  $V^m$  is called a stationary hypersurface. Then we have

**THEOREM 3.2.** *If  $R^{m+1}$  admits a homothetic Killing vector field  $\xi^i$  ( $\xi_{i;j} + \xi_{j;i} = 2Cg_{ij}$ ,  $C \neq 0$ ) and if  $V^m$  is a closed orientable hypersurface in  $R^{m+1}$ , then the first mean curvature  $H_1$  of  $V^m$  is constant if and only if  $V^m$  is a stationary hypersurface.*

*Proof.* Let  $V^m$  be given by (1.2) and suppose for simplicity that  $V(0) = 1$  and let  $V^m(\varepsilon)$  be a variation of  $V^m$ ; denote its total area and the total volume of the domain bounded by  $V^m(\varepsilon)$  by  $A(\varepsilon)$  and  $V(\varepsilon)$  respectively.  $V^m(\varepsilon)$  can be represented by

$$\bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \eta^i(u^\alpha)\varepsilon + \dots$$

for each value of  $\varepsilon$  near  $\varepsilon = 0$ , where  $\eta^i(u^\alpha) = (\partial \bar{x}^i / \partial \varepsilon)_{\varepsilon=0}$ . Then from (2.7) and (2.8) we have

$$\left( \frac{\delta V}{\partial \varepsilon} \right)_{\varepsilon=0} = - \int \dots \int_{V^m} \eta^i n_i dA, \quad \left( \frac{\delta A}{\partial \varepsilon} \right)_{\varepsilon=0} = - m \int \dots \int_{V^m} H_1 \eta^i n_i dA.$$

Sufficiency in Theorem 3.2. is similar by proved as in Theorem 3.1; that is, suppose  $H_1$  is constant and  $\bar{x}^i(u^\alpha, \varepsilon)$  is a volume preserving variation of  $V^m$  then  $(\delta V / \partial \varepsilon)_{\varepsilon=0} = - \int \dots \int_{V^m} \eta^i n_i dA = 0$  and hence

$$\left( \frac{\delta A}{\partial \varepsilon} \right)_{\varepsilon=0} = - m \int \dots \int_{V^m} H_1 \eta^i n_i dA = - m H_1 \int \dots \int_{V^m} \eta^i n_i dA = 0.$$

Conversely, suppose  $(\delta A / \partial \varepsilon)_{\varepsilon=0} = 0$  for every volume preserving variation. Then we must show that  $H_1$  is constant.

Let  $\varphi$  be an arbitrary function defined on  $V^m$  such that  $\int \dots \int_{V^m} \varphi dA = 0$ ; we wish to show first that  $\varphi$  is the normal component of a volume preserving variation. Consider the family of hypersurfaces

$$V^m(\varepsilon) : \bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \varphi n^i \varepsilon, \quad (3.1)$$

then let  $V(\varepsilon)$  denote the total volume of the domain bounded by the hypersurface  $V^m(\varepsilon)$ , then  $V(0) = V = 1$ ; now the normal component of  $(\partial \bar{x}^i / \partial \varepsilon)_{\varepsilon=0} = \varphi n^i$  is given by  $(\partial \bar{x}^i / \partial \varepsilon)_{\varepsilon=0} n_i = \varphi n^i n_i = \varphi$ . Hence, by virtue of (2.7) we have

$$\left( \frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int \dots \int_{V^m} \left( \frac{\partial \bar{x}^i}{\partial \varepsilon} \right)_{\varepsilon=0} n_i dA = - \int \dots \int_{V^m} \varphi dA = 0$$

by hypothesis. But the variation  $\bar{x}^i(u^\alpha, \varepsilon)$  need not be volume preserving.

However by hypothesis, our space  $R^{m+1}$  admits an infinitesimal homothetic transformation given by (1.1) with the additional condition

$$\xi_{i;j} + \xi_{j;i} = 2C g_{ij} \quad (C \neq 0, \text{ constant}). \quad (3.2)$$

Let us choose a coordinate system such that the path of the infinitesimal transformation is the new  $x^1$ -coordinate curve, that is, a coordinate system in which the vector  $\xi^i$  has the components  $\delta_1^i$  (where  $\delta_j^i$  denotes the Kronecker delta); then (1.1) becomes  $x'^i = x^i + \delta_1^i \delta \tau$  and  $R^{m+1}$  admits a one-parameter continuous group  $G$  of transformations given by

$$x'^i = x^i + \delta_1^i \tau. \quad (3.3)$$

Then in this new coordinate system, the condition (3.2) becomes as follows  $\partial g_{ij} / \partial x^1 = 2C g_{ij}$ . Therefore the metric tensor  $g_{ij}$  with respect to the new coordinate system has the form  $g_{ij} = f_{ij}(x^2, \dots, x^{m+1}) e^{2Cx^1}$ . Now we take the family of hypersurfaces

$$V^{*m}(\varepsilon) : x^{*i}(u^\alpha, \varepsilon) = \bar{x}^i(u^\alpha, \varepsilon) + \frac{1}{(m+1)C} \log \frac{1}{V(\varepsilon)} \delta_1^i; \quad (3.4)$$

we shall show that  $V^{*m}(\varepsilon)$  is a volume preserving variation. Let  $V^*(\varepsilon)$  be the total volume of the domain bounded by  $V^{*m}(\varepsilon)$  and let  $n^{*i}$  and  $dA^*$  be a normal vector and an area element of the hypersurface  $x^{*i}(u^\alpha, \varepsilon)$  respectively. Then from Corollary 2.1 and Corollary 2.2, we have

$$(m+1)C V^*(\varepsilon) = - \int \dots \int_{V^{*m}(\varepsilon)} \delta_1^i n_i^* dA^* = - \int \dots \int_{V^{*m}(\varepsilon)} n_1^* dA^*. \quad (3.5)$$

On the other hand, from (3.4) we have the relations

$$\begin{aligned} g_{ij}(x^*) &= f_{ij}(x^{*2}, \dots, x^{*(m+1)}) e^{2Cx^*1} \\ &= f_{ij}(\bar{x}^2, \dots, \bar{x}^{m+1}) e^{2C\bar{x}^1} \cdot e^{(2/m+1) \log(1/V(\varepsilon))} = g_{ij}(\bar{x}) e^{(2/m+1) \log(1/V(\varepsilon))}; \end{aligned}$$

thus we obtain

$$\sqrt{g(x^*)} = \sqrt{g(\bar{x})} e^{\log(1/V(\varepsilon))} = \frac{\sqrt{g(\bar{x})}}{V(\varepsilon)}. \quad (3.6)$$

Substituting (3.6) in (3.5) and making use of the relations

$$x^{*2}(u^\alpha, \varepsilon) = \bar{x}^2(u^\alpha, \varepsilon), \dots, x^{*(m+1)}(u^\alpha, \varepsilon) = \bar{x}^{m+1}(u^\alpha, \varepsilon),$$

we see that

$$\int_{V^{*(m)}(\varepsilon)} \dots \int n_1^* dA^* = \int_{V^m(\varepsilon)} \dots \int \frac{n_1(\varepsilon)}{V(\varepsilon)} dA(\varepsilon)$$

and

$$(m+1)C V^*(\varepsilon) = -\frac{1}{V(\varepsilon)} \int_{V^m(\varepsilon)} \dots \int \delta_1^i n_i(\varepsilon) dA(\varepsilon) = \frac{1}{V(\varepsilon)} (m+1)C V(\varepsilon) = (m+1)C.$$

Thus we have  $V^*(\varepsilon) = 1$ , therefore  $V^{*(m)}(\varepsilon)$  is a volume preserving variation of  $V^m$ .

Now, since  $(\delta V/\delta \varepsilon)_{\varepsilon=0} = 0$  it follows that

$$\left( \frac{\partial x^{*i}}{\partial \varepsilon} \right)_{\varepsilon=0} = \left( \frac{\partial \bar{x}^i}{\partial \varepsilon} \right)_{\varepsilon=0} = \varphi n^i,$$

and we have

$$\left( \frac{\partial x^{*i}}{\partial \varepsilon} \right)_{\varepsilon=0} n_i = \left( \frac{\partial \bar{x}^i}{\partial \varepsilon} \right)_{\varepsilon=0} n_i = \varphi.$$

Therefore  $\varphi$  is not only the normal component of  $(\partial \bar{x}^i/\partial \varepsilon)_{\varepsilon=0}$  but is also the normal component of  $(\partial x^{*i}/\partial \varepsilon)_{\varepsilon=0}$  and thus  $\varphi$  is the normal component of a volume preserving variation.

By hypothesis, since  $V^m$  is stationary, it follows that  $(\delta A/\delta \varepsilon)_{\varepsilon=0} = -m \int \dots \int_{V^m} H_1 \varphi dA = 0$ , thus  $\int \dots \int_{V^m} H_1 \varphi dA = 0$ . Also, if  $h$  is an arbitrary constant then  $\int \dots \int_{V^m} \varphi h dA = 0$  and hence for any function  $\varphi$  such that  $\int \dots \int_{V^m} \varphi dA = 0$  and for any constant  $h$ ,  $\int \dots \int_{V^m} (H_1 - h) \varphi dA = 0$ . Now let  $h$  be the mean value of  $H_1$ :  $h = (1/A) \int \dots \int_{V^m} H_1 dA$  then we have  $\int \dots \int_{V^m} (H_1 - h) dA = 0$ . Consequently we see  $\int \dots \int_{V^m} (H_1 - h)^2 dA = 0$ . Therefore  $H_1 \equiv h$ , which completes the proof.

From Theorem A and Theorem 3.2, we have the following corollary:

**COROLLARY 3.2** If  $R^{m+1}$  is an Euclidean space  $E^{m+1}$ , then a simple closed hypersurface with minimal hypersurface area enclosing a fixed volume is a hypersphere.

This may be called a form of the isoperimetric theorem in  $E^{m+1}$ .

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## REFERENCES

- [1] H. HOPF, *Lectures on Differential Geometry in the Large* (notes by J. W. Gray) (Stanford University 1956).
- [2] H. HOPF, *Lectures on selected topics in differential geometry in the large* (notes by Tilla S. Klotz) (New York University 1955).
- [3] Y. KATSURADA, *Generalized Minkowski Formulas for Closed Hypersurfaces in a Riemann Space*, Ann. di Mat. [Serie IV] 57 (1962) 283–294.
- [4] Y. KATSURADA, *On a Certain Property of Closed Hypersurfaces in an Einstein Space*, Comment. Math. Helv. 38, (1964) 165–171.
- [5] K. YANO, *The Theory of Lie Derivatives and its Applications*, Amsterdam 1957.
- [6] C. C. HSIUNG, *Some Integral Formulas for Closed Hypersurfaces*, Math. Scand. 2 (1954) 286–294.
- [7] A. D. ALEXANDROV, *A Characteristic Property of Spheres*, Ann. di Mat. [Serie IV] 58 (1962) 303–315.

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