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Autor: Katsurada, Yoshie
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On the Isoperimetric Problem in a Riemann Space

To Professor Heinz Hopf on his 70th birthday

By YOSHIE KATSURADA, Sapporo

Introduction

As well-known, the isoperimetric problem in an Euclidean space of two dimensions is to find the shortest simple closed curve enclosing a fixed area. The solution is a circle. The analogous problem in an Euclidean space of three dimensions is to find the simple closed surface with minimum area enclosing a fixed volume. Here again the classical answer is the sphere.

One knows (see, for instance, [1, 2]) that the closed surfaces with constant mean curvature are closely related to the isoperimetric problem, because of the following.

THEOREM. *Let S be a simple closed surface, then S has constant mean curvature H if and only if S is stationary with respect to the isoperimetric problem ([1], p. 75).*

In previous papers ([3, 4]), the author has investigated some properties of a closed orientable hypersurface with the first mean curvature $H_1 = \text{constant}$ in an $(m+1)$ -dimensional Riemann space R^{m+1} .

It is the aim of the present paper to generalize the above Theorem to hypersurfaces in R^{m+1} and to investigate the connection with the isoperimetric problem in R^{m+1} . In §1 some integral formulas for a closed orientable hypersurface which is the boundary of a domain in R^{m+1} are derived; §2 gives a variational interpretation for these formulas and for a formula (I) of Minkowski type in R^{m+1} ([3], p. 288). In §3 the main theorem is proved.

§ 1. Some integral formulas

We consider a Riemann space R^{m+1} ($m+1 \geq 3$) of class C^v ($v \geq 3$) which admits a one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x) \delta\tau \quad (1.1)$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a contravariant vector ξ). We suppose that the paths of these transformations cover R^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. If ξ is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. ([5], p. 32), then the group G is called isometric, homothetic, conformal, etc. respectively.

We now consider a domain D in R^{m+1} such that its boundary is a closed hyper-

surface V^m of class C^3 imbedded in R^{m+1} , locally given by

$$x^i = x^i(u^\alpha); \quad (1.2)$$

here and henceforth, Latin indices run from 1 to $m+1$ and Greek indices from 1 to m .

Let us consider a differential form of degree m at a point P of the domain D , defined by

$$((\xi, \underbrace{dx, \dots, dx}_m)) = \sqrt{g(\xi, dx, \dots, dx)} \quad (1.3)$$

where dx^k is a displacement in the domain D and g denotes the determinant of the metric tensor g_{ij} of R^{m+1} . Then the exterior differential of the differential form (1.3) divided by $m!$ becomes as follows

$$\frac{1}{m!} d((\xi, dx, \dots, dx)) = -\frac{1}{2} g^{ij} \mathcal{L}_\xi g_{ij} dV \quad (1.4)$$

where $\mathcal{L}_\xi g_{ij}$ is the Lie derivative of the tensor g_{ij} with respect to the infinitesimal point transformation (1.1), and dV means the volume element of D .

Integrating both members of (1.4) over the whole domain D , and applying Stokes' theorem, we have

$$-\frac{1}{2} \int_D \dots \int g^{ij} \mathcal{L}_\xi g_{ij} dV = \frac{1}{m!} \int_D \dots \int d((\xi, dx, \dots, dx)) = \frac{1}{m!} \int_{V^m} \dots \int ((\xi, dx, \dots, dx)) \quad (1.5)$$

V^m being the boundary of D . On the other hand, we can easily see the following relation $((\xi, dx, \dots, dx)) = \xi^i n_i m! dA$, where dx^k means a displacement along the hypersurface V^m , i.e., $dx^k = (\partial x^k / \partial u^\alpha) du^\alpha$, and n_i is a unit normal covariant vector at a point P of the hypersurface V^m and dA denotes the area element of V^m . Thus we obtain the integral formula

$$-\frac{1}{2} \int_D \dots \int g^{ij} \mathcal{L}_\xi g_{ij} dV = \int_{V^m} \dots \int \xi^i n_i dA \quad (\alpha).$$

Let the group G be conformal, that is, ξ^i satisfy the equation

$$\mathcal{L}_\xi g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\phi(x) g_{ij}$$

(cf. [5], p. 32), where the symbol ";" always means the covariant derivative, then (α) becomes

$$-(m+1) \int_D \dots \int \phi dV = \int_{V^m} \dots \int \xi^i n_i dA \quad (\alpha)_c.$$

Let G be homothetic, that is, $\phi \equiv C = \text{constant}$, then

$$-(m+1)C V = \int_{V^m} \cdots \int \xi^i n_i dA \quad (\alpha)_h$$

V being the total volume of D . Especially, if our space R^{m+1} is an Euclidean space E^{m+1} and if we take a point of D as origin of the euclidean coordinates x^i and attach to each point x the vector ξ^i with the components $\xi^i = x^i$ (i.e., the position vector of x), then the transformations (1.1) are homothetic, that is, $C = 1$, thus the formula $(\alpha)_h$ becomes the following well-known formula

$$(m+1)V = - \int_{V^m} \cdots \int x^i n_i dA.$$

In the case $m+1=3$, we have $3V = - \int_{V^2} x^i n_i dA$ (cf. [2], p. 18).

Furthermore in the Riemann space R^{m+1} , let G be isometric, that is, $C=0$, then we have

$$\int_{V^m} \cdots \int \xi^i n_i dA = 0 \quad (\alpha)_i.$$

By making use of the formula $(\alpha)_c$ and the formula $(I)_c$ of the previous paper ([4], p. 3), we have the following

THEOREM 1.1. *If D is a domain in R^{m+1} admitting a conformal Killing vector ξ (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$) and if its boundary V^m is a closed hypersurface with $H_1 = \text{constant}$, then it follows that*

$$(m+1)H_1 \int_D \cdots \int \phi dV = \int_{V^m} \cdots \int \phi dA \quad (1.6)$$

where H_1 means the first mean curvature of V^m .

Proof. Multiplying the formula $(\alpha)_c$ by $H_1 (= \text{const.})$, we obtain

$$-(m+1)H_1 \int_D \cdots \int \phi dV = H_1 \int_{V^m} \cdots \int \xi^i n_i dA.$$

By making use of the formula $(I)_c$ of the previous paper $H_1 \int_{V^m} \xi^i n_i dA = - \int_{V^m} \phi dA$ (cf. [4], p. 3), we see that $(m+1)H_1 \int_D \phi dV = \int_{V^m} \phi dA$.

COROLLARY. *If D is a domain in R^{m+1} admitting a homothetic Killing vector ξ (i.e., $\xi_{i;j} + \xi_{j;i} = 2C g_{ij}$) and if its boundary V^m is a closed hypersurface with $H_1 = \text{const.}$, then we have*

$$V = \frac{1}{m+1} \cdot \frac{A}{H_1} \quad (1.7)$$

where A is the total area of V^m .

Proof. Substituting $\phi=C$ ($=\text{const.}$) into both members of (1.6), we obtain easily (1.7).

Especially, if our space R^{m+1} is an Euclidean space E^{m+1} and if V^m is a hypersphere with radius γ , then the formula (1.7) becomes $V=\gamma \cdot A/m + 1$.

§ 2. On variational problems of integral formulas

In this section, we shall discuss the preceding integral formulas and the integral formulas of the previous paper ([4], p. 3) from the point of view of the calculus of variations.

We now consider a variation of a geometrical object in R^{m+1} , defined by

$$\bar{x}^i = x^i + \xi^i(x) \varepsilon \quad (2.1)$$

where ε is a parameter near $\varepsilon=0$; then substituting (1.2) into (2.1), we get a family $\bar{x}^i = \bar{x}^i(u^\alpha, \varepsilon)$ of admissible hypersurfaces of the form

$$\bar{x}^i = x^i(u^\alpha) + \xi^i(x^j(u^\alpha)) \varepsilon. \quad (2.2)$$

For each value of ε near $\varepsilon=0$, we thus obtain a domain $D(\varepsilon)$ with a boundary $V^m(\varepsilon)$, where $D(0)=D$, $V^m(0)=V^m$; let $V(\varepsilon)$ be the total volume of $D(\varepsilon)$. Now we have the following

THEOREM 2.1. *If $(\delta V/\delta \varepsilon)_{\varepsilon=0}$ is the first variation of the total volume of $D(\varepsilon)$ along D with respect to a direction ξ^i , then*

$$\left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = \frac{1}{2} \int \dots \int_D g^{ij} \mathcal{L}_\xi g_{ij} dV. \quad (2.3)$$

Proof. Let \bar{V} be the total volume of $D(\varepsilon)$, which is given by the integral form

$$\bar{V} = \int_{D(\varepsilon)} \dots \int \sqrt{\bar{g}}(d\bar{x}, \dots, d\bar{x})$$

where $\bar{g}=g(x, \varepsilon)$ and $d\bar{x}^i = dx^i + (\partial \xi^i / \partial x^l) dx^l$. For the first variation of \bar{V} along D we have

$$\begin{aligned} \frac{\delta V}{\delta \varepsilon} &= \int_{D(\varepsilon)} \dots \int \frac{\partial}{\partial \varepsilon} \sqrt{\bar{g}}(d\bar{x}, \dots, d\bar{x}) + \sqrt{\bar{g}} \frac{\partial}{\partial \varepsilon}(d\bar{x}, \dots, d\bar{x}), \\ \left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} &= \frac{1}{2} \int_D \dots \int \sqrt{g} g^{ij} \left(\frac{\partial g_{ij}}{\partial x^l} \xi^l + g_{lj} \frac{\partial \xi^l}{\partial x^i} + g_{li} \frac{\partial \xi^l}{\partial x^j} \right) (dx, \dots, dx) \\ &= \frac{1}{2} \int_D \dots \int g^{ij} \mathcal{L}_\xi g_{ij} dV \end{aligned}$$

because of $dV = \sqrt{g}(dx, \dots, dx)$ and $\mathcal{L}_\xi g_{ij} = (\partial g_{ij}/\partial x^l) \xi^l + g_{lj}(\partial \xi^l/\partial x^i) + g_{li}(\partial \xi^l/\partial x_j)$ (cf. [5], p. 4).

Therefore we evidently have the following

COROLLARY 2.1. The first variation of the total volume of $D(\varepsilon)$ along D , with respect to a direction ξ^i becomes as follows

$$\left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = (m+1) \int_D \cdots \int \phi \, dV, \quad \left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = (m+1) C V, \quad \text{or} \quad \left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = 0$$

according to ξ^i being a conformal Killing vector ($\mathcal{L}_\xi g_{ij} = 2\phi g_{ij}$), a homothetic Killing vector, or a Killing vector.

COROLLARY 2.2. The first variation of the total volume of $D(\varepsilon)$ along D , with respect to a direction ξ^i , is given by

$$\left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int_{V^m} \cdots \int \xi^i n_i \, dA. \quad (2.4)$$

The proof easily follows from the integral formula (α) and (2.3).

We consider next a closed orientable hypersurface V^m of class C^3 imbedded in R^{m+1} , locally given by (1.2). then we obtain a family $\bar{x}^i = x^i(u^\alpha, \varepsilon)$ of admissible hypersurfaces of the form (2.2). For each value of ε near $\varepsilon=0$, we have a hypersurface $V^m(\varepsilon)$, where $V^m(0) = V^m$, and we have a value $A(\varepsilon)$ of the total area of $V^m(\varepsilon)$. Then we shall prove the following theorem.

THEOREM 2.2. Let $(\delta A/\delta \varepsilon)_{\varepsilon=0}$ be the first variation of the total area of $V^m(\varepsilon)$ along V^m , with respect to a direction ξ^i , then

$$\left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = \frac{1}{2} \int_{V^m} \cdots \int \mathcal{L}_\xi g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} \, dA. \quad (2.5)$$

Proof. As well-known, the total area of $V^m(\varepsilon)$ is given by the form

$$A(\varepsilon) = \int_{V^m(\varepsilon)} \cdots \int \sqrt{\tilde{g}(\varepsilon)} (du, \dots, du)$$

where $\tilde{g}(\varepsilon)$ means the determinant of the metric tensor $g_{\alpha\beta}(\varepsilon)$ of the hypersurface $V^m(\varepsilon)$ (i.e., $g_{\alpha\beta}(\varepsilon) = g_{ij}(\bar{x})(\partial \bar{x}^i/\partial u^\alpha)(\partial \bar{x}^j/\partial u^\beta)$).

Differentiating the above integral form with respect to ε , we have

$$\frac{\delta A}{\delta \varepsilon} = \int_{V^m(\varepsilon)} \cdots \int \frac{\partial}{\partial \varepsilon} \sqrt{\tilde{g}(\varepsilon)} (du, \dots, du)$$

where u^α and ε are independent parameters.

On making use of the following results

$$\begin{aligned}\frac{\partial}{\partial \varepsilon} \sqrt{\tilde{g}} &= \frac{1}{2\sqrt{\tilde{g}}} \left\{ \frac{\partial \tilde{g}}{\partial \tilde{x}^k} \xi^k + \frac{\partial \tilde{g}}{\partial (\partial \tilde{x}^k / \partial u^\alpha)} \frac{\partial \xi^k}{\partial u^\alpha} \right\}, \\ \frac{\partial \tilde{g}}{\partial \tilde{x}^k} &= \frac{\partial \tilde{g}_{ij}}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial u^\alpha} \frac{\partial \tilde{x}^j}{\partial u^\beta} g^{\alpha\beta}(\varepsilon) \tilde{g}, \quad \frac{\partial \tilde{g}}{\partial (\partial \tilde{x}^k / \partial u^\alpha)} = 2 g_{kj}(\tilde{x}) \frac{\partial \tilde{x}^j}{\partial u^\beta} g^{\alpha\beta}(\varepsilon) \tilde{g}, \\ \frac{\partial \tilde{g}}{\partial (\partial \tilde{x}^k / \partial u^\alpha)} \frac{\partial \xi^k}{\partial u^\alpha} &= \left(g_{kj}(\tilde{x}) \frac{\partial \xi^k}{\partial x^i} + g_{ki}(\tilde{x}) \frac{\partial \xi^k}{\partial x^j} \right) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta}(\varepsilon) \tilde{g},\end{aligned}$$

we obtain

$$\left(\frac{\partial \sqrt{\tilde{g}}}{\partial \varepsilon} \right)_{\varepsilon=0} = \frac{1}{2} \mathcal{L}_\xi g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} d\sqrt{\tilde{g}} A.$$

Consequently for the first variation of the total area of $V^m(\varepsilon)$ along V^m , we can see

$$\left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = \frac{1}{2} \int_{V^m} \cdots \int \mathcal{L}_\xi g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA.$$

COROLLARY 2.3. The first variation of the total area of $V^m(\varepsilon)$ along V^m , with respect to a direction ξ^i , becomes as follows

$$\left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = m \int_{V^m} \cdots \int \phi dA, \quad \left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = m C A \quad \text{or} \quad \left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = 0$$

according to ξ^i being a conformal Killing vector, a homothetic Killing vector or a Killing vector.

From Theorem 2.2 and the formula (I) of the previous paper (cf. [4], p. 3), we can see easily the following

COROLLARY 2.4. The first variation of the total area $V^m(\varepsilon)$ along V^m with respect to a direction ξ^i , has the form

$$\frac{1}{m} \left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int_{V^m} \cdots \int H_1 \xi^i n_i dA. \quad (2.6)$$

If our space R^{m+1} is an Euclidean space E^{m+1} and if we take to each point x the vector $\xi^i(x)$ with the components $\xi^i = x^i$ (i.e., the position vector of x), then the vector ξ^i is a homothetic Killing vector with $C=1$, and $\xi^i n_i$ is the support function p for $x \in V^m$. In this case, the formula (2.6) becomes

$$\int_{V^m} \cdots \int H_1 p dA + A = 0,$$

this being nothing but the formula of Minkowski type of V^m in E^{m+1} given by C. C. HSIUNG (cf. [6], p. 286). Therefore we can see the formula (2.6):

$$\int_{V^m} \cdots \int H_1 \xi^i n_i dA + \frac{1}{m} \left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = 0$$

as a generalization of the formula of Minkowski type.

Remark 1. Although the vector field $\xi^i(x)$ is not defined on the whole Riemann space but defined on a certain domain including both D and V^m , all the preceding theorems are valid.

Remark 2. In case an arbitrary vector η^i is defined on the hypersurface V^m given by (1.2), we can find also the following formulas

$$\left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int_{V^m} \cdots \int \eta^i(u^\alpha) n_i dA \quad (2.7)$$

and

$$\left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = - m \int_{V^m} \cdots \int H_1 \eta^i(u^\alpha) n_i dA \quad (2.8)$$

for the first variation of the total volume of $D(\varepsilon)$ along D and the first variation of the total area of $V^m(\varepsilon)$ along V^m , by means of a family $\bar{x}^i = x^i(u^\alpha, \varepsilon)$ of the hypersurfaces of the form

$$\bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \eta^i(u^\alpha) \varepsilon.$$

§ 3. The isoperimetric problems

In this section, we shall prove the following theorems closely related to what may be called an isoperimetric problem in R^{m+1} .

If $(\delta A/\delta \varepsilon)_{\varepsilon=0} = 0$ for all variations with respect to a direction such that $(\delta V/\delta \varepsilon)_{\varepsilon=0} = 0$, then the hypersurface V^m is called a pseudo-stationary hypersurface.

THEOREM 3.1. *Let V^m be a closed orientable hypersurface in R^{m+1} . Then the first mean curvature of V^m is constant if and only if V^m is a pseudo-stationary hypersurface.*

Proof. Suppose H_1 is constant; if $(\delta V/\delta \varepsilon)_{\varepsilon=0} = 0$, then we get from (2.7)

$$\left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int_{V^m} \cdots \int \eta^i n_i dA = 0$$

and hence from (2.8)

$$\left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = - m \int_{V^m} \cdots \int H_1 \eta^i n_i dA = - m H_1 \int_{V^m} \cdots \int \eta^i n_i dA = 0.$$

Thus V^m is a pseudo-stationary hypersurface.

Conversely suppose $(\delta A/\delta \varepsilon)_{\varepsilon=0}=0$ for every variation with respect to a direction η^i such that $(\delta V/\delta \varepsilon)_{\varepsilon=0}=0$; we must prove that H_1 is constant. Let φ be an arbitrary function defined on V^m such that $\int_{V^m} \varphi dA = 0$. We wish to show first that φ is in fact the normal component of a variation vector η^i such that $(\delta V/\delta \varepsilon)_{\varepsilon=0}=0$. Let us consider the family of hypersurfaces $\tilde{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \varphi n^i \varepsilon$, then from (2.4) we see

$$\left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int_{V^m} \cdots \int \varphi n^i n_i dA = - \int_{V^m} \cdots \int \varphi dA = 0.$$

Thus φ is the normal component of a variation vector such that $(\delta V/\delta \varepsilon)_{\varepsilon=0}=0$.

By hypothesis, V^m is pseudo-stationary, therefore it follows that

$$\left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = -m \int_{V^m} \cdots \int H_1 \varphi dA = 0.$$

Thus we have $\int_{V^m} H_1 \varphi dA = 0$. Also if h is an arbitrary constant, we have $\int_{V^m} h \varphi dA = 0$, and hence for any function φ such that $\int_{V^m} \varphi dA = 0$ and for any constant h , we obtain

$$\int_{V^m} \cdots \int (H_1 - h) \varphi dA = 0.$$

Now let h be the mean value of H_1 :

$$h = \frac{1}{A} \int_{V^m} \cdots \int H_1 dA,$$

then we have

$$\begin{aligned} \int_{V^m} \cdots \int (H_1 - h) dA &= \int_{V^m} \cdots \int H_1 dA - h \int_{V^m} \cdots \int dA \\ &= \int_{V^m} \cdots \int H_1 dA - h \cdot A = \int_{V^m} \cdots \int H_1 dA - \int_{V^m} \cdots \int H_1 dA = 0. \end{aligned}$$

Consequently taking $H_1 - h$ for φ , we obtain

$$\int_{V^m} \cdots \int (H_1 - h)^l dA = 0.$$

Therefore $H_1 \equiv h$, which concludes the proof.

This theorem is nothing but a generalization of the same theorem in an Euclidean space given already in [2], p. 19, and this proof follows the same argument as in [2].

A. D. Alexandrov has already proved the following result in his paper ([7], p. 304), where in the case of positive curvature, R^{m+1} shall be a sphere and V^m contained in a hemisphere of R^{m+1} :

THEOREM A. *If R^{m+1} has constant curvature and if V^m is a simple closed hypersurface with $H_1 = \text{constant}$, then V^m is a hypersphere.*

From this result, we have (under the same assumptions as above):

COROLLARY 3.1. *If V^m is a simple closed hypersurface in R^{m+1} with constant curvature, then V^m is a hypersphere if and only if V^m is a pseudo-stationary hypersurface.*

Now in R^{m+1} , let S be the collection of all closed orientable hypersurfaces V^m enclosing a fixed volume. Then the total area A of V^m is a function on S . Let V^m be a fixed hypersurface and consider a one parameter family of continuous and differentiable variations of V^m , indexed by a parameter ε . Let $V^m(\varepsilon)$ denote the varied hypersurface. Then we require that $V^m(0) = V^m$ and that for each ε , $V^m(\varepsilon) \in S$ (i.e. these variations are volume preserving).

The total area $A(\varepsilon)$ of $V^m(\varepsilon)$ is a differentiable function of ε . If $(\delta A/\delta \varepsilon)_{\varepsilon=0} = 0$ for all volume preserving variations, then V^m is called a stationary hypersurface. Then we have

THEOREM 3.2. *If R^{m+1} admits a homothetic Killing vector field ξ^i ($\xi_{i;j} + \xi_{j;i} = 2Cg_{ij}$, $C \neq 0$) and if V^m is a closed orientable hypersurface in R^{m+1} , then the first mean curvature H_1 of V^m is constant if and only if V^m is a stationary hypersurface.*

Proof. Let V^m be given by (1.2) and suppose for simplicity that $V(0) = 1$ and let $V^m(\varepsilon)$ be a variation of V^m ; denote its total area and the total volume of the domain bounded by $V^m(\varepsilon)$ by $A(\varepsilon)$ and $V(\varepsilon)$ respectively. $V^m(\varepsilon)$ can be represented by

$$\bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \eta^i(u^\alpha)\varepsilon + \dots$$

for each value of ε near $\varepsilon = 0$, where $\eta^i(u^\alpha) = (\partial \bar{x}^i / \partial \varepsilon)_{\varepsilon=0}$. Then from (2.7) and (2.8) we have

$$\left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int_{V^m} \dots \int \eta^i n_i dA, \quad \left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = -m \int_{V^m} \dots \int H_1 \eta^i n_i dA.$$

Sufficiency in Theorem 3.2. is similar by proved as in Theorem 3.1; that is, suppose H_1 is constant and $\bar{x}^i(u^\alpha, \varepsilon)$ is a volume preserving variation of V^m then $(\delta V/\delta \varepsilon)_{\varepsilon=0} = - \int_{V^m} \dots \int \eta^i n_i dA = 0$ and hence

$$\left(\frac{\delta A}{\delta \varepsilon} \right)_{\varepsilon=0} = -m \int_{V^m} \dots \int H_1 \eta^i n_i dA = -m H_1 \int_{V^m} \dots \int \eta^i n_i dA = 0.$$

Conversely, suppose $(\delta A/\delta \varepsilon)_{\varepsilon=0} = 0$ for every volume preserving variation. Then we must show that H_1 is constant.

Let φ be an arbitrary function defined on V^m such that $\int \dots \int_{V^m} \varphi \, dA = 0$; we wish to show first that φ is the normal component of a volume preserving variation. Consider the family of hypersurfaces

$$V^m(\varepsilon) : \bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \varphi n^i \varepsilon, \quad (3.1)$$

then let $V(\varepsilon)$ denote the total volume of the domain bounded by the hypersurface $V^m(\varepsilon)$, then $V(0) = V = 1$; now the normal component of $(\partial \bar{x}^i / \partial \varepsilon)_{\varepsilon=0} = \varphi n^i$ is given by $(\partial \bar{x}^i / \partial \varepsilon)_{\varepsilon=0} n_i = \varphi n^i n_i = \varphi$. Hence, by virtue of (2.7) we have

$$\left(\frac{\delta V}{\delta \varepsilon} \right)_{\varepsilon=0} = - \int \dots \int_{V^m} \left(\frac{\partial \bar{x}^i}{\partial \varepsilon} \right)_{\varepsilon=0} n_i \, dA = - \int \dots \int_{V^m} \varphi \, dA = 0$$

by hypothesis. But the variation $\bar{x}^i(u^\alpha, \varepsilon)$ need not be volume preserving.

However by hypothesis, our space R^{m+1} admits an infinitesimal homothetic transformation given by (1.1) with the additional condition

$$\xi_{i;j} + \xi_{j;i} = 2C g_{ij} \quad (C \neq 0, \text{ constant}). \quad (3.2)$$

Let us choose a coordinate system such that the path of the infinitesimal transformation is the new x^1 -coordinate curve, that is, a coordinate system in which the vector ξ^i has the components δ_1^i (where δ_j^i denotes the Kronecker delta); then (1.1) becomes $x'^i = x^i + \delta_1^i \delta \tau$ and R^{m+1} admits a one-parameter continuous group G of transformations given by

$$x'^i = x^i + \delta_1^i \tau. \quad (3.3)$$

Then in this new coordinate system, the condition (3.2) becomes as follows $\partial g_{ij} / \partial x^1 = 2C g_{ij}$. Therefore the metric tensor g_{ij} with respect to the new coordinate system has the form $g_{ij} = f_{ij}(x^2, \dots, x^{m+1}) e^{2C x^1}$. Now we take the family of hypersurfaces

$$V^{*m}(\varepsilon) : x^{*i}(u^\alpha, \varepsilon) = \bar{x}^i(u^\alpha, \varepsilon) + \frac{1}{(m+1)C} \log \frac{1}{V(\varepsilon)} \delta_1^i; \quad (3.4)$$

we shall show that $V^{*m}(\varepsilon)$ is a volume preserving variation. Let $V^*(\varepsilon)$ be the total volume of the domain bounded by $V^{*m}(\varepsilon)$ and let n^{*i} and dA^* be a normal vector and an area element of the hypersurface $x^{*i}(u^\alpha, \varepsilon)$ respectively. Then from Corollary 2.1 and Corollary 2.2, we have

$$(m+1)C V^*(\varepsilon) = - \int \dots \int_{V^{*m}(\varepsilon)} \delta_1^i n_i^* \, dA^* = - \int \dots \int_{V^{*m}(\varepsilon)} n_1^* \, dA^*. \quad (3.5)$$

On the other hand, from (3.4) we have the relations

$$\begin{aligned} g_{ij}(x^*) &= f_{ij}(x^{*2}, \dots, x^{*m+1}) e^{2C x^1} \\ &= f_{ij}(\bar{x}^2, \dots, \bar{x}^{m+1}) e^{2C \bar{x}^1} \cdot e^{(2/m+1) \log(1/V(\varepsilon))} = g_{ij}(\bar{x}) e^{(2/m+1) \log(1/V(\varepsilon))}; \end{aligned}$$

thus we obtain

$$\sqrt{g(x^*)} = \sqrt{g(\bar{x})} e^{\log(1/V(\varepsilon))} = \frac{\sqrt{g(\bar{x})}}{V(\varepsilon)}. \quad (3.6)$$

Substituting (3.6) in (3.5) and making use of the relations

$$x^{*2}(u^\alpha, \varepsilon) = \bar{x}^2(u^\alpha, \varepsilon), \dots, x^{*m+1}(u^\alpha, \varepsilon) = \bar{x}^{m+1}(u^\alpha, \varepsilon),$$

we see that

$$\int_{V^{*m}(\varepsilon)} \dots \int n_1^* dA^* = \int_{V^m(\varepsilon)} \dots \int \frac{n_1(\varepsilon)}{V(\varepsilon)} dA(\varepsilon)$$

and

$$(m+1)C V^*(\varepsilon) = -\frac{1}{V(\varepsilon)} \int_{V^m(\varepsilon)} \dots \int \delta_1^i n_i(\varepsilon) dA(\varepsilon) = \frac{1}{V(\varepsilon)} (m+1)C V(\varepsilon) = (m+1)C.$$

Thus we have $V^*(\varepsilon) = 1$, therefore $V^{*m}(\varepsilon)$ is a volume preserving variation of V^m .

Now, since $(\delta V/\partial \varepsilon)_{\varepsilon=0} = 0$ it follows that

$$\left(\frac{\partial x^{*i}}{\partial \varepsilon} \right)_{\varepsilon=0} = \left(\frac{\partial \bar{x}^i}{\partial \varepsilon} \right)_{\varepsilon=0} = \varphi n^i,$$

and we have

$$\left(\frac{\partial x^{*i}}{\partial \varepsilon} \right)_{\varepsilon=0} n_i = \left(\frac{\partial \bar{x}^i}{\partial \varepsilon} \right)_{\varepsilon=0} n_i = \varphi.$$

Therefore φ is not only the normal component of $(\partial \bar{x}^i/\partial \varepsilon)_{\varepsilon=0}$ but is also the normal component of $(\partial x^{*i}/\partial \varepsilon)_{\varepsilon=0}$ and thus φ is the normal component of a volume preserving variation.

By hypothesis, since V^m is stationary, it follows that $(\delta A/\partial \varepsilon)_{\varepsilon=0} = -m \int \dots \int_{V^m} H_1 \varphi dA = 0$, thus $\int \dots \int_{V^m} H_1 \varphi dA = 0$. Also, if h is an arbitrary constant then $\int \dots \int_{V^m} \varphi h dA = 0$ and hence for any function φ such that $\int \dots \int_{V^m} \varphi dA = 0$ and for any constant h , $\int \dots \int_{V^m} (H_1 - h) \varphi dA = 0$. Now let h be the mean value of H_1 : $h = (1/A) \int \dots \int_{V^m} H_1 dA$ then we have $\int \dots \int_{V^m} (H_1 - h) dA = 0$. Consequently we see $\int \dots \int_{V^m} (H_1 - h)^2 dA = 0$. Therefore $H_1 \equiv h$, which completes the proof.

From Theorem A and Theorem 3.2, we have the following corollary:

COROLLARY 3.2 If R^{m+1} is an Euclidean space E^{m+1} , then a simple closed hypersurface with minimal hypersurface area enclosing a fixed volume is a hypersphere.

This may be called a form of the isoperimetric theorem in E^{m+1} .

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