Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 41 (1966-1967)

Artikel: Some homeomorphic sphere pairs that are combinatorially distinct.

Autor: Siebenmann, L. / Sondow, J.

DOI: https://doi.org/10.5169/seals-31383

Some homeomorphic sphere pairs that are combinatorially distinct

By L. SIEBENMANN and J. SONDOW[1])

§ 1. Introduction

We will establish the following improvement of a result of B. Mazur [8].

THEOREM A

For every dimension $n \geq 5$, there exist $(n-2)$-dimensional spheres K_1, K_2 piecewise linearly imbedded in the n-sphere S^n such that there exists a (topological) homeomorphism of pairs

$$h:(S^n, K_1) \to (S^n, K_2)$$

but no p.l. (= piecewise linear) homeomorphism of pairs $(S^n, K_1) \to (S^n, K_2)$. Further h can be p.l. on K_1 and on $S^n - p$ where p is a point on K_1.

Complement. Our construction will actually provide *infinitely* many such $(n-2)$-spheres so that the resulting pairs are all homeomorphic but combinatorially (i.e. piecewise linearly) distinct.

Remark 1. In all our examples the $(n-2)$-sphere is locally knotted at two points, and if $h \mid S^n - p$ is p.l., one can show that p is one of these two points. We do not know whether there exist locally flat p.l. manifold pairs that are homeomorphic but combinatorially distinct.

Remark 2. Five is the least dimension of combinatorially distinct polyhedra that are known to be homeomorphic, i.e. of known counterexamples to the Hauptvermutung (Stallings [16]). For $n=4$, Theorem A is undecided, and for $n=3$ it fails by Moise [11].

B. Mazur gave similar examples in [8] for dimensions $n \geq 23$, but the subpolyhedra K_i were not even manifolds. A version of Theorem A was initially established for $n \geq 6$ by the second author in using Reidemeister representation torsions to distinguish strongly h-cobordant knots [14][15]. Thus our purpose here is to point out a simple proof of Theorem A that uses only Whitehead torsions in the spirit of Stallings [16] and to give a device to accomplish the proof in dimension 5. This device (Construction 2.5 for invertible h-cobordisms of dimension ≥ 5) incidentally gives counterexamples to the Hauptvermutung in dimension 5 by invoking the s-cobordism theorem of Mazur rather than the engulfing technique of Stallings. Reference [16] explains why, as should our proof of Theorem A.

[1]) The first author was supported by the National Research Counsil of Canada.

As Mazur points out, Theorem A for dimension n disproves the hypothesis: B_n) Let $f: R^n \to R^n$ be a topological imbedding of euclidean n-space into itself that is p.l. on a closed, possibly infinite polyhedron $K \subset R^n$, and let $\varepsilon(x) > 0$ be a continuous function on R^n. Then there is p.l. imbedding $g: R^n \to R^n$ such that $f \mid K = g \mid K$ and $|f(x) - g(x)| < \varepsilon(x)$ for all x in R^n.

Now, Homma studies a strictly equivalent hypothesis in [3]. It is easy to show that B_n implies (cf. proof of Theorem 1 in [3]) a strong version of the Hauptvermutung for p.l. manifolds:

C_n) If $f: M_1^n \to M_2^n$ is a homeomorphism of closed p.l. manifolds and K is a finite subpolyhedron in M_1 such that $f \mid K$ is p.l. then f can be approximated by a p.l. homeomorphism g with $g \mid K = f \mid K$.

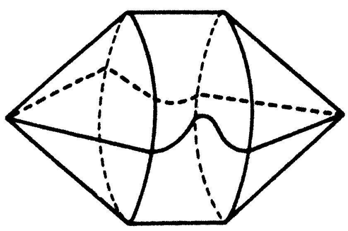
As this flatly contradicts Theorem A, C_n and B_n are false for $n \geq 5$. By Theorem 2 of [3], B_n also implies:

D_n) Every closed topological n-manifold can be triangulated as a p.l. manifold.

However, it would probably be wrong to accept Theorem A as evidence against either the Hauptvermutung for manifolds (e.g. C_n with $K = \emptyset$) or triangulability (e.g. D_n) because hypotheses weaker than B_n imply both. For example, weaken B_n to B'_n by adding the assumption that f is p.l. on a neighborhood of K. The contradictions vanish. Homma's arguments are easily adapted to show that B'_n implies the case of C_n where f is p.l. on a neighborhood of K, and implies D_n without qualification. To deal with separable n-manifolds with boundary, the appropriate hypothesis would be the conjunction of B'_{n-1} and B'_n; noncompactness gives no difficulties.

In spite of Mazur's remark [8, p. 289], B_3 is not a published theorem. However, it is said that Bing et al. have a proof. B'_3 is the well known result - c.f. Bing [2, Theorem 4].

Our examples are constructed from triangulated 'strong' h-cobordisms between smooth knots (§ 2.1) by adding the cone over each end of the h-cobordism. The diagram illustrates this for $n = 2$ and (unfortunately) codimension 1.



The Whitehead torsions of the h-cobordisms distinguish the examples combinatorially; invertibility of the h-cobordisms proves the examples are all topologically the suspension of one knot.

If an exposition of the s-cobordism theorem for p.l. manifolds were available we could work entirely with p.l. manifolds. The reader willing to grant this theorem can afford to ignore the technicalities entailed in using smooth objects and then triangulating.

A general reference for piecewise linear topology is [20]. S^n always denotes the standard n-sphere $\{\vec{x} \in R^n;\ |\vec{x}|=1\}$ with its natural differentiable structure and with a p.l. structure deriving from some Whitehead C^1 triangulation (cf. appendix).

§ 2. Strong Knot h-Cobordisms

Definition 2.1

Let M^{n-1} be a smoothly imbedded submanifold of $W^{n+1}=S^n\times[0,1]$ which is the image of a smooth imbedding $F: S^{n-2}\times[0,1]\to W$ and meets $S^n\times i$ in a knot $K_i\times i= =F(S^{n-2}\times i)$, transversely, $i=0,1$. Identify (S^n, K_i) naturally with $(S^n\times i, K_i\times i)$, $i=0,1$. We say (W, M) gives a *strong h-cobordism* $c=\{(W, M); (S^n, K_0), (S^n, K_1)\}$ from the knot (S^n, K_0) to the knot (S^n, K_1) if the inclusion $(S^m-K_i)\subsetneq(W-M)$ is a homotopy equivalence, $i=0,1$. (S^n, K_0) is called the *left end* of c, (S^n, K_1) the *right end*; and (W, M) may be written for c when no confusion is likely.

Notice that c has a well defined invariant, its torsion $\tau(c)$ lying in the Whitehead group $\mathrm{Wh}(\pi_1(S^n-K_0))$. It coincides with the torsion of the (relative) h-cobordism

$$d=(W-\mathring{T}; S^n\times 0-\mathring{T}, S^n\times 1-\mathring{T})$$

where $\mathring{T}$ is the open 2-disk bundle of a tubular neighborhood T of M in W, and $\mathrm{Wh}\,\pi_1(S^n-K_0)$ is naturally identified with $\mathrm{Wh}\,\pi_1(S^n\times 0-\mathring{T})$. This means that $\tau(c)$ is the Whitehead torsion of the homotopy equivalence $(S^n\times 0-\mathring{T})\subsetneq(W-\mathring{T})$, which in turn can be calculated using any Whitehead C^1 triangulation of $W-\mathring{T}$. For details concerning these invariants see Milnor [10].

Remark: When (as above) there exists, up to inner automorphism, a natural isomorphism of fundamental groups $f: \pi_1 X\to\pi_1 Y$, one can identify $\mathrm{Wh}\,\pi_1 X$ and $\mathrm{Wh}\,\pi_1 Y$ by $f_*=\mathrm{Wh}(f)$ (which is unique). The reader is warned that we will repeatedly do this without special apology.

By passing always from c to the relative h-cobordism

$$d=(W-\mathring{T}; S^n\times 0-\mathring{T}, S^n\times 1-\mathring{T})$$

one readily derives the following facts from the usual theory of h-cobordisms. (Recall that d qualifies as a "relative" h-cobordism because d gives a product cobordism from the boundary of its left end to the boundary of its right end - viz

$$\partial d=(\partial T; S^n\times 0\cap\partial T, S^n\times 1\cap\partial T)$$

such that $\partial T=T-\mathring{T}$ is diffeomorphic to the product $(S^n\times 0\cap\partial T)\times[0,1]$.)

(i) The (relative) s-cobordism theorem of Mazur [7] (see Barden [1] and [17] [21]) tells us that if $\tau(c)=0$ for $c=\{(W, M); (S^n, K_0), (S^n, K_1)\}$, and $(n+1)\geq 6$, then (and only then) (W, M) is diffeomorphic to $(S^n \times [0,1], K_0 \times [0,1])$, i.e. c is a *product cobordism.*

(ii) The torsion of the *dual* of c,

$$\bar{c} = \{(W,M);\ (S^n, K_1), (S^n, K_0)\},$$

obtained by interchanging the ends of c, is

$$\tau(\bar{c}) = (-1)^n \overline{\tau(c)}$$

where the bar over $\tau(c)$ denotes the involution of $\mathrm{Wh}\,\pi_1(S^n - K_0)$ induced by the involution $g \to g^{-1}$ of $\pi_1(S^n - K_0)$. See Milnor [10].

(iii) Suppose the right end of c is identified with the left end of another strong h-cobordism $c' = \{(W', M'); (S^n, K_0'), (S^n, K_1')\}$ so that $(W \cup W', M \cup M')$ gives a *composed strong h-cobordism* cc' from (S^n, K_0) to (S^n, K_1'). Then $\tau(cc') = \tau(c) + \tau(c')$. See Stallings [16], Milnor [10].

One says that c is *invertible* if there exist c' and c'' so that cc' and $c''c$ exist and are product cobordisms. Then observe that $c'' \sim c''(cc') \sim (c''c)c' \sim c'$ where $\sim$ denotes smooth equivalence.

(iv) For $(n+1) \geq 6$ there exist strong h-cobordisms (W^{n+1}, M) with prescribed left end (S^n, K) and prescribed torsion (Stallings [16], also [10]). Hence, in view of (i) and (iii), any strong h-cobordism (W^{n+1}, M) is invertible provided $n+1 \geq 6$.

The result of this section is

PROPOSITION 2.2

Let K^{n-2} be a $(n-2)$-sphere smoothly imbedded in S^n, $n \geq 4$, so that

$$\pi_1(S^n - K) \cong J \times G$$

where J is infinite cyclic and G is the binary icosahedral group of order 120. Then there exist infinitely many invertible strong h-cobordisms $c_0, c_1, c_2, \ldots$ with left end (S^n, K) such that when $i \neq j$, there exists no automorphism θ of $\pi_1(S^n - K)$ making $\theta_* \tau(c_i)$ equal to $\tau(c_j)$ or $\tau(\bar{c}_j)$.

OBSERVATION 2.3

Zeeman has constructed (S^n, K^{n-2}) as above for $n \geq 4$, by 'twist-spinning' a trefoil knot [19] c.f. Kervaire [6].

For the proof of Proposition 2.2 we need:

LEMMA 2.4

Let $\varphi: Z_5 \to J \times G$ be the inclusion of a 5-Sylow subgroup. Then $\varphi_* \mathrm{Wh}(Z_5) \subset \mathrm{Wh}(J \times G)$ is infinite cyclic and for any automorphism θ of $J \times G$, θ_* maps $\varphi_* \mathrm{Wh}(Z_5)$ to itself.

Proof of Lemma:

$\mathrm{Wh}(Z_5)$ is an infinite cyclic group and a generator α is represented by the unit

$(a+a^{-1}-1)$ in the group ring of $Z_5=\{a;\, a^5=1\}$, cf. [10]. To prove that $\beta=\varphi_*(\alpha)$ has infinite order, it will suffice to give a homomorphism $h\colon J\times G\to 0(3)$ so that $B=h(b)+h(b^{-1})-h(1)$, $b=\varphi(a)$, is a 3×3 matrix with determinant $\det B\neq\pm 1$. For by Milnor [10], h induces a homomorphism h_* from $\mathrm{Wh}(J\times G)$ to the multiplicative group of positive real numbers such that

$$h_*(\beta)=|\det B|$$

The homomorphism we choose is a composition $\mathrm{h}=\mathrm{h}_3\mathrm{h}_2\mathrm{h}_1$

$$J\times G\xrightarrow{h_1}G\xrightarrow{h_2}A_5\xrightarrow{h_3}0(3)$$

where h_1 is projection, h_2 is a '2-fold covering' homomorphism onto the group A_5 of 60 orientation preserving rotations of the icosahedron [19], and h_3 is an inclusion so chosen that $h(b)$ is a rotation of order 5 about the x_3-axis in R^3, i.e.

$$h(b)=\begin{pmatrix}\cos\xi & \sin\xi & 0\\ -\sin\xi & \cos\xi & 0\\ 0 & 0 & 1\end{pmatrix}$$

where $\xi=k2\pi/5$, k being 1, 2, 3, or 4.
Then

$$B=\begin{pmatrix}2\cos\xi & -1 & 0 & & 0\\ 0 & & 2\cos\xi & -1 & 0\\ 0 & & 0 & & 1\end{pmatrix}$$

which has determinant $\neq\pm 1$.

It remains to show that for any automorphism θ of $J\times G$, θ_* maps $\varphi_*\mathrm{Wh}(Z_5)$ onto itself. This is clear if θ maps $\varphi(Z_5)$ onto itself. In the general case $\theta(\varphi(Z_5))$ is another 5-Sylow subgroup; hence we can find an inner automorphism $\Psi(x)=g^{-1}xg$ such that $\Psi\theta$ maps $\varphi(Z_5)$ to itself. Since $\Psi_*=1$ the conclusion follows. This completes the lemma.

Proof of Proposition 2.2 for $n\geq 5$:

For $k=0, 1, 2, \ldots$ let

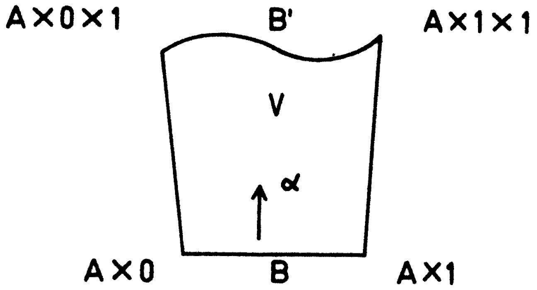
$$c_k=\{(W_k, M);\quad (S^n, K), (S^n, K_k)\}$$

be an invertible strong h-cobordism with torsion $k\beta$ as provided by (iv) where β generates the subgroup $\varphi_*\mathrm{Wh}(Z_5)\subset\mathrm{Wh}\,\pi_1(S^n-K)$ of Lemma 2.4. Take distinct $i, j\geq 0$. Then $\theta_*\tau(c_i)=\pm i\beta$ is not equal to $\tau(c_j)=j\beta$ or $\tau(\bar{c}_j)=\pm j\bar{\beta}=\pm j(\pm\beta)$. This completes the proof.

For the case $n=4$ we will use the following.

CONSTRUCTION 2.5

Suppose A^n, $n \geq 4$, is a smooth compact n-manifold, possibly with boundary and α an element of $\mathrm{Wh}(\pi_1 A)$. Now $B = A \times [0, 1]$ is a $(n+1)$-manifold (with corners along $BdA \times 0 \cup BdA \times 1$). We form a relative h-cobordism $c = (V; B, B')$ with torsion α.



For convenience let c be constructed by attaching 2-handles and then 3-handles to $B \times [0,1]$ along $Int(B \times 1)$, cf. [10], and identifying B with $B \times 0$. Then observe that B' gives a relative h-cobordism d from $A \times 0 \times 1$ to $A \times 1 \times 1$. Call c the *wedge* over B with torsion α and *d the end of the wedge.*

We assert that $\tau(d) = \alpha + (-1)^n \bar{\alpha}$. To see this consider the commutative diagram of inclusions

$$\begin{tikzcd} A \times 0 \times 1 \arrow[rr, "i_1"] \arrow[dr, "i_2"'] & & V \\ & B' \arrow[ur, "i_3"'] & \end{tikzcd}$$

By an addition theorem for Whitehead torsions of maps [10],

$$\tau(i_1) = \tau(i_2) + \tau(i_3) \qquad (*)$$

Now $\tau(i_2) = \tau(d)$ by definition of $\tau(d)$, and $\tau(i_3) = (-1)^{n+1}\bar{\alpha}$ by the duality theorem for h-cobordisms [10]. Further i_1 factorizes up to homotopy

$$A \times 0 \times 1 \rightarrow A \times 0 \hookrightarrow A \times [0,1] \hookrightarrow V$$

whence $\tau(i_1) = 0 + 0 + \tau(c) = \alpha$ by the addition theorem cited above. Substituting in $(*)$ we find $\tau(d) = \alpha + (-1)^n \bar{\alpha}$.

Secondly, we assert that the end d' of the wedge over $A \times [1,2]$ with torsion $-\alpha$ is an inverse for d, even when $n = 4$. To see this paste the wedges together along $A \times 1 \times [0,1]$ and behold a wedge over $C = A \times [0, 2]$ with torsion $\alpha + (-\alpha) = 0$. As $(n+2) \geq 6$, this wedge is a product, and so its end dd' is also. Indeed the relative form of the s-cobordism theorem says that there exists a diffeomorphism

$$C \times [0,1] \rightarrow V \cup V' = C \times [0,1] \cup \{\text{2- and 3-handles}\}$$

that extends the identity map on $C \times O \cup BdC \times [0,1]$. This proves the sharper assertion, used implicitly below, that the natural product structure for $\partial(dd')$ given by $BdA \times [0,2] \times 1$, extends to a product structure for dd'. Similarly (a copy of) d' is a left inverse for d.

Proof of Proposition 2.2. for $n=4$:

Apply the above construction with $A=(S^4 - Int\, N)$, where N is a tubular neighborhood of $K^2 \subset S^4$, and with $\alpha = k\beta$, where β is again a generator of $\varphi_* \mathrm{Wh}(Z_5) \subset \mathrm{Wh}(J \times G)$, and $k \geq 0$ is an integer fixed for the moment. Now the end

$$d = (B';\, A \times 0 \times 1,\, A \times 1 \times 1)$$

of the wedge over $A \times [0,1]$ with torsion α gives the product cobordism

$$(BdA \times [0,1] \times 1;\, BdA \times 0 \times 1,\, BdA \times 1 \times 1)$$

between the boundaries of the ends of d. Since $BdA = BdN$ we can form

$$W = B' \cup N \times [0,1]$$

identifying $BdA \times [0,1] \times 1$ with $BdN \times [0,1]$. Then W gives an h-cobordism between two copies of S^4. Hence W is a smooth homotopy 5-sphere Σ^5 with the interiors of two disjoint smooth 5-disks removed. Since one knows Σ^5 is S^5 [11], W is diffeomorphic to $S^4 \times [0,1]$. We conclude that the pair (W, M), where M was $K \times [0,1] \subset N \times [0,1]$, gives a strong h-cobordism c_k from the knot (S^4, K) to (a copy of) itself.

One easily checks that the inverse given above for d provides an inverse for c_k.

Next consider the torsion

$$\tau(c_k) = \alpha + \bar{\alpha} = k(\beta + \bar{\beta}) \in \mathrm{Wh}\,\pi_1(S^4 - K)$$

Since β comes from the unit $(b + b^{-1} - 1)$ which is invariant under $b \to b^{-1}$, we have $\beta = \bar{\beta}$ and $\tau(c_k) = 2k\beta$. So, when k takes distinct values $i, j \geq 0$, we can show

$$\theta_* \tau(c_i) \neq \tau(c_j) \quad \text{or} \quad \tau(\bar{c}_j)$$

just as when $n \geq 5$. This completes Proposition 2.2.

§ 3. Proof of Theorem A

1) THE CONSTRUCTION

Let $c_0, c_1, c_2, \ldots$ be an infinite sequence of strong h-cobordisms provided by Proposition 2.2 and Observation 2.3.

We write

$$c_k = \{(W^n, M);\, (S^{n-1}, L^{n-3}),\, (S^{n-1}, L_k^{n-3})\}$$

making a notational shift from n to $n-1$ and K to L. Thus (S^{n-1}, L) is a fixed smooth knot with $n \geq 5$ and group $\pi_1(S^{n-1} - L) = J \times G$.

Now give (W_k, M_k) a Whitehead C^1-triangulation such that a smooth product neighborhood T_k of M_k becomes a p.l. product neighborhood. To do this one can spread the triangulation from M_k to T_k to W_k using Whitehead [18] or Munkres [13]. Thus W_k becomes a p.l. manifold, and M_k becomes a p.l. submanifold with regular neighborhood T_k. Think of W_k as a topological manifold with both a smoothness structure and a p.l. structure. Then from the definition of torsions (c.f. § 2.1) and the uniqueness theorem for regular neighborhoods (Hudson and Zeeman [4], also [20]) we conclude that if

$$f:(W_i, M_i)\to(W_j, M_j)$$

were a p.l. homeomorphism and θ the automorphism of $\pi_1(S^{n-1}-L)$ induced by $f\mid S^{n-1}-L$, then $\theta_*\tau(c_i)$ would be either $\tau(c_j)$ or $\tau(\bar{c}_j)$ according as f maps the left end (S^{n-1}, L) of c_i either to the left end or to the right end of c_j. Thus Proposition 2.2 shows that no such f exists when $i\neq j$.

Next note that by Whitehead's C^1-triangulation uniqueness theorem, W_k is p.l. homeomorphic to $S^{n-1}\times[0,1]$, and M_k is p.l. homeomorphic to $S^{n-3}\times[0,1]$. Thus adding the cone over each end of the triangulated cobordism (W_k, M_k) we produce a p.l. pair (S^n, K_k) where K_k is a $(n-2)$-sphere p.l. imbedded in S^n and locally knotted at the two cone points. These pairs (S^n, K_k), $k=0, 1, 2, \ldots$, form our infinite collection of examples for dimension n.

2) Distinguishing the pairs combinatorially

Suppose for the sake of argument that

$$g:(S^n, K_i)\to(S^n, K_j), \qquad i\neq j,$$

is a p.l. homeomorphism and choose subdivisions so that g is simplicial. The two locally knotted points of K_i must be carried in some order to those of K_j. Excise the open stars of all locally knotted points. One proves with the help of the relative regular neighborhood uniqueness theorem of Hudson and Zeeman [4, Theorem 3][1] and the isotopy extension theorem [5] that what remains is a p.l. equivalence of a copy of (W_i, M_i) with a copy of (W_j, M_j), which, as we have observed, is impossible. For a way of avoiding the regular neighborhood argument see [9, p. 58].

3) Finding the homeomorphisms

Finally we show that there exists a homeomorphism of pairs

$$(S^n, K_i)\to(S^n, K_j)$$

[1]) The definition of relative regular neighborhood in [4, p. 722] requires the extra condition, which with the notation used there, would read "(5). There exists a simplicial subdivision of $(N, X_\natural, Y_\natural)$ with respect to which $\mathrm{lk}(A, N)$ collapses to $\mathrm{lk}(A, X_\natural)$ for each simplex A in $Y_\natural$." A counterexample of Ralph Tindell will appear in Bull. Amer. Math. Soc.; for corrected proofs see the thesis of Lawrence S. Hush, Florida State University.

that is p.l. on K_i and on $S^n - p_i$ where $p_i \in K_i$ is one of the two points at which K_i is locally knotted in S^n.

Let $i, j \geq 0$ be thought of as fixed and $k \geq 0$ as generic. Introduce the symbols $\approx$ and $\equiv$ for diffeomorphism and p.l. homeomorphism respectively. Form (W_k', M_k') from (W_k, M_k) by attaching a collar $(S^{n-1}, L_k) \times [0,1)$ naturally at the right end and give (W_k', M_k') a smoothness structure using a smooth collar of (S^{n-1}, L_k) in (W_k, M_k) as in composing cobordisms. Since all the cobordisms c_k are invertible and have (S^{n-1}, L) as left end, the formal infinite product argument in [16] yields:

Lemma 3.1

For any $k \geq 0$, (W_k', M_k') is diffeomorphic to $(S^{n-1}, L) \times [0,1)$.
Proof: Let $e = (S^{n-1}, L) \times [0,1]$ and $e_k = (S^{n-1}, L_k) \times [0,1]$. Then

$$\begin{aligned}(W', M_k') &\approx c_k e_k e_k \dots \approx c_k (c_k^{-1} c_k)(c_k^{-1} c_k) \dots \\ &\approx (c_k c_k^{-1})(c_k c_k^{-1}) \dots \approx ee \dots \\ &\approx (S^{n-1}, L) \times [0,1) \text{ as required}.\end{aligned}$$

Let $c(S^{n-1}, L) = (cS^{n-1}, cL) \subset (S^n, K_k)$ denote the cone on the left end of c_k, let $c(S^{n-1}, L_k) \subset (S^n, K_k)$ denote the cone on the right end, and let p_k be the vertex of the cone $c(S^{n-1}, L_k)$. Let (W_k'', M_k'') be (S^n, K_k) with *Int* $c(S^{n-1}, L)$ and (p_k, p_k) deleted, or equivalently (W_k, M_k) with $[c(S^{n-1}, L_k) - (p_k, p_k)] \equiv (S^{n-1}, L_k) \times [0,1)$ added.

Now observe that $(W_k', M_k') = (W_k, M_k) \cup (S^{n-1}, L_k) \times [0,1)$ receives a well defined p.l. structure from (W_k, M_k) and that this p.l. structure clearly gives a C^1-triangulation of (W_k', M_k') as a smooth pair. Further there is a natural identification of (W_k', M_k') with (W_k'', M_k'') that is a p.l. homeomorphism. Since $(W_i', M_i') \approx (W_j', M_j')$ by Lemma 3.1, the uniqueness theorem for C^1-triangulations of pairs (see Appendix) shows that $(W_i', M_i') \equiv (W_j', M_j')$. Hence $(W_i'', M_i'') \equiv (W_j'', M_j'')$, i.e. there exists a p.l. homeomorphism

$$G : (W_i'', M_i'') \to (W_j'', M_j'').$$

Extend G to a homeomorphism

$$H : (S^n, K_i) \to (S^n, K_j)$$

by setting $H(p_i) = p_j$ and setting $H | c(S^{n-1}, L)$ equal to the cone on the restriction of G to (S^{n-1}, L). Then H is p.l. on the complement of p_i and it remains to show that H may be chosen so that $H | K_i$ is p.l.

Choose any extension of $H | cL$ to a p.l. homeomorphism of $(n-3)$-spheres

$$h : K_i \to K_j.$$

We claim H can be improved so that $H | K_i = h$. This will certainly be the case if we can always replace G by a p.l. homeomorphism

$$G' : (W_i'', M_i'') \to (W_j'', M_j'')$$

that gives any prescribed p.l. homeomorphism $M_i'' \to M_j''$ coinciding with G on $B\,d M_i'' = L$. Since $(W_k'', M_k'') \equiv (W_0'', M_0'') \equiv (S^{n-1}, L) \times [0,1)$, the problem reduces to the ad hoc

LEMMA 3.2

Suppose (S^{n-1}, L) is a C^1-triangulated smooth knot. Any p.l. homeomorphism F of $L \times [0,1)$ onto itself extends to a p.l. homeomorphism F of $(S^{n-1}, L) \times [0,1)$ onto itself.

Proof of Lemma:

$L \times [0,1)$ admits a p.l. product neighborhood $T \equiv M \times D^2$ where M abbreviates $L \times [0,1)$ and D^2 is the p.l. 2-disk. (Since L has a smooth product neighborhood, this is clear for a suitably constructed C^1 triangulation. By the uniqueness theorem of the appendix it holds for any C^1 triangulation.) We put $F|T = f \times 1_{D^2}$. Now there exists a p.l. isotopy of f to the identity, e.g. by Alexander's device [5, p. 70]. On a p.l. collar neighborhood of $\partial T \equiv M \times B\,d\,D^2$ in the complement of $\mathring{T} \equiv M \times \text{Int } D^2$ set F equal to the product of this isotopy with 1_{BdD^2}. On the rest of $S^{n-1} \times [0,1]$, F can be the identity.

The proof of Theorem A is now complete.

§ 4. Appendix: On C^r Triangulation of Pairs

Let (W, M) be a smooth manifold pair where M is a smooth, properly imbedded submanifold of W such that M meets BdW in BdM, transversely. A C^r *triangulation* of (W, M), $r \geq 1$ an integer or ∞, is a homeomorphism

$$h : (K, L) \to (W, M)$$

of a simplicial pair (K, L) onto (W, M), such that the restriction of h to each closed simplex of K is a non-singular C^r imbedding. Whitehead's uniqueness theorem [18] applied to local charts shows that the p.l. structure that h gives to W or to M is a p.l. manifold structure.

Although Whitehead's existence and uniqueness theorems [18] are usually stated only for individual manifolds, *they actually hold for pairs.* Thus any smooth pair admits a C^r triangulation, $r \geq 1$, and secondly if $h_i : (K_i, L_i) \to (W, M)$, $i = 1,2$, are two C^r triangulations, $r \geq 1$, there exists a simplicial subdivision

$$(K_i', L_i') \quad \text{of} \quad (K_i, L_i), \quad i = 1,2,$$

and a simplicial homeomorphism

$$h' : (K_1', L_1') \to (K_2', L_2').$$

Since we made use of the latter fact in § 3 to deduce the existence of a p.l. homeomorphism from the existence of a diffeomorphism, we indicate how one can derive these theorems by following the argument in Munkres [13]. Considering the proofs of [13, § 10.5, § 10.6] one sees that it suffices to complement the basic approximation theorem in [13] as follows:

THEOREM: Munkres [13, § 10.4]

Let M be a nonbounded C^r submanifold of R^n, $r \geq 1$. Let $f: K \to M$ and $g: L \to M$ be C^r imbeddings whose images are closed in M. Given $\delta(x) > 0$, continuous on the disjoint union of K and L, there are δ-approximations $f': K' \to M$ and $g': L' \to M$ to f and g respectively, which intersect in a full subcomplex such that their union is a C^r imbedding.

Explanations: C^r imbeddings are defined in [13, p. 76]; K', L' denote subdivisions of the simplicial complexes K, L; approximation is in the strong C^1 topology [13, p. 78]; for intersection in a full subcomplex see [13, p. 95].
We add

COMPLEMENT

The theorem remains true if M has a boundary. Also, suppose N^n is a C^r, properly imbedded n-submanifold of M that meets BdM in BdN, transversely. Then f' can be chosen so that, when a simplex of K is mapped by f into BdM, respectively into N, it will also be mapped there by f'. A parallel statement holds for g'.

The complement is proved by approximating f and g using only C^r co-ordinate charts (U, h) on M^m such that $h: U \to R^m$ maps U into $R^m_+ = \{\vec{x} \in R_m;\ x^m \geq 0\}$, $U \cap BdM$ into $R^{m-1} = \{\vec{x} \in R^m;\ x_m = 0\}$, and $U \cap N^n$ into $R^n_0 = \{\vec{x} \in R^m;\ x_1 = \cdots = x_{m-n} = 0\}$, then observing that the necessary extension holds for the basic local approximation lemmas [13, §§ 9.7, 9.8], cf. [13, Exercise (b), p. 101]. Roughly stated, all the little adjustments to f and g, as specified in local charts by these lemmas, will never move a simplex out of R^m_+, R^{m-1} or R^n_0, hence will yield maps to M respecting BdM and N as the complement asserts.
Remark: In a similar way one can treat manifolds with corners.

REFERENCES

[1] D. BARDEN, *The structure of manifolds*, Thesis, Cambridge U. (1963).
[2] R. H. BING, *Locally tame sets are tame*, Ann. of Math. *59* (1954), 145–158.
[3] T. HOMMA, *On Hauptvermutung and triangulation of n-manifolds*, Yokohama Math. J. (2) *12* (1963), 51–56.
[4] J. F. P. HUDSON and E. C. ZEEMAN, *On regular neighborhoods*, Proc. London Math. Soc. (3) *14* (1964), 719–745.
[5] J. F. P. HUDSON and E. C. ZEEMAN, *On combinatorial isotopy*, Publications Mathématiques, No. 19, Institut des Hautes Etudes Scientifiques (1964).
[6] M. KERVAIRE, *On higher dimensional knots*, Differential and Combinatorial Topology, Princeton U. Press, Princeton N.J., U.S.A. (1965), pp. 105–119.

[7] B. MAZUR, *Relative neighborhoods and the theorems of Smale*, Ann. of Math. *77* (1963), 232–249.

[8] B. MAZUR, *Combinatorial equivalence versus topological equivalence*, Trans. Amer. Math. Soc. *111* (1964), 288–316.

[9] J. W. MILNOR, *Two complexes which are homeomorphic but combinatorially distinct*, Ann. of Math., *74* (1961), 575–590.

[10] J. MILNOR, *Whitehead Torsion*, mimeographed notes, Princeton University (October, 1964), to appear in a revised form in Bull. Amer. Math. Soc. 72 (1966), 358–426.

[11] J. MILNOR, *Lectures on the h-cobordism theorem*, notes by L. Siebenmann and J. Sondow, Princeton Mathematical Notes, Princeton N.J., U.S.A. (1965).

[12] E. MOISE, *Affine Structure in 3-manifolds VIII*, Ann. of Math. *59* (1954), 159–170.

[13] J. MUNKRES, *Elementary differential topology*, Annals of Math. Study 54, Princeton University Press, Princeton N.J., U.S.A. 1963.

[14] J. SONDOW, Notices Amer. Math. Soc. (5) *12*, p. 561, no. 625–58.

[15] J. SONDOW, *Disproof of the Hauptvermutung for manifold-pairs*, Thesis, Princeton University Princeton N.J., U.S.A. 1965.

[16] J. STALLINGS, *On infinite processes leading to differentiability in the complement of a point*, Differential and Combinatorial Topology, Princeton U. Press, Princeton, N.J., U.S.A. (1965) pp. 245–254.

[17] C. T. C. WALL, *Differential Topology, Part IV, Theory of handle decomposition*, mimeographed, Cambridge U. (1964).

[18] J. H. C. WHITEHEAD, *On C^1-complexes*, Ann. of Math. 41 (1940), 809–824.

[19] E. C. ZEEMAN, *Twisting spun knots*, Trans. Amer. Math. Soc. (3) *115* (1965), 471–495.

[20] E. C. ZEEMAN, *Seminar on Combinatorial Topology*, Institut des Hautes Études Scientifiques (1963) (mimeographed notes).

[21] M. KERVAIRE, *Le théorème de Barden–Mazur–Stallings*, Comment. Math. Helv., 40 (1965), 31–42.

Mathematical Institute of Oxford University
University of Paris

Received March 15, 1966