

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 41 (1966-1967)

Artikel: Some results on functions holomorphic in the unit disk.
Autor: Lappan, Peter
DOI: <https://doi.org/10.5169/seals-31379>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 12.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Some results on functions holomorphic in the unit disk

By PETER LAPPAN, Michigan State University

1. Let D and C denote the unit disk and unit circle, respectively, let f denote a complex-valued function defined in D , and let W denote the extended complex plane. By a path γ in D we mean the image of the interval $0 \leq x < 1$ under a continuous function g . A path γ is called an *asymptotic path* if (1) $|g(t)| \rightarrow 1$ as $t \rightarrow 1$ and (2) there exists a number $w \in W$ such that $f(g(t)) \rightarrow w$ as $t \rightarrow 1$. The number w is called the *asymptotic value* for the asymptotic path γ . If γ is a path, the set $C \cap \bar{\gamma}$ is called the end of γ , and we say that γ ends in $C \cap \bar{\gamma}$. It is clear that the end of an asymptotic path must be either a point or an arc of C . A path γ is called a *point asymptotic path* if γ is an asymptotic path which ends in a single point.

The following sets will be considered:

- (i) $C_\gamma(f, \zeta)$, where ζ is a point of C and γ is a path which ends in ζ , denotes the set $\{w \in W: \text{there exists a sequence of points } \{z_n\} \text{ in } \gamma \text{ such that } |z_n| \rightarrow 1 \text{ and } f(z_n) \rightarrow w\}$;
- (ii) $\prod(f, \zeta)$, where ζ is a point of C , denotes the intersection $\bigcap_\gamma C_\gamma(f, \zeta)$, where the intersection is taken over all paths γ which end at ζ ;
- (iii) $\prod_\infty(f) = \{\zeta \in C: \infty \in \prod(f, \zeta)\}$;
- (iv) $\Gamma(f) = \{w \in W: \text{there exists an asymptotic path } \gamma \text{ for which the corresponding asymptotic value is } w\}$;
- (v) $A(f) = \{\zeta \in C: \text{there exists an asymptotic path } \gamma \text{ for which the end contains the point } \zeta\}$;
- (vi) $A_P(f) = \{\zeta \in C: \text{there exists a point asymptotic path for which the end is } \zeta\}$.

The sets $\Gamma(f)$, $A(f)$, and $A_P(f)$ have been studied by many persons, with two of the more complete treatments being given by Collingwood and Cartwright [3] and MacLane [5]. The main focus of this paper will be results concerning $\prod_\infty(f)$. We first prove that if f is a holomorphic function then $\prod_\infty(f) - \overline{A(f)}$ is an open subset of C . Next, it is shown that if f is a continuous function in the extended sense, then $\prod_\infty(f)$ is a measurable subset of C . Finally, it is proved that if f is a normal holomorphic function then $\prod_\infty(f)$ is nowhere dense in C . We conclude with some unsolved questions relating to $\prod_\infty(f)$.

2. We begin by proving a lemma.

LEMMA. *Let f be a function holomorphic in D , let β be a subarc of C with endpoints at ζ_1 and ζ_2 , and let $\zeta_1 \notin \prod_\infty(f)$ and $\zeta_2 \notin \prod_\infty(f)$. Then there exists an asymptotic path for which the end is a subset of β .*

Proof. Since $\zeta_1 \notin \prod_\infty(f)$ and $\zeta_2 \notin \prod_\infty(f)$, there exist paths γ_1 and γ_2 leading from

0 to ζ_1 and ζ_2 , respectively, such that $\gamma_1 \cap \gamma_2 = \{0\}$ and f is bounded on $\gamma_1 \cup \gamma_2$. Let H be the region bounded by $\gamma_1 \cup \gamma_2 \cup \beta$. If f is bounded on H then, by Fatou's Theorem [6, p. 5], f has point asymptotic paths to almost every point of β . If f is unbounded in H , then there exists a point $z_0 \in H$ such that $|f(z_0)|$ is greater than the bound of $|f(z)|$ on $\gamma_1 \cup \gamma_2$. Let L be the ray described by $\{w: w = t f(z_0), t \geq 1\}$. The component of $f^{-1}(L)$ which contains z_0 is an asymptotic path in H with its end contained in β (since $f^{-1}(L) \cap (\gamma_1 \cup \gamma_2) = \emptyset$). Thus the lemma is proved.

THEOREM 1. *Let f be a function holomorphic in D . Then $\prod_{\infty}(f) - \overline{A(f)}$ is an open subset of C .*

Proof. Let $\zeta \in \prod_{\infty}(f) - \overline{A(f)}$. Then there exists a neighborhood N of ζ such that $N \cap C \cap A(f) = \emptyset$. Suppose that ζ_1 and ζ_2 are two points of $N \cap C$ such that $\zeta_1 \notin \prod_{\infty}(f)$ and $\zeta_2 \notin \prod_{\infty}(f)$. If β is the subarc of $N \cap C$ with ζ_1 and ζ_2 as endpoints, then $A(f) \cap \beta \neq \emptyset$ according to the Lemma. But this would violate the condition that $N \cap C \cap A(f) = \emptyset$. Thus $N \cap C$ may contain at most one point which is not in $\prod_{\infty}(f)$. But this means that ζ is an interior point of $\prod_{\infty}(f)$, and the theorem is proved.

THEOREM 2. *Let f be a function holomorphic in D . Then the complement of $\prod_{\infty}(f) \cup \overline{A(f)}$ in C is a finite (or empty) set.*

Proof. Let E be the complement of $\prod_{\infty}(f) \cup \overline{A(f)}$ in C . Then $C = \prod_{\infty}(f) \cup \overline{A(f)} \cup E$. If $\zeta \in E$, there exists a neighborhood N of ζ such that $N \cap C \cap A(f) = \emptyset$. Suppose there exists a point $\zeta' \in E \cap N \cap C$, $\zeta' \neq \zeta$. Then by the Lemma we have $A(f) \cap N \cap C \neq \emptyset$, in violation of the choice of N . Thus ζ must be an isolated point of E . Therefore, each point of E must be an isolated point, and E is a finite set.

THEOREM 3. *If f is a holomorphic function in D , then $\prod_{\infty}(f) \cup A(f)$ is a dense subset of C .*

Theorem 3 is an immediate consequence of Theorem 2.

We note that Theorem 3 need not be true when f is a meromorphic function in D , as is illustrated by the Schwarz triangle functions, for which both $\prod_{\infty}(f)$ and $A(f)$ are empty.

3. We now let f be a continuous complex-valued function in the extended sense.

THEOREM 4. *If f is a continuous function in D , then $\prod_{\infty}(f)$ is a measurable set.*

Proof. We will show that $C - \prod_{\infty}(f)$ is a measurable set.

If $\zeta \in C - \prod_{\infty}(f)$, there exists an integer n such that ζ is an accessible boundary point of $A(n) = \{z \in D: |f(z)| < n\}$. For each n , $A(n)$ has a finite or a countable number of components $\{A(n, i): i = 1, 2, 3, \dots\}$.

Let $B(n) = \{z \in D: |f(z)| > n\}$ and let $\{B(n, j): j = 1, 2, 3, \dots\}$ be the components of $B(n)$. Let $E(n, i, j)$ be the set of points of $C \cap \overline{B(n, j)}$ which are accessible from within

$A(n, i)$. Let $K(n, j)$ be the set of all points in $C - \overline{B(n, j)}$. By a result of Kaczynski [4, Lemma 1, p. 590], $E(n, i, j)$ contains at most two points, so that $E(n) = \bigcup_{i,j} E(n, i, j)$ is a countable subset of C . But $K(n, j)$ is an open subset of C , and $K(n) = \bigcap_j K(n, j)$ is a G_δ set. Then $E(n) \cup K(n)$ is a measurable set. But

$$C - \prod_{\infty}(f) = \bigcup_n [E(n) \cup K(n)]$$

and thus $C - \prod_{\infty}(f)$ is a measurable set, and therefore $\prod_{\infty}(f)$ is also a measurable subset of C .

We have already noted that $\prod_{\infty}(f)$ may be empty and thus have measure zero, where f is a meromorphic function. Likewise, f may be holomorphic and $\prod_{\infty}(f)$ may equal C , as in the case of annular functions in the sense of Bagemihl and Erdős [1].

4. We now consider the case where f is a normal holomorphic function.

THEOREM 5. *If f is a normal holomorphic function in D , then $\prod_{\infty}(f)$ is nowhere dense in C .*

Proof. Suppose there exists an arc α of C such that $\alpha \subset \prod_{\infty}(f)$. Let β be a subarc of α with endpoints ζ_1 and ζ_2 in the interior of α . Let S_1 and S_2 be the radii to ζ_1 and ζ_2 , respectively, and let H be the sector of D bounded by $S_1 \cup S_2 \cup \beta$. For each n , let $D_n = \{z \in \bar{H} \cap D : |f(z)| < |f(0)| + n\}$, and let F_n be the component of D_n which contains 0. Since $\beta \subset \prod_{\infty}(f)$, we must have $F_n \cap C = \emptyset$ for each n . However, $H \subset \bigcup_n F_n$. For each n , the boundary of F_n contains a component which meets both S_1 and S_2 . Thus for each n there exists a Jordan arc J_n leading from a point on S_1 to a point on S_2 such that $|f(z)| > n - 1$ for $z \in J_n$, and $J_n \subset F_n$. The sequence $\{J_n\}$ forms a Koebe sequence of arcs relative to β such that $f(z) \rightarrow \infty$ along $\{J_n\}$. By a result of Bagemihl and Seidel [2, Theorem 1, p. 10], f must be identically ∞ and hence not holomorphic. Thus the theorem is proved.

We remark that Theorem 5 remains true when f is a normal meromorphic function. To prove this, we need only to modify the proof above by choosing S_1 and S_2 which do not contain poles of f , and by showing that the sequence $\{J_n\}$ does not allow limit points in D . For if $\{J_n\}$ had a limit point in D , then $\{J_n\}$ would have uncountably many such limit points in D , and f would be identically ∞ .

We further remark that Theorem 5 is valid if f is assumed to be of bounded characteristic, but not necessarily normal. However, an example of MacLane [5, Example 3, p. 57] shows that Theorem 5 may fail if the assumption that f is normal is removed, even though $A_P(f)$ may be dense in C .

5. The following questions concerning $\prod_{\infty}(f)$ are still unanswered.

QUESTION 1. *If f is a normal holomorphic function in D , can $\prod_{\infty}(f)$ have positive measure in C ?*

QUESTION 2. *If f is a holomorphic or meromorphic function in D which is the sum of two normal functions, must $\prod_{\infty}(f)$ be nowhere dense in C ?*

REFERENCES

- [1] F. BAGEMIHLE and P. ERDÖS, *A problem concerning the zeros of a certain kind of holomorphic function in the unit disk*, J. reine angew. Math. 214/215 (1964), 340–344.
- [2] F. BAGEMIHLE and W. SEIDEL, *Koebe arcs and Fatou points of normal functions*, Comment. Math. Helvetici 36 (1961), 9–18.
- [3] E. F. COLLINGWOOD and M. L. CARTWRIGHT, *Boundary theorems for a function meromorphic in the unit circle*, Acta Math. 87 (1952), 83–146.
- [4] T. J. KACZYNSKI, *Boundary functions for functions defined in a disk*, J. Math. and Mechanics 14 (1965), 589–612.
- [5] G. R. MACLANE, *Asymptotic values of holomorphic functions*, Rice University Studies, Vol. 49, No. 1, Winter 1963.
- [6] K. NOSHIRO, *Cluster sets*, Berlin–Göttingen–Heidelberg, 1960.

Michigan State University

Received February 16, 1966