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## Some results on functions holomorphic in the unit disk

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1. Let  $D$  and  $C$  denote the unit disk and unit circle, respectively, let  $f$  denote a complex-valued function defined in  $D$ , and let  $W$  denote the extended complex plane. By a path  $\gamma$  in  $D$  we mean the image of the interval  $0 \leq x < 1$  under a continuous function  $g$ . A path  $\gamma$  is called an *asymptotic path* if (1)  $|g(t)| \rightarrow 1$  as  $t \rightarrow 1$  and (2) there exists a number  $w \in W$  such that  $f(g(t)) \rightarrow w$  as  $t \rightarrow 1$ . The number  $w$  is called the *asymptotic value* for the asymptotic path  $\gamma$ . If  $\gamma$  is a path, the set  $C \cap \bar{\gamma}$  is called the end of  $\gamma$ , and we say that  $\gamma$  ends in  $C \cap \bar{\gamma}$ . It is clear that the end of an asymptotic path must be either a point or an arc of  $C$ . A path  $\gamma$  is called a *point asymptotic path* if  $\gamma$  is an asymptotic path which ends in a single point.

The following sets will be considered:

- (i)  $C_\gamma(f, \zeta)$ , where  $\zeta$  is a point of  $C$  and  $\gamma$  is a path which ends in  $\zeta$ , denotes the set  $\{w \in W : \text{there exists a sequence of points } \{z_n\} \text{ in } \gamma \text{ such that } |z_n| \rightarrow 1 \text{ and } f(z_n) \rightarrow w\}$ ;
- (ii)  $\prod(f, \zeta)$ , where  $\zeta$  is a point of  $C$ , denotes the intersection  $\bigcap_\gamma C_\gamma(f, \zeta)$ , where the intersection is taken over all paths  $\gamma$  which end at  $\zeta$ ;
- (iii)  $\prod_\infty(f) = \{\zeta \in C : \infty \in \prod(f, \zeta)\}$ ;
- (iv)  $\Gamma(f) = \{w \in W : \text{there exists an asymptotic path } \gamma \text{ for which the corresponding asymptotic value is } w\}$ ;
- (v)  $A(f) = \{\zeta \in C : \text{there exists an asymptotic path } \gamma \text{ for which the end contains the point } \zeta\}$ ;
- (vi)  $A_p(f) = \{\zeta \in C : \text{there exists a point asymptotic path for which the end is } \zeta\}$ .

The sets  $\Gamma(f)$ ,  $A(f)$ , and  $A_p(f)$  have been studied by many persons, with two of the more complete treatments being given by Collingwood and Cartwright [3] and MacLane [5]. The main focus of this paper will be results concerning  $\prod_\infty(f)$ . We first prove that if  $f$  is a holomorphic function then  $\prod_\infty(f) - \overline{A(f)}$  is an open subset of  $C$ . Next, it is shown that if  $f$  is a continuous function in the extended sense, then  $\prod_\infty(f)$  is a measurable subset of  $C$ . Finally, it is proved that if  $f$  is a normal holomorphic function then  $\prod_\infty(f)$  is nowhere dense in  $C$ . We conclude with some unsolved questions relating to  $\prod_\infty(f)$ .

2. We begin by proving a lemma.

**LEMMA.** *Let  $f$  be a function holomorphic in  $D$ , let  $\beta$  be a subarc of  $C$  with endpoints at  $\zeta_1$  and  $\zeta_2$ , and let  $\zeta_1 \notin \prod_\infty(f)$  and  $\zeta_2 \notin \prod_\infty(f)$ . Then there exists an asymptotic path for which the end is a subset of  $\beta$ .*

*Proof.* Since  $\zeta_1 \notin \prod_\infty(f)$  and  $\zeta_2 \notin \prod_\infty(f)$ , there exist paths  $\gamma_1$  and  $\gamma_2$  leading from

0 to  $\zeta_1$  and  $\zeta_2$ , respectively, such that  $\gamma_1 \cap \gamma_2 = \{0\}$  and  $f$  is bounded on  $\gamma_1 \cup \gamma_2$ . Let  $H$  be the region bounded by  $\gamma_1 \cup \gamma_2 \cup \beta$ . If  $f$  is bounded on  $H$  then, by Fatou's Theorem [6, p. 5],  $f$  has point asymptotic paths to almost every point of  $\beta$ . If  $f$  is unbounded in  $H$ , then there exists a point  $z_0 \in H$  such that  $|f(z_0)|$  is greater than the bound of  $|f(z)|$  on  $\gamma_1 \cup \gamma_2$ . Let  $L$  be the ray described by  $\{w: w = t f(z_0), t \geq 1\}$ . The component of  $f^{-1}(L)$  which contains  $z_0$  is an asymptotic path in  $H$  with its end contained in  $\beta$  (since  $f^{-1}(L) \cap (\gamma_1 \cup \gamma_2) = \emptyset$ ). Thus the lemma is proved.

**THEOREM 1.** *Let  $f$  be a function holomorphic in  $D$ . Then  $\prod_{\infty}(f) - \overline{A(f)}$  is an open subset of  $C$ .*

*Proof.* Let  $\zeta \in \prod_{\infty}(f) - \overline{A(f)}$ . Then there exists a neighborhood  $N$  of  $\zeta$  such that  $N \cap C \cap A(f) = \emptyset$ . Suppose that  $\zeta_1$  and  $\zeta_2$  are two points of  $N \cap C$  such that  $\zeta_1 \notin \prod_{\infty}(f)$  and  $\zeta_2 \notin \prod_{\infty}(f)$ . If  $\beta$  is the subarc of  $N \cap C$  with  $\zeta_1$  and  $\zeta_2$  as endpoints, then  $A(f) \cap \beta \neq \emptyset$  according to the Lemma. But this would violate the condition that  $N \cap C \cap A(f) = \emptyset$ . Thus  $N \cap C$  may contain at most one point which is not in  $\prod_{\infty}(f)$ . But this means that  $\zeta$  is an interior point of  $\prod_{\infty}(f)$ , and the theorem is proved.

**THEOREM 2.** *Let  $f$  be a function holomorphic in  $D$ . Then the complement of  $\prod_{\infty}(f) \cup \overline{A(f)}$  in  $C$  is a finite (or empty) set.*

*Proof.* Let  $E$  be the complement of  $\prod_{\infty}(f) \cup \overline{A(f)}$  in  $C$ . Then  $C = \prod_{\infty}(f) \cup \overline{A(f)} \cup E$ . If  $\zeta \in E$ , there exists a neighborhood  $N$  of  $\zeta$  such that  $N \cap C \cap A(f) = \emptyset$ . Suppose there exists a point  $\zeta' \in E \cap N \cap C$ ,  $\zeta' \neq \zeta$ . Then by the Lemma we have  $A(f) \cap N \cap C \neq \emptyset$ , in violation of the choice of  $N$ . Thus  $\zeta$  must be an isolated point of  $E$ . Therefore, each point of  $E$  must be an isolated point, and  $E$  is a finite set.

**THEOREM 3.** *If  $f$  is a holomorphic function in  $D$ , then  $\prod_{\infty}(f) \cup A(f)$  is a dense subset of  $C$ .*

Theorem 3 is an immediate consequence of Theorem 2.

We note that Theorem 3 need not be true when  $f$  is a meromorphic function in  $D$ , as is illustrated by the Schwarz triangle functions, for which both  $\prod_{\infty}(f)$  and  $A(f)$  are empty.

3. We now let  $f$  be a continuous complex-valued function in the extended sense.

**THEOREM 4.** *If  $f$  is a continuous function in  $D$ , then  $\prod_{\infty}(f)$  is a measurable set.*

*Proof.* We will show that  $C - \prod_{\infty}(f)$  is a measurable set.

If  $\zeta \in C - \prod_{\infty}(f)$ , there exists an integer  $n$  such that  $\zeta$  is an accessible boundary point of  $A(n) = \{z \in D: |f(z)| < n\}$ . For each  $n$ ,  $A(n)$  has a finite or a countable number of components  $\{A(n, i): i = 1, 2, 3, \dots\}$ .

Let  $B(n) = \{z \in D: |f(z)| > n\}$  and let  $\{B(n, j): j = 1, 2, 3, \dots\}$  be the components of  $B(n)$ . Let  $E(n, i, j)$  be the set of points of  $C \cap B(n, j)$  which are accessible from within

$A(n, i)$ . Let  $K(n, j)$  be the set of all points in  $C - \overline{B(n, j)}$ . By a result of Kaczynski [4, Lemma 1, p. 590],  $E(n, i, j)$  contains at most two points, so that  $E(n) = \bigcup_{i, j} E(n, i, j)$  is a countable subset of  $C$ . But  $K(n, j)$  is an open subset of  $C$ , and  $K(n) = \bigcap_j K(n, j)$  is a  $G_\delta$  set. Then  $E(n) \cup K(n)$  is a measurable set. But

$$C - \prod_{\infty}(f) = \bigcup_n [E(n) \cup K(n)]$$

and thus  $C - \prod_{\infty}(f)$  is a measurable set, and therefore  $\prod_{\infty}(f)$  is also a measurable subset of  $C$ .

We have already noted that  $\prod_{\infty}(f)$  may be empty and thus have measure zero, where  $f$  is a meromorphic function. Likewise,  $f$  may be holomorphic and  $\prod_{\infty}(f)$  may equal  $C$ , as in the case of annular functions in the sense of Bagemihl and Erdős [1].

4. We now consider the case where  $f$  is a normal holomorphic function.

**THEOREM 5.** *If  $f$  is a normal holomorphic function in  $D$ , then  $\prod_{\infty}(f)$  is nowhere dense in  $C$ .*

*Proof.* Suppose there exists an arc  $\alpha$  of  $C$  such that  $\alpha \subset \prod_{\infty}(f)$ . Let  $\beta$  be a subarc of  $\alpha$  with endpoints  $\zeta_1$  and  $\zeta_2$  in the interior of  $\alpha$ . Let  $S_1$  and  $S_2$  be the radii to  $\zeta_1$  and  $\zeta_2$ , respectively, and let  $H$  be the sector of  $D$  bounded by  $S_1 \cup S_2 \cup \beta$ . For each  $n$ , let  $D_n = \{z \in \bar{H} \cap D : |f(z)| < |f(0)| + n\}$ , and let  $F_n$  be the component of  $D_n$  which contains 0. Since  $\beta \subset \prod_{\infty}(f)$ , we must have  $F_n \cap C = \emptyset$  for each  $n$ . However,  $H \subset \bigcup_n F_n$ . For each  $n$ , the boundary of  $F_n$  contains a component which meets both  $S_1$  and  $S_2$ . Thus for each  $n$  there exists a Jordan arc  $J_n$  leading from a point on  $S_1$  to a point on  $S_2$  such that  $|f(z)| > n - 1$  for  $z \in J_n$ , and  $J_n \subset F_n$ . The sequence  $\{J_n\}$  forms a Koebe sequence of arcs relative to  $\beta$  such that  $f(z) \rightarrow \infty$  along  $\{J_n\}$ . By a result of Bagemihl and Seidel [2, Theorem 1, p. 10],  $f$  must be identically  $\infty$  and hence not holomorphic. Thus the theorem is proved.

We remark that Theorem 5 remains true when  $f$  is a normal meromorphic function. To prove this, we need only to modify the proof above by choosing  $S_1$  and  $S_2$  which do not contain poles of  $f$ , and by showing that the sequence  $\{J_n\}$  does not allow limit points in  $D$ . For if  $\{J_n\}$  had a limit point in  $D$ , then  $\{J_n\}$  would have uncountably many such limit points in  $D$ , and  $f$  would be identically  $\infty$ .

We further remark that Theorem 5 is valid if  $f$  is assumed to be of bounded characteristic, but not necessarily normal. However, an example of MacLane [5, Example 3, p. 57] shows that Theorem 5 may fail if the assumption that  $f$  is normal is removed, even though  $A_p(f)$  may be dense in  $C$ .

5. The following questions concerning  $\prod_{\infty}(f)$  are still unanswered.

**QUESTION 1.** *If  $f$  is a normal holomorphic function in  $D$ , can  $\prod_{\infty}(f)$  have positive measure in  $C$ ?*

QUESTION 2. *If  $f$  is a holomorphic or meromorphic function in  $D$  which is the sum of two normal functions, must  $\prod_{\infty}(f)$  be nowhere dense in  $C$ ?*

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