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# Actions of $R^n$ on manifolds

by HAROLD ROSENBERG

We shall be concerned with smooth manifolds  $V^n$ , compact and without boundary, and actions of  $R^{n-1}$  on  $V$  all of whose orbits are  $n-1$  dimensional. The rank of  $V$  is the largest  $k$  such that there is an action of  $R^k$  on  $V$  with  $k$  dimensional orbits; this is the same as the maximal number of linearly independent vector fields on  $V$  which pairwise commute. Elon Lima has proved the rank of  $S^3$  is one [1], and the author proved the rank of  $S^2 \times S^1$  is one [4]. One of our results is a generalization of Lima's theorem: the rank of a simply connected closed  $n$  manifold is less than  $n-1$ . Unfortunately, the author knows of no  $n$ -dimensional sphere whose rank is greater than one.

We also consider  $M \times S^1$  where  $M$  is a closed two-dimensional manifold of genus greater than one. Our results are not complete; we do not know the rank of this space. We do prove, however, that if there is a locally free action of  $R^2$  on  $M \times S^1$ , then it must have a torus orbit, embedded in a nontrivial way.<sup>1)</sup>

## Definitions and Notation

An action  $\Phi$  of a Lie Group  $G$  on  $V$  is a differentiable map  $\Phi: G \times V \rightarrow V$  such that (i)  $\Phi(gh, x) = \Phi(g, \Phi(h, x))$  for all  $g, h \in G$  and  $x \in V$ , and (ii)  $\Phi(e, x) = x$  for  $x \in V$ ,  $e$  the identity of  $G$ . Given  $x \in V$ , the isotropy subgroup of  $x$  is  $H_x = \{g \in G / \Phi_g(x) = x\}$ , it is a closed subgroup of  $V$ . The orbit or leaf of  $x$  is  $\{\Phi_g(x) / g \in G\}$ . The action  $\Phi$  induces a  $1-1$  continuous map of  $G/H_x$  onto  $L_x$ , the orbit of  $x$ .

If  $X_1, \dots, X_k$  are vector fields on  $V$ , we say they pairwise *commute* if  $[X_i, X_j] \equiv 0$  for all  $i$  and  $j$ . Let  $V$  be a closed manifold and  $\xi^1, \dots, \xi^k$  the integral curves of  $X_1, \dots, X_k$  respectively. We know  $[X_i, X_j] \equiv 0$  is equivalent to  $\xi_s^i \xi_t^j = \xi_t^j \xi_s^i$  for all real numbers  $s$  and  $t$ .

When  $G = R^K$ , an action of  $G$  on  $V$  is equivalent to  $K$  commuting vector field (we assume  $V$  is closed); the relation is

$$\Phi(t, x) = (\xi_{t_1}^1 \circ \xi_{t_2}^2 \circ \dots \circ \xi_{t_k}^k)(x), \quad t = (t_1, \dots, t_k) \in R^k.$$

We call  $\Phi$  a *locally free* action if all the orbits are  $K$ -dimensional.

Suppose  $n=3$  and  $k=2$ . The orbits of  $x$  are classified by their isotropy subgroups  $H_x$  and we have the following possibilities. If the dimension of  $H_x$  is two, then  $H_x = R^2$  and  $L_x = X$ . When  $H_x$  has dimension one we have  $H_x = L + nv$ ,  $L$  a line through the origin and  $v \in R^2$ ,  $n=0, \pm 1, \pm 2, \dots$ .  $L_x$  is then a line or circle (i.e.,  $1-1$  continuous

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<sup>1)</sup> Conversations with Elon Lima and André Haefliger were very useful in the preparation of this paper.

image of) depending on the direction of  $v$ . The case dimension  $H_x=0$  gives three possible orbits. When  $H_x=Zu$ ,  $Z$  the group of integers,  $u \in R^2$ , we have  $L_x=R^2$  or a cylinder depending on whether  $u=0$  or  $u \neq 0$ . If  $H_x=Zu+Zv$  with  $u$  and  $v$  independent, then  $H_x$  is a torus.

### 1. The Existence of Compact Leaves

**THEOREM 1.1.** (Reeb [2]). *Let  $V$  be a closed Riemannian manifold and  $\omega$  a closed one form on  $V$  satisfying  $\|\omega\|=1$ . Let  $F$  be the foliation of  $V$  defined by  $\omega=0$ . Then the leaves of  $F$  are homeomorphic and if  $L$  is one leaf, there is a covering map  $p: R \times L \rightarrow V$ .*

*Proof.* Since  $\|\omega\|=1$ , the foliation is oriented, and we may choose a unit vector field on  $V$  orthogonal to the foliation.

The orthogonal trajectories to a leaf  $F$  are geodesics [3]. Let  $\Psi_s(x)$  be a parametrization by arc length of the orthogonal trajectory through  $x$ . For each  $x$ , there is a neighborhood  $U$  of  $x$ , where we may define a smooth function  $s(y)$  by  $s(y)$  = the distance of the point  $y$  from the leaf containing  $x$ . Our assumptions imply  $\omega=ds$  locally.

If  $L$  is a leaf of  $F$  and  $s$  a real number,  $\omega$  vanishes on  $\Psi_s(L)$ . Thus  $\Psi_s$  carries leaves into leaves. The set  $\{\Psi_s(L) | s \in R\}$  is open and closed in  $V$ , hence all of  $V$ . This proves the first assertion.

Let  $x_0 \in V$ , and  $H$  be the subgroup of  $\pi_1(V, x_0)$  of homotopy classes representable by closed curves  $h$  at  $x_0$  such that

$$\int_h \omega = 0$$

Here we use the hypothesis  $d\omega=0$ .

Let  $W$  be the connected covering space of  $V$  over  $H$ . On  $W$  we have the one form  $\omega^*=p^*\omega$  and a foliation  $F_0$  defined by  $\omega^*=0$ .  $W$  inherits a Riemannian metric such that  $\|\omega^*\|=1$ ;  $\omega^*$  is never zero, and  $d\omega^*=0$ .

Let  $a$  be a closed curve in  $W$  based at some point in  $p^{-1}(x_0)$ . Since

$$\int_a \omega^* = \int_{pa} \omega$$

and  $pa$  represents an element of  $H$ , we have

$$\int_a \omega^* = 0.$$

It follows easily that the integral of  $\omega^*$  about any closed curve in  $W$  is zero. Thus  $\omega^*=df$  for some smooth function  $f$  on  $W$ . The level surfaces of  $f$  are precisely the leaves of  $F_0$ . Each orthogonal trajectory to  $F_0$  is an embedding of  $R$  in  $W$  and each

leaf meets an orthogonal trajectory in precisely one point. Hence  $W$  is homeomorphic to  $R \times L_0$ , where  $L_0 \in F_0$ , and for each  $t$ ,  $t \times L_0$  corresponds to a leaf of  $F_0$ .

We observe that  $L_0$  is homeomorphic to  $p(L_0) = L$ , a leaf of  $F$ . There is a map  $L \rightarrow L_0$  defined as follows: fix  $x_0 \in L$  and  $\bar{x} \in L_0$  such that  $px = x_0$ . For  $x \in L$ , let  $h$  be a path in  $L$  from  $x_0$  to  $x$ . Lift  $h$  to a path  $a$  in  $L_0$  starting at  $\bar{x}$ . We map  $L \rightarrow L_0$  by sending  $x$  to  $a(1)$ , the endpoint of  $a$ . This map does not depend on the path  $h$ , since closed paths in  $L$  lift to closed paths in  $L_0$ . Thus  $V$  may be covered by  $R \times L$ .

**THEOREM 1.2** (Sacksteder [5]). *Let  $\Phi$  be a locally free action of  $R^{n-1}$  on a closed  $n$  manifold  $V$ , such that no orbit is compact. There is a Riemannian metric on  $V$  and a closed non-vanishing one form  $\omega$  of norm one, such that the foliation defined by  $\omega = 0$  is the same as the foliation defined by  $\Phi$ . This foliation admits a simple closed curve as an orthogonal trajectory.*

**COROLLARY 1.3:** *Let  $V$  be a closed  $n$  manifold with non-Abelian fundamental group. Then each locally free action of  $R^{n-1}$  on  $V$  has a non-simply connected leaf.*

*Proof.* Suppose the orbits of  $\Phi$  are simply connected. Then theorems 1.1 and 1.2 imply  $V$  is covered by  $R^n$  and  $H = \{[a] \in \pi_1(V) \mid \int_a \omega = 0\}$  is isomorphic to  $\pi_1(R^n)$  hence trivial. But  $H$  contains the commutator subgroup of  $V$ , hence  $\pi_1(V)$  is abelian.

**COROLLARY 1.4:** *Let  $\Phi$  be a locally free action of  $R^2$  on  $M \times S^1$  where  $M$  is a closed 2-dimensional manifold of genus greater than one. Then  $\Phi$  has a compact orbit (a torus).*

*Proof.* Since  $\pi_1(M \times S^1)$  is not abelian we know all the orbits of  $\Phi$  cannot be  $R^2$ . If  $\Phi$  has no compact orbit, all of the orbits are the one to one continuous image of  $R \times S^1$ , and each orbit is dense in  $M \times S^1$ . Let  $X$  and  $Y$  be linearly independent commuting vector fields on  $M \times S^1$  such that  $X$  and  $Y$  span the orbits of  $\Phi$ . Let  $x_0 \in V = M \times S^1$ . The isotropy subgroup of  $R^2$  at  $x_0$  is a discrete group on one generator; hence, we may find real numbers  $a, b, c, d$  such that the vector fields  $X' = aX + bY$ ,  $Y' = cX + dY$  are linearly independent and the  $X'$  orbit through  $x_0$  is a simple closed curve  $\gamma$ . Let  $\xi_t$  and  $\eta_\tau$  be the integral curves of  $X'$  and  $Y'$ . Because  $X'$  and  $Y'$  commute, we have  $\xi_t \eta_\tau = \eta_\tau \xi_t$  for all  $t$  and  $\tau$ . Thus  $\eta_\tau(\gamma)$  is also a simple closed curve for all  $\tau$ . Since the  $\Phi$  orbit of  $x_0$  is dense in  $V$ , it follows from continuity that all the integral curves of  $X'$  are simple closed curves. Moreover, the foliation of  $V$  induced by  $\Phi$  may be assumed oriented which implies the integral curves of  $X'$  have the same period. Consider the quotient space  $Y$  of  $V$  obtained by identifying each integral curve of  $X'$  to a point.  $Y$  is a closed two-dimensional orientable manifold. By choosing a non-zero normal vector field to the orbits of  $\Phi$  we obtain a non-zero vector field on  $Y$ ; hence  $Y$  must be a two-dimensional torus. But this means  $M \times S^1$  is a circle bundle over a two torus which is easily seen to be a contradiction. Simply consider the homotopy exact sequence of this fibre bundle. Thus some orbit of  $\Phi$  is compact.



**THEOREM 1.4.** *Let  $\Phi$  be a locally free action of  $R^{n-1}$  on a closed  $n$  manifold  $V$  and assume  $\Phi$  has no compact orbits. There is a covering map  $p: R^{n-1} \times S^1 \rightarrow V$ .*

*Proof.* We may apply 1.2 to obtain a metric on  $V$  and closed non-vanishing one form  $\omega$  of norm one which defines the foliation induced by  $\Phi$ . Let  $j: I \rightarrow V$  be a parametrization by arc length of the closed orthogonal trajectory through  $x_0$ ; i.e.,  $j(0)=j(1)=x_0$ ,  $j(t_1) \neq j(t_2)$  if  $t_1 \neq t_2$ ,  $0 < t_1, t_2 < 1$  and  $j(I)$  is orthogonal to  $\Phi$ . It is no loss of generality to assume this orbit has length one.

Let  $L$  be the  $\Phi$  orbit of  $x_0$ . By 1.1 we know  $V$  is covered by  $R \times L$ . If  $L$  is not simply connected, then  $L = R^{n-i} \times T^{i-1}$  where  $T^{i-1}$  is the  $i-1$  dimensional torus and  $i > 1$ . In this case  $R \times L$  is covered by  $R^{n-1} \times S^1$ . So we may assume  $L$  is the one to one continuous image of  $R^{n-1}$  which implies each orbit of  $\Phi$  is of the same type. We state in [4] that these assumptions imply  $V$  is covered by  $R^{n-1} \times S^1$ . Since this was stated without proof, we give the proof here.

Let  $H$  be the subgroup of  $\pi_1(V, x_0)$  generated by the homotopy class of  $j$ . Let  $W$  be the connected covering space of  $V$  over  $H$  with covering map  $p$ . We will prove  $W$  is homeomorphic to  $R^{n-1} \times S^1$ .

We may think of  $W$  as the quotient space of the space of paths  $h: I \rightarrow V$  starting at  $x_0$  where  $h_1$  is identified with  $h_2$  if  $h_1(1)=h_2(1)$  and  $h_1 h_2^{-1}$  represents an element of  $H$ . Parametrize  $j$  by arc length so that the distance of  $j(t)$  to  $x_0$  is  $t$ .

Define a path  $h(\tau)$  at  $x_0$  by  $h(\tau)(t)=j(t\tau)$ ,  $0 \leq \tau \leq 1$ . Let  $U(\tau)=(h(\tau))$ =equivalence class of  $h(\tau)$  in  $W$ . We have  $U(0)=U(1)$  since  $h(1)=j$ ,  $h(0)=C_{x_0}$ =constant path at  $x_0$ , and  $h(1)h(0)^{-1}=j$  represents an element of  $H$ . Also  $U(\tau_1) \neq U(\tau_2)$  for  $\tau_1 \neq \tau_2$ ,  $0 < \tau_2, \tau_2 < 1$ , since  $h(\tau_1) \neq h(\tau_2)$ . Hence  $U$  is a simple closed curve in  $W$  such that  $pU=j$ .

Let  $\Phi_0$  be a lifting of the action  $\Phi$  to an action on  $W$ ; that is,  $p\Phi_0=\Phi(1 \times p)$ ,  $1$ =the identity map of  $R^{n-1}$ . The orbits of  $\Phi_0$  cover the orbits of  $\Phi$  hence they are also the one to one continuous image of  $R^{n-1}$ . To complete the proof we will show each orbit of  $\Phi_0$  intersects the image of  $U$  in precisely one point.

Suppose some orbits  $A$  of  $\Phi_0$  meets  $U$  in two points  $(h(\tau_1))$  and  $(h(\tau_2))$ . Let  $\mu: I \rightarrow A$  be a path joining  $(h(\tau_1))$  to  $(h(\tau_2))$ ;  $p\mu=\beta$  is a path from  $j(\tau_1)$  to  $j(\tau_2)$  contained in the orbit  $pA$ .

For  $0 \leq \tau \leq 1$ , define  $\eta(\tau): I \rightarrow V$  by

$$\eta(\tau)(t) = \begin{cases} j(2t\tau_1), & t \leq \frac{1}{2} \\ \beta(\tau(2t-1)), & t \geq \frac{1}{2} \end{cases}$$

Then  $\eta(0)=h(\tau_1) \circ C_{j(\tau_1)}$ ,  $\eta(1)=h(\tau_1) \circ \beta$  so that  $\eta h(\tau_1)^{-1}$  is homotopic to  $C_{x_0}$ . Let  $f$  be the path in  $W$ ,  $f(\tau)=(\eta(\tau))$ . We have  $pf(\tau)=\eta(\tau)(1)=\beta(\tau)$  and  $f(0)=(\eta(0))=(h(\tau_1))$ . Since  $p\mu=\beta$  and  $\mu(0)=(h(\tau_1))$ , we have  $\mu=f$ ; in particular  $\mu(1)=f(1)$ ,  $(h(\tau_2))=(\eta(1))=(h(\tau_1)\beta)$  so that  $h(\tau_1)\beta h(\tau_2)^{-1}$  represents an element of  $H$ . Hence

$$\int_W$$

$h(\tau_1)\beta h(\tau_2)^{-1}$

is an integer multiple of  $\int_j w$ . However,

$$\int_{h(\tau_1)\beta h(\tau_2)^{-1}} w = \int_{h(\tau_1)} w - \int_{h(\tau_2)} w + \int_{\beta} w = \tau_1 - \tau_2$$

i.e.,  $\int_{\beta} w = 0$  since  $\beta$  lies in one leaf. Consequently,  $\tau_1 = \tau_2$  or  $\tau_1 = 1, \tau_2 = 0$ . In any case  $(h(\tau_1)) = (h(\tau_2))$  and  $A$  meets  $U$  in at most one point.

Now we will show  $A$  meets  $U$  in at least one point. Let  $(h)$  be a point of  $A$ . We shall construct a map  $G: I \times I \rightarrow V$  satisfying:  $G(1, t) = h(t)$ ,  $G(0, t) = h(a)(t)$  for some real number  $a$ ,  $G(s, 0) = x_0$  and  $G(s, 1)$  is in the orbit through  $h(1)$  for  $0 \leq s \leq 1$ . The map  $s \rightarrow (G(s, \cdot))$  is then a path in  $A$  joining  $(h)$  to  $(h(a))$ ; where  $G(s, \cdot)$  means the map  $G(s, \cdot)(t) = G(s, t)$ . Since  $(ha)$  is a point of  $V$  this will complete the proof. Observe that a curve  $h$  in  $V$  is homotopic to a curve consisting of segments such that each segment is an arc of an orthogonal trajectory or is entirely contained in one leaf. Therefore we may assume there exists numbers  $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1$  such that for each  $i$ , the arc  $h[t_i, t_{i+1}]$  is either a segment of an orthogonal trajectory or is contained in one leaf.

Let  $L$  be a leaf of  $\Phi$  and  $x \in L$ ;  $C(t)$  a curve in  $L$  starting at  $x$ . The orthogonal trajectories are infinitely extendable, hence for any positive number  $s_0$ , the orthogonal trajectories of length  $s_0$  along  $C$  define a map  $F: I \times [0, s_0] \rightarrow V$  such that for fixed  $t$ ,  $F(t, s)$  is an orthogonal trajectory with  $F(t, 0) = C(t)$ , and  $F(t, s)$  is the point a distance  $s$  from  $C(t)$  along the orthogonal trajectory through  $C(t)$ . Moreover, the metric on  $V$  guarantees the points  $F(t, s)$ , for fixed  $s$ , are contained in the leaf through  $F(0, s)$ .

Now  $G$  is defined as follows. We may assume  $h[t_0, t_1]$  is contained in the leaf  $L$  through  $x_0$ , and  $h[t_1, t_2]$  is an orthogonal arc. Let  $C$  be the path  $h[t_0, t_1]$  and  $s_0$  the length of  $h[t_1, t_2]$ . Apply the last paragraph to obtain a map  $F_1: I \times [0, s_0] \rightarrow V$  such that  $F_1(0, s) = j(s)$ ,  $F_1(1, s) = h(t_1 + s)$  and  $F_1(t, s_0)$  is in the orbit through  $j(s_0)$  for  $0 \leq t \leq 1$ . Repeat this construction with  $C$  the curve  $F_1(t, s_0)$  followed by  $h[t_2, t_3]$ . Induction on  $k$  yields the desired map  $G$ . This completes the proof of 1.4.

**COROLLARY 1.5.** *Let  $V$  be a closed  $n$  manifold which cannot be covered by  $R^{n-1} \times S^1$ . Then a locally free action of  $R^{n-1}$  on  $V$  has a compact orbit.*

**LEMMA 1.6.** Let  $D = \{(x_1, x_2, 0, \dots, 0) \in R^n | x_1^2 + x_2^2 \leq 1\}$ ,  $\{e_1, \dots, e_{n-1}\}$  the  $n-1$  frame on  $\partial D$  defined as follows:  $e_1(x_1, x_2, 0, \dots, 0) = (-x_2, x_1, 0, \dots, 0)$ ,  $e_2 = (0, 0, 1, 0, \dots, 0)$ ,  $\dots$ ,  $e_{n-1} = (0, 0, \dots, 0, 1)$ . Then  $\{e_1, \dots, e_{n-1}\}$  does not extend to an  $n-1$  frame on  $D$ .

The frame  $\{e_1, \dots, e_{n-1}\}$  represents the nonzero element of  $\pi_1(SO(n))$ . This is proved in Chevalley's book on Lie Groups.

**THEOREM 1.7.** *Let  $V$  be a simply connected closed  $n$  manifold. The rank of  $V$  is less than  $n-1$ .*

*Proof.* The case  $n=3$  has been proved by Lima [1], and  $n=4$  is trivial since a simply connected 4 manifold does not admit a foliation of codimension one; it does not admit a nonzero vector field. So we assume  $n \geq 5$ .

Let  $\Phi$  be a locally free action of  $R^{n-1}$  on  $V$ . According to 1.5,  $\Phi$  has a torus orbit  $T$ . Since  $V$  is simply connected,  $i: T \subset V$ , induces the zero homomorphism. Thus there is a simple closed curve  $C$  on  $T$  which bounds an embedded two-dimensional disk  $D$  in  $V$  such that  $D$  is transverse to  $T$ , (here we use  $n \geq 5$ ). But this contradicts 1.6, (cf. [1]).

## 2. Locally Free Actions of $R^2$ on $M \times S^1$

(2.1) Let  $D$  be a two-dimensional disk with  $k$  contours in the interior of  $D$ . Let  $V = D \times I$  and  $S$  be an embedded sphere in  $V$ . Then  $S$  bounds an embedded ball.

*Proof.* For  $k=0$  this is Schoenflies Theorem. We consider the case  $k=1$ . Let  $C$  be an embedding of  $[0,1]$  in  $D$  with one endpoint on  $\partial D$ , the other on the contour, and interior  $C \subset \text{interior } D$ . If  $S \cap A \neq \emptyset$ ,  $A = C \times I$ , then we may cut  $V$  along  $A$  to obtain a 3 ball; this is the case  $k=0$ . Assume then, that  $S \cap A = \emptyset$  and the intersection is transverse. This is no loss of generality since  $S$  may be approximated by an embedded sphere which is transverse to  $A$  and then there is a diffeomorphism of  $V$  sending one sphere onto the other. Let  $a_1, \dots, a_k$  be the simple closed curves in  $S \cap A$ . Choose  $a_j$  so that  $a_j$  bounds a disk  $E$  on  $S$  and  $E$  contains no  $a_i$  in its interior.  $A$  is homeomorphic to  $I \times I$  so  $a_j$  bounds a disk  $F$  on  $A$ . Consider the sphere  $E \cup F$ . For our purposes this sphere is disjoint from  $A$ , i.e.,  $E \cup F$  bounds a ball  $B$  in  $V$ . Now by an isotopy of  $B$  across  $A$  we obtain a sphere  $S_0$  which intersects  $A$  in the curves  $a_1 \cup \dots \cup a_j \cup \dots \cup a_k$  (cf. [4] for details). Continuing we see  $S$  is isotopic to a sphere which does not intersect  $A$ , hence bounds a ball. The general case is just as easy.

Suppose there are  $k$  contours with  $k > 1$ . Let  $C$  be an embedding of  $I$  in  $D$  with both endpoints on distinct contours and interior  $C \subset \text{interior } D$ . If  $S \cap A = \emptyset$ ,  $A = C \times I$ , then by cutting  $V$  along  $A$  we reduce the problem to  $k-1$  contours. Otherwise we take the intersection to be transverse and displace  $S$  off  $A$  as above.

(2.2) Let  $M$  be a closed two-dimensional orientable manifold of connectivity  $h > 1$ . Let  $S$  be a sphere embedded in  $M \times S^1$ . Then  $S$  bounds an embedded ball in  $M \times S^1$ .

*Proof.* Let  $a_1, \dots, a_k$ ,  $k = (h+1)/2$  be simple closed curves on  $M$  as indicated in figure 1.

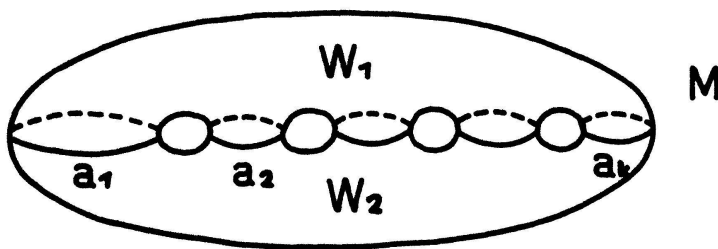


Fig. 1

Denote by  $A_i = a_i \times S^1$ , and  $A = A_1 \cup \dots \cup A_k$ .  $A$  separates  $V$  into two connected components  $E_1$  and  $E_2$ ;  $E_1 = W_1 \times S^1$ ,  $E_2 = W_2 \times S^1$ , where  $W_1, W_2$  are the connected components of  $M - (a_1 \cup \dots \cup a_k)$ .  $W_1$  and  $W_2$  are disks with  $k-1$  contours. We may think of  $M \times S^1$  as the quotient space of  $M \times I$  where  $(x, 0)$  is identified with  $(x, 1)$ , and we identify  $M$  with  $M \times 0 \in M \times S^1$ .

Suppose  $S$  is embedded in  $V$  so that  $S$  is disjoint from  $M$ . If  $S$  is also disjoint from  $A$  then  $S$  is contained in  $E_1$  or  $E_2$ . Assume  $S \subset E_1$ . We have  $E_1 = W_1 \times I$  where  $W_1 \times 0$  is identified with  $W_1 \times 1$ . Since  $S \cap M = \emptyset$ ,  $S$  is really contained in a subspace of  $V$  homeomorphic to  $W_1 \times I$  and by (2.1),  $S$  bounds a ball in this subspace, hence in  $V$ . Otherwise we may assume  $S$  meets  $A$  transversally. Let  $b$  be a simple closed curve in  $S \cap A$  such that  $b$  bounds a disk  $E$  on  $S$  whose interior is disjoint from  $A$ . Since  $S \cap M = \emptyset$ ,  $b$  bounds a disk  $F$  contained in  $a_i \times I$  for some  $i$ . Then  $F \cup E$  is a sphere contained in  $W_1 \times I$  or  $W_2 \times I$  hence  $F \cup E$  bounds a ball. Now by displacing  $E$  across this ball we see that  $S$  is isotopic to a sphere having one less circle of intersection with  $A$ . Continuing in this way, we obtain a sphere isotopic to  $S$  whose intersection with  $A \cap M$  is void hence this sphere bounds a ball and  $S$  also bounds a ball.

It remains to consider the case  $S \cap M \neq \emptyset$ . Let  $S$  meet  $M$  transversally, and  $b$  be a simple closed curve in  $S \cap M$  which bounds a disk  $E$  on  $S$  whose interior is disjoint from  $M$ . Since the inclusion of  $M$  in  $V$  induces a monomorphism of  $\pi_1(M)$  into  $\pi_1(V)$ ,  $b$  must be null homotopic on  $M$  hence  $b$  bounds a disk  $F$  on  $M$ . The sphere  $E \cup F$  is (for all practical purposes) disjoint from  $M$  hence bounds a ball in  $V$ . Then  $S$  may be displaced in  $V$  to a sphere having one less intersection curve with  $M$  and iterating the process removes  $S$  from  $M$  entirely. This completes the proof of 2.2.

(2.3) *Let  $T$  be a torus embedded in the interior of  $M \times I$  where  $M$  is a closed orientable two-dimensional manifold of genus greater than one. Then  $T$  separates  $M \times I$  into two connected components. Moreover  $M \times 0$  and  $M \times I$  are contained in the same connected component.*

*Proof.* Let  $i$  be the inclusion map of  $T$  into  $M \times I$ . The map  $i_*: H_2(T) \rightarrow H_2(M \times I)$  is zero since  $M \times I$  may be retracted onto  $M \times 0 = M$ , and  $M$  has genus greater than one so any map of  $T$  to  $M$  has degree zero. We must compute  $H_0(M \times I - T)$  (all homology and cohomology groups are with  $Z_2$  coefficients). By Lefschetz Duality  $H_0(M \times I - T)$  is isomorphic to  $H^3(M \times I; T)$ . Consider the exact sequence in cohomology:

$$H^2(M \times I) \rightarrow H^2(T) \rightarrow H^3(M \times I; T) \rightarrow H^3(M \times I) \rightarrow H^3(T)$$

The first map is zero since it is the transpose of  $i_*$  and the last group is zero. The second and fourth groups are  $Z_2$ , hence  $H^3(M \times I; T) = Z_2 + Z_2$ . This proves the first part of 2.3.

Now we will prove  $M \times 0$  and  $M \times 1$  are in the same component. Let  $a_1$  and  $a_2$  be simple closed curves on  $M \times 0$ , as in 2.2.

Let  $T$  intersect  $a_1 \times I$  and  $a_2 \times I$  transversally. If  $T$  is disjoint from  $a_1 \times I$  or  $a_2 \times I$  then we may find a curve from  $M \times 0$  to  $M \times 1$  not meeting  $T$ . Assume then that  $T \cap (a_1 \times I) = b_1 \cup \dots \cup b_k$ ,  $T \cap (a_2 \times I) = c_1 \cup \dots \cup c_l$ , where the  $b_i$ 's and  $c_j$ 's are pairwise disjoint simple closed curves.

If each  $b_i$ , or each  $c_j$ , is null homotopic in  $M \times I$ , then we can join  $a_1 \times 0$  to  $a_1 \times 1$  (or  $a_2 \times 0$  to  $a_2 \times 1$ ) by arcs in  $a_1 \times I - T$  (or  $a_2 \times I - T$ ). So we may suppose there is a  $b_i$  and  $c_j$  such that  $b_i$  and  $c_j$  are not homotopically trivial. Clearly  $b_i$  is homotopic to  $a_1$  and  $c_j$  to  $a_2$ . Now  $b_i$  and  $c_j$  are disjoint simple closed curves on the torus  $T$  and both represent generators of  $\pi_1(T)$ , hence  $b_i$  and  $c_j$  are the boundary circles of a cylinder on  $T$ . This implies  $a_1$  is homotopic to  $a_2$  in  $M$  which is a contradiction. Thus  $M \times 0$  and  $M \times 1$  are in the same connected component of  $M \times I - T$ .

(2.4) Let  $T$  be a torus embedded in  $M \times S^1$  where  $M$  is a closed orientable two manifold of genus greater than one. If  $T \cap (M \times x_0) = \emptyset$  for some  $x_0 \in S^1$ , then  $T$  separates  $M \times S^1$  into two connected components  $A$  and  $B$ . If  $h$  and  $g$  are the inclusion maps of  $T$  into  $A$  and  $B$  respectively, then  $h_*: \pi_1(T) \rightarrow \pi_1(A)$  or  $g_*: \pi_1(T) \rightarrow \pi_1(B)$  has a nonzero kernel.

It remains to establish the latter assertion of 2.4. First we need an algebraic fact whose proof may be found in Kurosh, volume two.

(2.5) Let  $G_1$ ,  $G_2$  and  $H$  be groups such that there are subgroups  $H_1$  and  $H_2$  of  $G_1$ ,  $G_2$  respectively each isomorphic to  $H$ . Denote by  $G_1 * H G_2$  the free product of  $G_1$  and  $G_2$  with  $H$  amalgamated. Every element of  $G_1 * H G_2$  can be written uniquely in the form

$$h \bar{a}_1 \bar{a}_2 \dots \bar{a}_n$$

where  $h \in H$ ,  $n \geq 0$ ,  $\bar{a}_i$  is a coset representative, other than the unit element, of a right coset of  $H_i$  in  $G_i$ ,  $i=1,2$ , and adjacent representatives  $\bar{a}_i, \bar{a}_{i+1}$ ,  $i=1, \dots, n-1$ , lie in distinct  $G_i$ 's.

From this it follows easily that the center of  $G_1 * H G_2$  is contained in  $H$ .

*Proof of 2.4.* Suppose that  $h_*$  and  $g_*$  are both monomorphisms. Let  $G_1 = \pi_1(A)$ ,  $H_1 = h_* \pi_1(T)$ ,  $G_2 = \pi_1(B)$ ,  $H_2 = g_* \pi_1(T)$  and  $H = \pi_1(T)$ . According to Van Kampen's Theorem and (2.3) we have

$$\pi_1(M \times S^1) = G_1 * H G_2.$$

Since  $\pi_1(S^1)$  is contained in the center of  $\pi_1(M \times S^1)$  and the center of  $G_1 * H G_2$  is contained in  $H$ , we have  $\pi_1(S^1)$  contained in  $\pi_1(T)$ . But  $T$  is disjoint from  $M \times x_0$  for some  $x_0 \in S^1$ , hence no curve on  $T$  can represent a generator of  $\pi_1(S^1)$ . Thus  $h_*$  or  $g_*$  is not a monomorphism.

(3.1) Let  $\Phi$  be a locally free action of  $R^2$  on  $M \times S^1$  with  $M$  a closed two manifold

of genus greater than one. Then  $\Phi$  has a compact orbit, and each compact orbit of  $\Phi$  intersects  $M$ .

This follows immediately from 1.4, 2.4 and [1].

(3.2) If  $T$  is a compact orbit of  $\Phi$ , then  $T \cap M$  contains a curve which is a generator of  $\pi_1(T)$ .

*Proof.* Assume  $T$  is transverse to  $M$  and each curve in  $T \cap M$  is trivial in  $\pi_1(T)$ . Let  $b$  be such a curve. Then  $b$  bounds a disk  $E$  on  $T$ , hence also bounds a disk  $F$  on  $M$  and the sphere  $E \cup F$  bounds a ball in  $M \times S^1$  by 2.2. Thus the intersection curve  $b$  may be removed from  $m$  by an isotopy of  $M \times S^1$  and all intersection curves may be so removed. This gives rise to a new action which is locally free and has a compact orbit disjoint from  $M$ . But 3.1. contradicts this.

#### REFERENCES

- [1] E. LIMA, *Commuting Vector Fields on  $S^3$* , Annals Math. 81 (1965), 70–81.
- [2] G. REEB, *Sur certaines propriétés topologiques des variétés feuilletées*, (Paris 1952).
- [3] B. REINHART, *Foliated manifolds with bundle-like-metrics*, Annals Math. 69 (1959), 119–131.
- [4] H. ROSENBERG, *The rank of  $S^2 \times S^1$* , American J. of Math. 87 (1965), 11–24.
- [5] R. SACKSTEDER, *Foliations and pseudogroups*, American J. of Math. 87 (1965), 79–102.

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