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# The spherical derivative of integral and meromorphic functions

by J. CLUNIE and W. K. HAYMAN

## 1. Introduction

In a recent paper LEHTO and VIRTANEN [2] introduced the spherical derivative

$$\varrho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

as a measure of the growth of  $f(z)$  near an isolated singularity. This point of view was further pursued by LEHTO [1]. If the singularity is taken to be at  $z = \infty$  then LEHTO obtained the following results.

**Theorem A.** *Suppose that  $f(z)$  is meromorphic for  $R < |z| < \infty$ , and has an essential singularity at  $z = \infty$ . Then*

$$\limsup_{z \rightarrow \infty} |z| \varrho(f(z)) \geq \frac{1}{2}. \quad (1.2)$$

*Equality holds for functions of the form*

$$f(z) = \prod_1^{\infty} \frac{a_v - z}{a_v + z}, \quad (1.3)$$

*where  $a_v$  is a sequence of complex numbers such that*

$$\left| \frac{a_{v+1}}{a_v} \right| \rightarrow \infty \quad (v \rightarrow \infty). \quad (1.4)$$

**Theorem B.** *If  $f(z)$  satisfies the hypotheses of Theorem A and in addition  $f(z)$  is regular near  $z = \infty$ , then (1.2) can be replaced by*

$$\limsup_{z \rightarrow \infty} |z| \varrho(f(z)) = \infty. \quad (1.5)$$

Following LEHTO, we denote by  $h(r)$  a positive function such that  $h(r) = o(r)$  ( $r \rightarrow \infty$ ). The connection between  $\varrho(f(z))$  and PICARD's Theorem is strikingly brought out by the following result of LEHTO [1].

**Theorem C.** Let  $f(z)$  be meromorphic for  $R < |z| < \infty$ . If for a sequence  $\{z_\nu\}$ ,  $\lim_{\nu \rightarrow \infty} z_\nu = \infty$  and

$$\lim_{\nu \rightarrow \infty} h(|z_\nu|) \varrho(f(z_\nu)) = \infty \quad (1.6)$$

then PICARD's Theorem holds for  $f(z)$  in the union of any infinite subsequence of the discs

$$C_\nu = \{z : |z - z_\nu| < \epsilon h(|z_\nu|)\} \quad (1.7)$$

for each  $\epsilon > 0$ .

Conversely if there exist discs (1.7) such that PICARD's Theorem is true in every union  $\bigcup_{k=1}^{\infty} C_{\nu_k}$  for every  $\epsilon > 0$  then (1.6) is satisfied. (V. GAVRILOV has pointed out to us that the converse must be modified here. (1.6) is satisfied for a sequence  $z'_\nu$  instead of  $z_\nu$ , where  $|z'_\nu - z_\nu| = o\{h(|z_\nu|)\}$ . This condition is also sufficient for the existence of the disks (1.7)).

In particular it follows that if  $f(z)$  has an essential singularity at  $z = \infty$  then  $f(z)$  possesses a JULIA direction provided that

$$\limsup_{z \rightarrow \infty} |z| \varrho(f(z)) = \infty. \quad (1.8)$$

From Theorem B we see that every transcendental integral function possesses a JULIA direction. If (1.8) is not satisfied there is not, in general, a JULIA direction as the examples (1.3) show if  $a_\nu > 0$ .

## 2. Some further results for meromorphic functions

Our aim in this paper is to obtain some extensions of Theorems A and B. We may suppose without loss of generality that  $f(z)$  is meromorphic in the whole plane. First we consider whether or not a restriction on the growth of  $f(z)$  as defined by its order imposes any restriction on  $\varrho(f(z))$ , or conversely. For meromorphic functions no restriction on  $\varrho(f(z))$  is implied by a restriction on the growth of the characteristic  $T(r, f)$ . Consider, for instance,

$$f(z) = \frac{\prod_{n=1}^{\infty} (1 - z/a_n)}{\prod_{n=1}^{\infty} (1 - z/b_n)}$$

where  $\sum |a_n|^{-1}$ ,  $\sum |b_n|^{-1}$  converge. Since  $f(a_n) = 0$ ,  $f(b_n) = \infty$  it follows that

$$\int \varrho(f(z)) |dz| \geq \pi,$$

where the integral is taken along the segment  $\Gamma_n$  joining  $a_n$  to  $b_n$ . In particular

$$\varrho(f(z_n)) \geq \frac{\pi}{|b_n - a_n|}$$

for some point  $z_n$  on  $\Gamma_n$ . By choosing  $a_n, b_n$  close enough together we can make the right hand side bigger than any preassigned function of  $|z_n|$ .

On the other hand a result in the opposite direction is possible. It is convenient to set

$$\mu(r, f) = \sup_{|z|=r} \varrho(f(z)).$$

Suppose that for  $r > r_0$  we have

$$\mu(r, f) < K r^\sigma. \quad (2.1)$$

By Theorem A this is only possible when  $\sigma > -1$  or when  $\sigma = -1$  and  $K \geq \frac{1}{2}$ . In the usual notation of NEVANLINNA Theory,

$$T_0(r, f) = \int_0^r \frac{S(t, f)}{t} dt$$

where

$$\begin{aligned} S(r, f) &= \frac{1}{\pi} \int_0^r \int_0^{2\pi} \varrho^2(f(te^{i\varphi})) t dtd\varphi \\ &\leq 2 \int_0^r \mu^2(t, f) t dt. \end{aligned}$$

Thus if  $\sigma = -1$  in (2.1),

$$S(r, f) = O(\log r), \quad T_0(r, f) = O(\log^2 r). \quad (2.2)$$

The examples (1.3) with  $a_\nu = A^\nu$  ( $A > 1$ ) show that the order of magnitude in (2.2) cannot be sharpened.

If (2.1) is satisfied with  $\sigma > -1$  we obtain

$$S(r, f) = O(r^{2\sigma+2}), \quad T_0(r, f) = O(r^{2\sigma+2}). \quad (2.3)$$

Hence a meromorphic function of proper order  $k > 0$  cannot satisfy (2.1) for any  $\sigma < \frac{k}{2} - 1$ . The implication from (2.1) to (2.3) is sharp as our first theorem shows.

**Theorem 1.** Suppose that  $0 < \lambda < \infty$  and that

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n^{\lambda n} - z^n}. \quad (2.4)$$

Then  $f(z)$  has perfectly regular growth of order  $2/\lambda$  and satisfies (2.1) with  $\sigma = \frac{1}{\lambda} - 1$ .



The function  $f(z)$  has poles at the points  $z = n^\lambda e^{\frac{2v\pi i}{n}}$  ( $v = 0, 1, \dots, n-1$ ;  $n \geq 1$ ). The number of poles in  $|z| \leq r$  is  $\frac{1}{2}p(p+1)$  where  $p$  is the largest integer such that  $p^\lambda \leq r$ , i.e.  $p = [r^{1/\lambda}]$ . Thus  $n(r, f)$ , the number of poles of  $f(z)$  in  $z \leq r$ , satisfies

$$n(r, f) \sim \frac{1}{2}p^2 \sim \frac{1}{2}r^{2/\lambda} (r \rightarrow \infty),$$

and so

$$N(r, f) = \int_0^r \frac{n(t, f)}{t} dt \sim \frac{\lambda}{4} r^{2/\lambda} (r \rightarrow \infty). \quad (2.5)$$

We now estimate  $|f(z)|$ . Assume that

$$(p - \frac{3}{4})^\lambda \leq |z| \leq (p + \frac{3}{4})^\lambda, \quad (2.6)$$

where  $p$  is a positive integer.  $A(\lambda)$  denotes a positive constant depending only on  $\lambda$  and is not necessarily the same at each occurrence. Let  $n$  be an integer satisfying  $n > p$  and put  $n = p + v$  so that  $v \geq 1$ . We have, in the range (2.6),

$$\begin{aligned} \left| \frac{z}{n^\lambda} \right|^n &\leq \left( \frac{n - v + \frac{3}{4}}{n} \right)^{\lambda n} = \left\{ 1 - \frac{(v - \frac{3}{4})}{n} \right\}^{\lambda n} \\ &\leq e^{-(v - \frac{3}{4})\lambda}. \end{aligned}$$

Hence, when  $z$  lies in the range (2.6),

$$\left| \sum_{n=p+1}^{\infty} \frac{(-1)^n z^n}{n^{\lambda n} - z^n} \right| \leq \sum_{v=1}^{\infty} \frac{e^{-(v - \frac{3}{4})\lambda}}{1 - e^{-(v - \frac{3}{4})\lambda}} = A(\lambda). \quad (2.7)$$

When  $1 \leq n < p$  and  $z$  lies in the range (2.6) then, if  $n = p - v$  with  $v \geq 1$ ,

$$\begin{aligned} \left| \frac{z}{n^\lambda} \right|^n &\geq \left( \frac{n + v - \frac{3}{4}}{n} \right)^{\lambda n} \geq \left( 1 + \frac{v - \frac{3}{4}}{n} \right)^{\lambda n} \\ &\geq \left( 1 + \frac{v - \frac{3}{4}}{k} \right)^{\lambda k} \quad (n \geq k). \end{aligned} \quad (2.8)$$

Now

$$\frac{(-1)^n z^n}{n^{\lambda n} - z^n} = (-1)^{n+1} + \frac{(-1)^n n^{\lambda n}}{n^{\lambda n} - z^n}$$

and so if we choose  $k$  in (2.8) to be  $\left[ \frac{2}{\lambda} \right] + 1$  so that  $\lambda k > 2$ , assuming that  $p > \left[ \frac{2}{\lambda} \right] + 1$ , we find that in the range (2.6)

$$\begin{aligned}
\left| \sum_{n=1}^{p-1} \frac{(-1)^n z^n}{n^{\lambda n} - z^n} \right| &\leq 1 + \left| \sum_{n=1}^{p-1} \frac{(-1)^n n^{\lambda n}}{n^{\lambda n} - z^n} \right| \\
&\leq 1 + \sum_{n=1}^{k-1} \frac{1}{\left(\frac{|z|}{n}\right)^{\lambda n} - 1} + \sum_{n=1}^{\infty} \frac{1}{\left(1 + \frac{p - \frac{3}{4}}{k}\right)^2 - 1} = A(\lambda).
\end{aligned}$$

From this and (2.7) we obtain

$$\left| f(z) - \frac{(-1)^p z^p}{p^{\lambda p} - z^p} \right| \leq A(\lambda) \quad (2.9)$$

in the range (2.6) for  $p > \left\lceil \frac{2}{\lambda} \right\rceil + 1$ . It is easy to see that consequently (2.9) holds in the range (2.6) for  $p \geq 1$ .

If  $|z| = t$  and (2.6) is satisfied then using (2.9) we see, in the notation of NEVANLINNA Theory, that

$$\begin{aligned}
m(t, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(te^{i\theta})| d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{t^p}{p^{\lambda p} - t^p e^{i p \theta}} \right| d\theta + A(\lambda) \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{\sin p\theta} \right| d\theta + A(\lambda) \\
&= A(\lambda).
\end{aligned}$$

From this and (2.5) we deduce that

$$T(r, f) = m(r, f) + N(r, f) \sim \frac{\lambda}{4} r^{2/\lambda}, \quad (r \rightarrow \infty)$$

so that  $f(z)$  is of perfectly regular growth, order  $\frac{2}{\lambda}$  and type  $\frac{\lambda}{4}$ .

It remains to be proved that  $f(z)$  satisfies (2.1) with  $\sigma = \frac{1}{\lambda} - 1$ .

We have

$$\begin{aligned}
f'(z) &= \sum_{n=1}^{\infty} (-1)^n \frac{n^{\lambda n+1} z^{n-1}}{(n^{\lambda n} - z^n)^2} \\
&= (-1)^p \frac{p^{\lambda p+1} z^{p-1}}{(p^{\lambda p} - z^p)^2} + f'_p(z), \quad \text{say,}
\end{aligned}$$

where  $f_p(z)$  is defined by the series for  $f(z)$  with the  $p$ th term omitted. Now, by the above,  $f_p(z)$  is regular and bounded by  $A(\lambda)$  in  $(p - 3/4)^\lambda \leq |z| \leq (p + 3/4)^\lambda$

and each point in  $(p - 1/2)^\lambda \leq |z| \leq (p + 1/2)^\lambda$  is the centre of a disc which lies in the larger annulus with radius  $\frac{p^{\lambda-1}}{A(\lambda)}$ . Hence, from CAUCHY's integral,

$$|f'_p(z)| \leq A(\lambda) p^{1-\lambda} < A(\lambda) |z|^{1/\lambda-1},$$

for

$$(p - 1/2)^\lambda \leq |z| \leq (p + 1/2)^\lambda \quad (p \geq 1). \quad (2.10)$$

Therefore in the range (2.10),

$$\begin{aligned} |f'(z)| &\leq \left| \frac{p^{\lambda p+1} z^{p-1}}{(p^{\lambda p} - z^p)^2} \right| + A(\lambda) |z|^{\frac{1}{\lambda}-1} \\ &= \frac{p^{\lambda p+1}}{|z|^{p+1}} \left| \left( \frac{z^p}{p^{\lambda p} - z^p} \right)^2 \right| + A(\lambda) |z|^{\frac{1}{\lambda}-1} \\ &\leq A(\lambda) \frac{p^{\lambda p+1}}{|z|^{p+1}} (1 + |f(z)|^2) + A(\lambda) |z|^{\frac{1}{\lambda}-1} \end{aligned}$$

by (2.9). Consequently, in the range (2.10),

$$\begin{aligned} \frac{|f'(z)|}{1 + |f(z)|^2} &\leq A(\lambda) \frac{p}{|z|} + A(\lambda) |z|^{1/\lambda-1} \\ &< A(\lambda) |z|^{1/\lambda-1}. \end{aligned}$$

Since the ranges (2.10) cover all the plane apart from a disc, the proof of the theorem is complete.

### 3. Positive theorems for integral functions

The remainder of the paper will be devoted to obtaining improvements of Theorem *B* and to showing that these are best possible. We assume without loss of generality that  $f(z)$  is an integral function. It will also be assumed that  $f(z)$  is always transcendental. In this section we state our positive theorems.

**Theorem 2.** *If  $f(z)$  is an integral function of proper order  $\sigma$  ( $0 \leq \sigma \leq \infty$ ), then*

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\log M(r, f)} \geq A_0(\sigma + 1), \quad (3.1)$$

where  $A_0$  is an absolute constant. In particular

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\log r} = \infty. \quad (3.2)$$

Inequality (3.2) sharpens (1.5) which is equivalent to

$$\limsup_{r \rightarrow \infty} r \mu(r, f) = \infty.$$

**Theorem 3.** *If  $f(z)$  is an integral function satisfying (2.1) for all large  $r$  with  $-1 < \sigma < \infty$ , then for large  $r$*

$$\log M(r, f) < \frac{A_1 K}{\sigma + 1} r^{\sigma+1}, \quad (3.3)$$

where  $A_1 = 25e \log 2$ .

It follows from (1.5) that the restriction  $\sigma > -1$  is necessary in Theorem 3. The theorem shows that for integral functions (2.1) implies that

$$T(r, f) = O(r^{\sigma+1}).$$

This is significantly stronger than (2.3) which is the best possible result for meromorphic functions by Theorem 1. Note that if  $f(z)$  is of perfectly regular growth then Theorem 3 is a consequence of Theorem 2.

As we shall see later, if  $f(z)$  is an integral function such that the growth of  $\log M(r, f)$  is properly of the order of  $\log^2 r$  in the sense that

$$0 < \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{\log^2 r} < \infty,$$

then no improvement of (3.2) is possible. On the other hand our next theorems show that if  $\log M(r, f) \neq O(\log^2 r)$  or  $\log M(r, f) = o(\log^2 r)$  then we can improve (3.2), the improvement depending on how large or how small  $\frac{\log M(r, f)}{\log^2 r}$  becomes respectively. However, there is no sharp difference in the behaviour of  $\mu(r, f)$  as we pass from one of the above classes of functions to another. By this we mean that if  $\varphi(r) \rightarrow \infty (r \rightarrow \infty)$ , then there is an  $f(z)$  from each of the above classes such that

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\varphi(r) \log r} < \infty.$$

Before stating our next theorem we give an indication of how one arrives at an improvement of (3.2) if  $\log M(r, f) \neq O(\log^K r)$  for  $K$  suitably large. If

$\mu(r, f) < K \frac{\log^2 r}{r}$  for large  $r$  then, from the inequality involving  $T_0(r, f)$  and  $\mu(r, f)$  in § 2, it follows that

$$T_0(r, f) = O(\log^6 r).$$

Hence if  $\log M(r, f) \neq O(\log^6 r)$  we see that (3.2) can be improved to

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\log^2 r} = \infty.$$

Our next result gives the improvement of (3.2) for functions  $f(z)$  such that  $\log M(r, f) \neq O(\log^2 r)$ , but  $\log M(r, f) = O(\log^6 r)$ .

**Theorem 4.** *If  $f(z)$  is an integral function and  $\varphi(r) \nearrow \infty$  ( $r \nearrow \infty$ ) and*

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{\varphi(r) \log^\alpha r} > 0, \quad \log M(r, f) = O(\log^{\alpha+1} r), \quad (3.4)$$

where  $2 \leq \alpha < \infty$ , then

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\varphi(r) \log^{\alpha-1} r} > 0. \quad (3.5)$$

When  $\alpha = 2$  in (3.4) then (3.5) is the improved form of (3.2). For functions such that  $\log M(r, f) \neq O(\log^3 r)$ ,  $\log M(r, f) = O(\log^6 r)$  take  $\varphi(r) = \{\log(r+1)\}^{1/2}$  and choose  $\alpha$  so that both conditions (3.4) are satisfied and  $\alpha \geq 2.5$ . The improved form of (3.2) is then

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{(\log r)^2} > 0.$$

To deal with functions such that  $\log M(r, f) = o(\log^2 r)$  we have the following result.

**Theorem 5.** *If  $\varphi(r)$  is increasing and  $f(z)$  is an integral function such that*

$$\log M(r, f) = O\left\{\frac{\log^2 r}{\varphi(r)}\right\} \quad (r \rightarrow \infty) \quad (3.6)$$

then

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\varphi(r) \log r} = \infty. \quad (3.7)$$

#### 4. Proofs of the positive theorems

4.1. We require a number of preliminary lemmas.

**Lemma 1.** *Let  $f(z) = a_0 + a_1(z - z_0) + \dots$  be regular in  $|z - z_0| \leq \delta$  and satisfy  $|f(z)| \geq 1$  there. Then*

$$|a_1| \leq \frac{2|a_0| \log |a_0|}{\delta}, \quad (4.1)$$

and for  $|z - z_0| \leq r < \delta$

$$|a_0|^{\frac{\delta-r}{\delta+r}} \leq |f(z)| \leq a_0^{\frac{\delta+r}{\delta-r}}. \quad (4.2)$$

If further  $|f(z_1)| = 1$  for some  $z_1$  with  $|z_1 - z_0| = \delta$  then for some  $z$  on the segment joining  $z_0$  to  $z_1$

$$\varrho(f(z)) \geq \frac{\log |a_0|}{10\delta \log 2} \geq \frac{|a_1|}{20|a_0| \log 2}. \quad (4.3)$$

(4.1) and (4.2) are classical.

Suppose that

$$|f(z_0 + \delta e^{i\varphi})| = 1 \quad (z_1 = z_0 + \delta e^{i\varphi}).$$

If

$$|f(z_0 + \varrho e^{i\varphi})| \leq 2 \quad (0 \leq \varrho \leq \delta) \quad (4.4)$$

then  $|a_0| \leq 2$  and

$$\begin{aligned} |a_0| - 1 &\leq |f(z_0 + \delta e^{i\varphi}) - f(z_0)| \leq \int_0^\delta |f'(z_0 + te^{i\varphi})| dt \\ &\leq \delta \max_{0 \leq t \leq \delta} |f'(z_0 + te^{i\varphi})|. \end{aligned}$$

If  $\zeta = z_0 + t_0 e^{i\varphi}$  is a point where the maximum on the right is attained then,

$$|f'(\zeta)| \geq \frac{|a_0| - 1}{\delta} \geq \frac{\log |a_0|}{\delta}$$

and so

$$\varrho(f(\zeta)) = \frac{|f'(\zeta)|}{1 + |f(\zeta)|^2} \geq \frac{|f'(\zeta)|}{5} \geq \frac{\log |a_0|}{5\delta}.$$

Hence the first inequality of (4.3) is true in this case.

If (4.4) is false let  $\varrho$  be the largest number with  $0 \leq \varrho < \delta$  such that  $|f(z_0 + \varrho e^{i\varphi})| = 2$ . Take  $\zeta = z_0 + t_1 e^{i\varphi}$  to be a point for which  $|f'(z)|$  is greatest when  $z = z_0 + te^{i\varphi}$  ( $\varrho \leq t \leq \delta$ ). Then  $|f(\zeta)| \leq 2$  and so

$$\frac{|f'(\zeta)|}{1 + |f(\zeta)|^2} \geq \frac{|f'(\zeta)|}{5}.$$

Also

$$1 \leq |f(z_0 + \delta e^{i\varphi}) - f(z_0 + \varrho e^{i\varphi})| \leq \int_{\varrho}^{\delta} |f'(z_0 + te^{i\varphi})| dt \\ \leq (\delta - \varrho) |f'(\zeta)|.$$

Further, by (4.2) and the fact that  $|f(z_0 + \varrho e^{i\varphi})| = 2$ , we have

$$|a_0|^{\frac{\delta-\varrho}{\delta+\varrho}} \leq 2,$$

and hence

$$\delta - \varrho \leq \frac{(\delta + \varrho) \log 2}{\log |a_0|} \leq \frac{2\delta \log 2}{\log |a_0|}.$$

From the above it follows that

$$\varrho(f(\zeta)) = \frac{|f'(\zeta)|}{1 + |f(\zeta)|^2} \geq \frac{|f'(\zeta)|}{5} \geq \frac{1}{5(\delta - \varrho)} \geq \frac{\log |a_0|}{10\delta \log 2}.$$

This completes the proof of the first inequality of (4.3). The second follows immediately from (4.1).

**Lemma 2.** *Suppose that  $f(z)$  is an integral function such that for some  $r_1 > 0$*

$$\min_{|z|=r_1} |f(z)| = 1, \quad (4.5)$$

and that

$$|f(z)| > 1 \quad (r_1 < |z| < 3r_1). \quad (4.6)$$

Then for some  $r$  satisfying  $r_1 < r < 2r_1$  we have

$$\mu(r, f) > \frac{e^{-4\pi} \log M(r, f)}{10r \log 2}. \quad (4.7)$$

In particular if the conditions are satisfied for arbitrarily large  $r_1$  then,

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\log M(r, f)} \geq \frac{e^{-4\pi}}{10 \log 2}. \quad (4.8)$$

Let  $r_0 = 2r_1$  and let  $z_0 = r_0 e^{i\vartheta_0}$  be such that

$$|f(z_0)| = M(r_0, f).$$

There is a  $\vartheta_1$  with  $|\vartheta_1 - \vartheta_0| \leq \pi$  such that

$$|f(r_1 e^{i\vartheta_1})| = 1.$$

For each  $\zeta$ , with  $|\zeta| = r_0$ ,  $|f(z)| > 1$  for  $|z - \zeta| < r_1 = \frac{r_0}{2}$  and so (4.1) gives

$$\frac{|f'(\zeta)|}{|f(\zeta)| \log |f(\zeta)|} \leq \frac{4}{r_0}.$$

Thus

$$\left| \frac{\partial}{\partial \vartheta} \log \log |f(r_0 e^{i\vartheta})| \right| \leq 4$$

and so

$$\left| \log \frac{\log |f(r_0 e^{i\vartheta_1})|}{\log |f(r_0 e^{i\vartheta_0})|} \right| \leq 4\pi,$$

from which it follows that

$$\log |f(r_0 e^{i\vartheta_1})| \geq e^{-4\pi} \log |f(r_0 e^{i\vartheta_0})| = e^{-4\pi} \log M(r_0, f).$$

In the closed disc  $|z - r_0 e^{i\vartheta_0}| \leq \frac{r_0}{2}$  we have  $|f(z)| \geq 1$  and, at the point  $z_1 = r_1 e^{i\vartheta_1}$  on the boundary,  $|f(z_1)| = 1$ . Consequently, by (4.3) with  $\delta = \frac{r_0}{2}$ , there is a point  $\xi$  on the segment joining  $r_0 e^{i\vartheta_1}$  to  $z_1$  for which

$$\varrho(f(\xi)) \geq \frac{\log |f(r_0 e^{i\vartheta_1})|}{5r_0 \log 2} \geq \frac{e^{-4\pi} \log M(r_0, f)}{5r_0 \log 2}.$$

If  $|\xi| = r$ , then  $\frac{r_0}{2} \leq r \leq r_0$  and hence we deduce that

$$\mu(r, f) \geq \frac{e^{-4\pi} \log M(r, f)}{10r \log 2}.$$

This proves Lemma 2.

The next lemma is required to cope with possible irregularities in the growth of  $\log M(r, f)$ .

**Lemma 3.** *Suppose that  $\varphi(r)$  ( $r_0 \leq r < \infty$ ) is continuous, positive and strictly increasing with a sectionally continuous locally bounded derivative  $\varphi'(r)$ . [At points of discontinuity we define  $\varphi'(r)$  as the limit from the left.] Suppose that for positive  $\alpha, \beta$*

$$\limsup_{r \rightarrow \infty} \frac{\varphi(r)}{r^\alpha} > \beta. \quad (4.9)$$

*Then given  $\alpha'$  ( $0 < \alpha' < \alpha$ ) there exist arbitrarily large  $r$  for which the following are satisfied:*



$$\frac{\varphi(r)}{r^\alpha} \geq \beta e^{-5}; \quad (4.10)$$

$$\frac{\varphi'(r)}{\varphi(r)} \geq \frac{\alpha'}{r}; \quad (4.11)$$

$$\varphi \left\{ r + 2 \frac{\varphi(r)}{\varphi'(r)} \right\} < e^4 \varphi(r). \quad (4.12)$$

We assume that  $\varphi'(r)$  is never zero. This really involves no loss of generality. First of all we show that there are arbitrarily large values of  $r$  such that (4.11) and

$$\frac{\varphi(r)}{r^\alpha} \geq \beta \quad (4.10)'$$

are satisfied. Now  $\frac{\varphi(r)}{r^{\alpha'}}$  is unbounded as  $r \rightarrow \infty$  and so for arbitrarily large  $r$  it must be locally nondecreasing. For such  $r$ ,

$$\frac{d}{dr} \left\{ \frac{\varphi(r)}{r^{\alpha'}} \right\} = \frac{\varphi(r)}{r^{\alpha'}} \left\{ \frac{\varphi'(r)}{\varphi(r)} - \frac{\alpha'}{r} \right\} \geq 0$$

and so (4.11) is satisfied. If for all large  $r$ ,  $\varphi(r) \geq \beta r^\alpha$  then we obtain the desired result. Otherwise there are arbitrarily large values of  $r$  such that  $\varphi(r) < \beta r^\alpha$ . From (4.9) there is a smallest  $R > r$  such that  $\varphi(R) = \beta R^\alpha$ . But then  $\frac{\varphi(r)}{r^\alpha}$  is nondecreasing at  $R$  and so  $\frac{\varphi'(R)}{\varphi(R)} \geq \frac{\alpha}{R}$ , as in the previous argument, and  $\frac{\varphi(R)}{R^\alpha} = \beta$ . Hence the result.

Now set  $h = h(r) = 2 \frac{\varphi(r)}{\varphi'(r)}$  and note that

$$\log \varphi(r + h) - \log \varphi(r) = \int_r^{r+h} \frac{\varphi'(t)}{\varphi(t)} dt \leq h \max_{r \leq t \leq r+h} \frac{\varphi'(t)}{\varphi(t)}.$$

Consequently if (4.12) is false for  $r = r_0$  there is an  $r_1$  such that  $r_0 < r_1 \leq r_0 + h(r_0)$  and

$$\frac{\varphi'(r_1)}{\varphi(r_1)} \geq \frac{4}{h(r_0)} = 2 \frac{\varphi'(r_0)}{\varphi(r_0)}.$$

Suppose that  $r_0, r_1, \dots, r_n$  have been defined in this way so that (4.12) is false for  $r = r_\nu$  ( $0 \leq \nu \leq n$ ) and

$$r_\nu < r_{\nu+1} \leq r_\nu + 2 \frac{\varphi(r_\nu)}{\varphi'(r_\nu)} \quad (0 \leq \nu \leq n-1),$$

$$\frac{\varphi'(r_{\nu+1})}{\varphi(r_{\nu+1})} \geq 2 \frac{\varphi'(r_\nu)}{\varphi(r_\nu)} \quad (0 \leq \nu \leq n-1).$$

Then we can define  $r_{n+1}$  so that

$$\frac{\varphi'(r_{n+1})}{\varphi(r_{n+1})} \geq 2 \frac{\varphi'(r_n)}{\varphi(r_n)}, \quad r_n < r_{n+1} \leq r_n + 2 \frac{\varphi(r_n)}{\varphi'(r_n)}.$$

If this process continued indefinitely then we should have

$$\frac{\varphi'(r_n)}{\varphi(r_n)} \rightarrow \infty \quad (r \rightarrow \infty)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} (r_{n+1} - r_n) &\leq 2 \sum_{n=0}^{\infty} \frac{\varphi(r_n)}{\varphi'(r_n)} \\ &\leq 2 \frac{\varphi(r_0)}{\varphi'(r_0)} \sum_{n=0}^{\infty} 2^{-n} \\ &= 4 \frac{\varphi(r_0)}{\varphi'(r_0)}. \end{aligned}$$

Thus  $r_n$  would tend to a finite limit and so  $\frac{\varphi'(r_n)}{\varphi(r_n)} \rightarrow \infty$ . This contradiction shows that the construction of the  $r_n$  must terminate after a finite number of steps.

Take now as  $r_0$  a value such that (4.10)' and (4.11) are satisfied for  $r = r_0$ . If (4.12) is not satisfied for  $r = r_0$  then there is a sequence  $r_0, r_1, \dots, r_N$  as above such that it is not satisfied for  $r = r_n$  ( $0 \leq n \leq N-1$ ) but it is satisfied for  $r = r_N$ . Then for  $0 \leq n < N$ ,

$$\frac{\varphi'(r_{n+1})}{\varphi(r_{n+1})} \geq 2 \frac{\varphi'(r_n)}{\varphi(r_n)} \geq 2^{n+1} \frac{\varphi'(r_0)}{\varphi(r_0)}$$

and so

$$\begin{aligned} r_N - r_0 &= \sum_{n=0}^{N-1} (r_{n+1} - r_n) \leq 2 \frac{\varphi(r_0)}{\varphi'(r_0)} \sum_{n=0}^{N-1} \frac{1}{2^n} \\ &< 4 \frac{\varphi(r_0)}{\varphi'(r_0)} \\ &< 4 \frac{r_0}{\alpha'} \end{aligned}$$

by (4.11). Hence if  $\alpha'$  is near enough to  $\alpha$ ,

$$r_N < r_0 \left( 1 + \frac{4}{\alpha'} \right) \leq r_0 (1 + 5/\alpha).$$

Since (4.10)' holds for  $r = r_0$ ,

$$\varphi(r_N) \geq \varphi(r_0) \geq \beta r_0^\alpha \geq \beta r_N^\alpha (1 + 5/\alpha)^{-\alpha} > \beta e^{-5} r_N^\alpha.$$

Also

$$\frac{\varphi'(r_N)}{\varphi(r_N)} \geq \frac{\varphi'(r_0)}{\varphi(r_0)} \geq \frac{\alpha'}{r_0} \geq \frac{\alpha'}{r_N}.$$

Hence the proof of Lemma 3 is complete.

#### 4.2. Proofs of Theorems 2 and 3 for $\sigma \geq 6$ .

Suppose now that  $f(z)$  is an integral function of order  $\sigma \geq 6$ . We apply Lemma 3 with  $\sigma > \alpha' > 5$  to  $\varphi(r) = \log M(r, f)$  so that for some arbitrarily large  $r$ , (4.10), (4.11) and (4.12) hold simultaneously. For such an  $r$  there is a point  $z_0 = re^{i\theta}$  so that [see e.g. 3, Lemma 2, p. 136.]

$$\begin{aligned} |f(z_0)| &= M(r, f), \\ \left| \frac{f'(z_0)}{f(z_0)} \right| &= \varphi'(r). \end{aligned}$$

It now follows from Lemma 1 that if  $\delta = \delta(r)$  is the radius of the largest disc with centre  $z_0$  in which  $|f(z)| > 1$  then, by (4.1),

$$\delta(r) \leq 2 \frac{|f(z_0)| \log |f(z_0)|}{|f'(z_0)|} = 2 \frac{\varphi(r)}{\varphi'(r)} \leq \frac{2r}{\alpha'} < \frac{2}{5} r.$$

By (4.3) there is a point  $z$  with  $|z - z_0| < \delta(r)$  and

$$\begin{aligned} \varrho(f(z)) &\geq \frac{\log |f(z_0)|}{10 \delta(r) \log 2} \\ &= \frac{\varphi(r)}{10 \delta(r) \log 2} \\ &\geq \frac{\alpha' \varphi(r)}{20 r \log 2}. \end{aligned} \tag{4.13}$$

If  $|z| = R$ , then  $R < r + \delta(r)$  and so, by (4.12),

$$\varphi(R) \leq \varphi(r + \delta(r)) \leq \varphi \left( r + 2 \frac{\varphi(r)}{\varphi'(r)} \right) \leq e^4 \varphi(r).$$

Hence, since also  $R > r - \delta(r) > 3/5 r$ ,

$$\begin{aligned}\mu(R, f) &\geq \varrho(f(z)) \geq \frac{\alpha' e^{-4} \varphi(R)}{20(2R) \log 2} \\ &= \frac{\alpha' e^{-4} \log M(R, f)}{40R \log 2}.\end{aligned}$$

From  $R > \frac{3}{5} r$  it follows that as  $r \rightarrow \infty$  then  $R \rightarrow \infty$  and so we arrive at

$$\limsup_{R \rightarrow \infty} \frac{R \mu(R, f)}{\log M(R, f)} \geq \frac{\sigma e^{-4}}{40 \log 2},$$

since  $\alpha'$  can be taken as near to  $\sigma$  as we please. This proves (3.1) and so Theorem 2.

We next prove Theorem 3 for  $\sigma \geq 5$ . Suppose in fact that (3.3) is false for some arbitrarily large  $r$  where  $A_1$  is some positive constant. We may apply Lemma 3 as before with  $\alpha = \sigma + 1$ ,  $\alpha' = \sigma$  and any quantity  $\beta$  such that

$$0 < \beta < \frac{A_1 K}{\sigma + 1}. \quad (4.14)$$

Then (4.13) yields for some  $z$  with  $|z| = R$

$$\varrho(f(z)) \geq \frac{\sigma \varphi(r)}{20r \log 2} \geq \frac{\sigma \beta e^{-5} r^\sigma}{20 \log 2}. \quad (4.15)$$

Also

$$|z| = R < r + \delta(r) \leq r + 2 \frac{\varphi(r)}{\varphi'(r)} \leq r \left(1 + \frac{2}{\sigma}\right)$$

by (4.11). Therefore

$$R^\sigma \leq r^\sigma \left(1 + \frac{2}{\sigma}\right)^\sigma \leq e^2 r^\sigma.$$

Then (4.15) shows that

$$\mu(R, f) \geq \frac{\sigma \beta e^{-7}}{20 \log 2} R^\sigma$$

for arbitrarily large values of  $R$ . From (4.14) we see that

$$\frac{\sigma A_1 K}{\sigma + 1} \frac{e^{-7}}{20 \log 2} \leq K,$$

and so

$$A_1 \leq \frac{\sigma + 1}{\sigma} 20 e^7 \log 2 < 25 e^7 \log 2.$$

Consequently it is only for such  $A_1$  that the result of the theorem is false. Hence it must be true with  $A_1 = 25 e^7 \log 2$ . This proves (3.3) for  $\sigma \geq 5$ .

### 4.3. Completion of proof of Theorem 3

Suppose that the hypotheses of Theorem 3 hold with  $-1 < \sigma < 5$ . Let  $n$  be a positive integer such that

$$n(\sigma + 1) \geq 6 \quad (4.16)$$

and consider  $F(z) = f(z^n)$ . Then for all large  $r$  we have

$$\varrho(F(z)) = \frac{|F'(z)|}{1 + |F(z)|^2} = \frac{n r^{n-1} |f'(z^n)|}{1 + |f(z^n)|^2} < K n r^{n-1} r^{n\sigma} \quad (|z| = r)$$

by (2.1). Hence  $F(z)$  satisfies (2.1) with  $Kn$  in place of  $K$  and  $n(\sigma + 1) - 1$  in place of  $\sigma$ . In view of (4.16) we can apply the previous result to  $F(z)$  and obtain

$$\log M(r, F) \leq \frac{A_1 K n r^{n(\sigma+1)}}{n(\sigma + 1)} = \frac{A_1 K}{\sigma + 1} r^{n(\sigma+1)}.$$

As  $M(r, F) = M(r^n, f)$  this completes the proof of Theorem 3.

### 4.4. Completion of proof of Theorem 2

We assume that  $f(z)$  is of order  $\sigma < 6$  and consider  $F(z) = f(z^{12})$ . Since, as above,

$$\varrho(F(z)) = 12 |z|^{11} \varrho(f(z^{11}))$$

and  $F(z)$  is of order  $12\sigma$  it follows that if (3.1) holds for  $F(z)$  then

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\log M(r, f)} \geq \frac{1}{12} A_0(12\sigma + 1)$$

and this is the result for  $f(z)$  if  $A_0$  is adjusted. Consequently it is sufficient for  $\sigma < 6$  to prove the theorem for  $F(z)$ .

Now for some constant  $A_2$  we have

$$\log M(4r, F) \leq A_2 \log M(r, F) \quad (4.17)$$

for arbitrarily large values of  $r$ . Otherwise for some  $r_0$  we find that

$$\log M(4^n r_0, F) \geq A_2^n \log M(r_0, F) \quad (n \geq 1)$$

so that the order of  $F(z)$  is at least  $\frac{\log A_2}{\log 4}$ . This is impossible if  $A_2 \geq 4^{72}$  as  $F(z)$  is of order less than 72.

We consider arbitrarily large  $r$  for which (4.17) is true. If for an infinite sequence of such  $r$ ,  $|f(z)| \geq 1$  ( $r \leq |z| \leq 3r$ ) then the result follows from Lemma 2. Hence we assume always that for some  $R$  in  $r \leq R \leq 3r$  there is a  $z$  on  $|z| = R$  where  $|f(z)| < 1$ . From the periodic nature of  $F(z)$  we see that there is a disc  $S(R)$  centred on  $\zeta$  where  $|\zeta| = R$ ,  $|F(\zeta)| = M(R, F)$  such that  $|F(z)| \geq 1$  in  $S(R)$ ,  $|F(z)| = 1$  at some boundary point and the radius of  $S(R)$  does not exceed  $\frac{\pi R}{12}$ . By Lemma 1 it follows that

$$\mu(t, F) \geq \frac{12 \log M(R, F)}{10\pi R \log 2},$$

for some  $t$  satisfying  $R - \frac{\pi R}{12} < t < R + \frac{\pi R}{12}$ , so that  $\frac{2}{3}R < t < \frac{4}{3}R$ . If  $t \leq R$  then we get

$$\begin{aligned} \mu(t, R) &\geq \frac{12 \log M(t, F)}{10\pi \cdot \frac{2}{3}t \log 2} \\ &= \frac{4 \log M(t, F)}{5\pi t \log 2}. \end{aligned}$$

If  $t > R$  then, since  $R \leq 3r$ ,  $t < 4r$  and so, using (4.17) we have

$$\begin{aligned} \mu(t, F) &\geq \frac{12 \log M(t, F)}{A_2 10\pi t \log 2} \\ &= \frac{6 \log M(t, F)}{5A_2 \pi t \log 2}. \end{aligned}$$

As  $t > \frac{2}{3}R \geq \frac{2}{3}r$  it follows that one of the above inequalities must hold for arbitrarily large  $t$ . Hence the proof of Theorem 2 is complete.

#### 4.5. Proof of Theorem 4

For any function  $f(z)$  of order less than 1 with  $f(0) \neq 0$  we have the well known inequalities [see e.g. 4, p. 28]

$$\int_0^r \frac{n(t)}{t} dt \leq \log \left( \frac{M(r, f)}{|f(0)|} \right) \leq \int_0^r \frac{n(t)}{t} dt + r \int_r^\infty \frac{n(t)}{t^2} dt, \quad (4.18)$$

where  $n(t)$  is the number of zeros of  $f(z)$  in  $|z| \leq t$ . The restriction  $f(0) \neq 0$

clearly involves no loss of generality. From the second condition of (3.4) and the left hand inequality of (4.18) it follows that

$$n(r) = O(\log^\alpha r). \quad (4.19)$$

From (4.19) we find that

$$r \int_r^\infty \frac{n(t)}{t^2} dt = O(\log^\alpha r). \quad (4.20)$$

Hence for  $r$  such that  $\log M(r, f) > \eta \varphi(r) \log^\alpha r$ , where  $\eta$  is some positive constant implied in the first condition of (3.4), we obtain, from (4.18) and (4.20),

$$\log M(r, f) = \{1 + o(1)\} \int_0^r \frac{n(t)}{t} dt. \quad (4.21)$$

Assume now that we are dealing with values  $r$  of the above kind. By a known result we have for some  $R$  in  $\left(\frac{r}{4}, \frac{r}{2}\right)$ ,  $\log |f(z)| > H \log M(R, f)$  ( $|z| = R$ ) where, here and elsewhere,  $H$  depends only on  $f(z)$  [5, pp. 64–65]. For sufficiently large  $r$  let  $R'$  be the smallest number such that  $|f(z)| > 1$  ( $R' < |z| < R$ ). We deal with two cases: a)  $R' > \frac{r}{12}$ ; b)  $R' \leq \frac{r}{12}$  for arbitrarily large values of  $R'$ . It is clear that in fact  $R'$  does take arbitrarily large values.

**Case a).** If  $|f(\zeta)| = 1$  ( $\zeta = R' e^{i\varphi}$ ) we consider the largest disc  $D$  centred on  $R e^{i\varphi}$  in which  $|f(z)| > 1$ . The radius of  $D$  is at most  $\frac{r}{2} - \frac{r}{12} = \frac{5}{12} r$  and so  $D$  lies in  $|z| < \frac{r}{2} + \frac{5}{12} r < r$ . By Lemma 1, (4.3), for some  $t$  in  $\frac{r}{12} < t < r$  we have

$$\mu(t, f) > \frac{H \log M(R, f)}{r}.$$

From (4.18), (4.19) and (4.21) it follows that

$$\begin{aligned} \log M\left(\frac{r}{12}, f\right) &> H \log M(r, f) - \int_{r/12}^r \frac{n(t)}{t} dr + O(\log^\alpha r) \\ &> H \log M(r, f) + O(\log^\alpha r) \\ &= H(1 + o(1)) \log M(r, f). \end{aligned}$$

Hence we see that

$$\begin{aligned}\mu(t, f) &> H \frac{\varphi(r) \log^\alpha r}{r} \\ &> H \frac{\varphi(t) \log^\alpha t}{t},\end{aligned}$$

for arbitrarily large values of  $t$ . This proves the theorem in this case.

**Case b).** In this case  $|f(z)| > 1$  ( $R' < |z| < 3R'$ ) and  $|f(\zeta)| = 1$  ( $\zeta = R' e^{i\varphi}$ ). We see from the proof of Lemma 2 that

$$\mu(t, f) > H \frac{\log M(2R', f)}{R'} \quad (4.22)$$

for some  $t$  satisfying  $R' < t < 2R'$ . Now from (4.19) and (4.21)

$$\begin{aligned}n\left(\frac{r}{4}\right) \log r &> H \int_0^{r/4} \frac{n(t)}{t} dt = H \left( \int_0^r \frac{n(t)}{t} dt - \int_{r/4}^r \frac{n(t)}{t} dt \right) \\ &> H \varphi(r) \log^\alpha r - H \log^\alpha r\end{aligned}$$

and so

$$n\left(\frac{r}{4}\right) > H \varphi(r) \log^{\alpha-1} r.$$

But  $\left(R', \frac{r}{4}\right)$  is free from zeros and so

$$n(R') > H \varphi(r) \log^{\alpha-1} r.$$

Hence, by (4.18),

$$\begin{aligned}\frac{\log M(2R', f)}{|f(0)|} &\geq \int_{R'}^{2R'} \frac{n(t)}{t} dt = n(R') \log 2 \\ &> H \varphi(r) \log^{\alpha-1} r.\end{aligned}$$

Therefore we find that in (4.22),

$$\mu(t, f) > \frac{H \varphi(t) \log^{\alpha-1} t}{t}.$$

Since this holds for arbitrarily large values of  $t$  the theorem is proved in this case.

#### 4.6. Proof of Theorem 5

From the left hand inequality of (4.18) we get

$$\begin{aligned}n(r) \log r &\leq \int_r^{r^2} \frac{n(t)}{t} dt \leq \log M(r^2, f) \\ &= O\left\{\frac{\log^2 r}{\varphi(r^2)}\right\}\end{aligned}$$



and so, since  $\varphi(r)$  is increasing,

$$n(r) = O \left\{ \frac{\log r}{\varphi(r)} \right\}. \quad (4.23)$$

Using (4.23) we obtain

$$\begin{aligned} r \int_r^\infty \frac{n(t)}{t^2} dt &= O \left\{ \frac{1}{\varphi(r)} \cdot r \int_r^\infty \frac{\log t}{t^2} dt \right\} \\ &= O \left\{ \frac{\log r}{\varphi(r)} \right\}. \end{aligned} \quad (4.24)$$

Hence if we put  $\beta(r) = \eta \sqrt{\frac{\log r}{\varphi(r) \log M(r)}}$ , where  $\eta > 0$  and depends on  $f(z)$ , then, by a known result [5, pp. 64–65], in  $r(1 - \beta(r)) < |z| < r(1 + \beta(r))$

$$\log |f(z)| > H \log M(|z|, f)$$

outside a set of circles the sum of whose radii is at most  $Hr\beta^2(r)$ .

Consider now values of  $r$  such that  $f(z)$  has a zero on  $|z| = r$ . Let  $z_0 = re^{i\theta_0}$  be such a zero. Then from the above, if  $r$  is large enough, for some  $R$  satisfying  $r - Hr\beta^2(r) < R < r$  we have

$$\log |f(Re^{i\theta_0})| > H \log M(R, f).$$

Let  $D$  be the disc with centre  $Re^{i\theta_0}$  in which  $|f(z)| > 1$ , assuming  $r$  is sufficiently large, with  $|f(z)| = 1$  somewhere on the boundary. Then, by Lemma 1 and the above for some  $z$  in this disc

$$\varrho(f(z)) > \frac{H \log M(R, f)}{r\beta^2(r)}. \quad (4.25)$$

Now as  $\beta(r) \rightarrow 0$  as  $r \rightarrow \infty$  it follows that for large  $r$ ,  $\frac{r}{2} < R < r$  and so

$$\begin{aligned} \log M(R, f) &= \{1 + o(1)\} \int_0^R \frac{n(t)}{t} dt \\ &> \{1 + o(1)\} \left\{ \log M(r, f) - \int_R^r \frac{n(t)}{t} dt \right\} \\ &= \{1 + o(1)\} \{ \log M(r, f) + O(\log r) \} \\ &= \{1 + o(1)\} \log M(r, f), \end{aligned}$$

where we have used (4.23), (4.24), (4.18) and the obvious result that  $\log r = o(\log M(r, f))$ . Hence, from (4.25),

$$\begin{aligned} \varrho(f(z)) &> \frac{H \log M(r, f)}{r \beta^2(r)} \\ &= \frac{H \varphi(r) \log r}{\eta^2 r} \left\{ \frac{\log M(r, f)}{\log r} \right\}^2. \end{aligned}$$

Now in (4.25),  $\frac{r}{2} < |z| < r$  for large  $r$  and so if  $|z| = t$  then for large  $r$  we find that

$$\mu(t, f) > H \frac{\varphi(t) \log t}{\eta^2 t} \left( \frac{\log M(r, f)}{\log r} \right)^2$$

since  $\varphi(t)$  is increasing. As the final factor above tends to  $\infty$  with  $r$  and the inequality holds for some arbitrarily large  $t$  this proves Theorem 5.

## 5. Counter examples

The first theorem shows that (3.2) is best possible and that the properties of  $f(z)$  referred to in §3 preceding Theorem 4 do in fact hold.

**Theorem 6.** *Given  $\varphi(r) \nearrow \infty$  ( $r \nearrow \infty$ ) there is a sequence of increasing integers  $k_n$  such that if*

$$\begin{aligned} f(z) &= \prod_1^\infty \left( 1 - \frac{z}{2^{k_n}} \right)^{k_n}, \quad f_1(z) = \prod_1^\infty \left( 1 - \frac{z}{2^{nk_n}} \right)^{k_n}, \\ f_2(z) &= \prod_1^\infty \left( 1 - \frac{z}{2^{k_n/n}} \right)^{k_n} \end{aligned}$$

then for  $g(z) = f(z)$ ,  $f_1(z)$  or  $f_2(z)$

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, g)}{\varphi(r) \log r} < \infty.$$

The sequence  $\{k_n\}$  will be seen later to satisfy  $\frac{k_{n+1}}{k_n} \geq 4$  and in this case it is easy to verify that

$$0 < \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{\log^2 r} < \infty, \quad \log M(r, f_1) = o(\log^2 r), \quad \log M(r, f_2) \neq O(\log^2 r).$$

The next theorem shows that Theorem 2 is best possible

**Theorem 7.** *Given  $\sigma$  ( $0 \leq \sigma < \infty$ ) there is an integral function of proper order  $\sigma$  and very regular growth when  $\sigma > 0$  such that*

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\log M(r, f)} < C(\sigma + 1)$$

for some absolute constant  $C$ .

### 5.1. Proof of Theorem 6

The proof of the theorem requires a number of lemmas. We assume that besides any other conditions that the integers  $k_n$  will be required to satisfy, that they will always satisfy

$$\frac{k_{n+1}}{k_n} \geq 4 \quad (n > 1), \quad k_1 \geq 2. \quad (5.1)$$

We confine our attention to  $f(z)$ . The proofs for  $f_1(z)$  and  $f_2(z)$  are similar.

**Lemma 4.** *On  $|z| = 2^{k_{n+1}}$  and on  $|z| = 2^{k_{n-1}}$ ,*

$$|f(z)| > H|z|.$$

On  $|z| = 2^{k_{n+1}}$  we have

$$|f(z)| \geq \prod_{m=1}^n \left( \frac{2^{k_{n+1}}}{2^{k_m}} - 1 \right)^{k_m} \cdot \prod_{m=n+1}^{\infty} \left( 1 - \frac{2^{k_{n+1}}}{2^{k_m}} \right)^{k_m}.$$

From (5.1) each factor in the first product is at least 1 and so

$$\begin{aligned} \prod_{m=1}^n \left( \frac{2^{k_{n+1}}}{2^{k_m}} - 1 \right)^{k_m} &\geq \left( \frac{2^{k_{n+1}}}{2^{k_1}} - 1 \right)^{k_1} \\ &> H \cdot 2^{k_{n+1}} = H|z|. \end{aligned} \quad (5.2)$$

Also, from (5.1),

$$\begin{aligned} \prod_{m=n+1}^{\infty} \left( 1 - \frac{2^{k_{n+1}}}{2^{k_m}} \right)^{k_m} &> \prod_{m=n+1}^{\infty} \left( 1 - 2^{-\frac{k_m}{2}} \right)^{k_m} \\ &> H. \end{aligned} \quad (5.3)$$

From (5.2) and (5.3) the lemma follows for  $|z| = 2^{k_{n+1}}$ .

In dealing with  $|z| = 2^{k_{n-1}}$  we assume for convenience that  $n > 1$ . This clearly involves no loss of generality. On  $|z| = 2^{k_{n-1}}$  we have

$$|f(z)| \geq \prod_{m=1}^{n-1} \left( \frac{2^{k_n}}{2^{k_{m+1}}} - 1 \right)^{k_m} \cdot 2^{-k_n} \prod_{m=n+1}^{\infty} \left( 1 - \frac{2^{k_n}}{2^{k_{m+1}}} \right)^{k_m}.$$

By (5.1) each factor in the first product is at least 1 and so

$$\prod_{m=1}^{n-1} \left( \frac{2^{k_n}}{2^{k_{m+1}}} - 1 \right)^{k_m} > \left( \frac{2^{k_n}}{2^{k_1+1}} - 1 \right)^{k_1} > H \cdot 2^{2k_n-1} \quad (5.4)$$

since  $k_1 \geq 2$ . As before

$$\prod_{n=m+1}^{\infty} \left( 1 - \frac{2^{k_n}}{2^{k_{m+1}}} \right)^{k_m} > H. \quad (5.5)$$

Hence on  $|z| = 2^{k_n-1}$ , by (5.4) and (5.5),

$$\begin{aligned} |f(z)| &> H \cdot 2^{2k_n-1} \cdot 2^{-k_n} \\ &= H 2^{k_n-1} = H |z|. \end{aligned}$$

Hence the lemma follows for  $|z| = 2^{k_n-1}$ .

We see from Lemma 4 that when  $z$  is large the regions in which  $|f(z)| < 1$  are disjoint, with one in each annulus  $2^{k_n-1} < |z| < 2^{k_n+1}$ . Denote these by  $D_n$ . Clearly  $D_n$  contains the zero at  $z = 2^{k_n}$ .

**Lemma 5.** *If the  $k_n$  increase sufficiently rapidly then on the boundary of  $D_n$  when  $n$  is large*

$$H 2^{k_n-k_1-k_2-\dots-k_{n-1}} < |z - 2^{k_n}| < H 2^{k_n-k_1-\dots-k_{n-1}}.$$

We have

$$|f(z)| = \prod_{m=1}^{n-1} \left| 1 - \frac{z}{2^{k_m}} \right|^{k_m} \left( \frac{|z - 2^{k_n}|}{2^{k_n}} \right)^{k_n} \cdot \prod_{m=n+1}^{\infty} \left| 1 - \frac{z}{2^{k_m}} \right|^{k_m}.$$

Now on the boundary of  $D_n$

$$\prod_{m=1}^{n-1} \left| 1 - \frac{z}{2^{k_m}} \right|^{k_m} = \frac{|z|^{k_1+\dots+k_{n-1}}}{2^{k_1^2+k_2^2+\dots+k_{n-1}^2}} \prod_{m=1}^{n-1} \left| 1 - \frac{2^{k_m}}{z} \right|^{k_m}. \quad (5.6)$$

When  $n$  is large then  $2^{k_n-1} < |z| < 2^{k_n+1}$  by Lemma 4 and so, if the  $k_n$  increase sufficiently rapidly to ensure that the final product in (5.6) lies between  $\frac{1}{H}$  and  $H$ , we obtain on the boundary of  $D_n$ ,

$$H \cdot \frac{2^{(k_n-1)(k_1+\dots+k_{n-1})}}{2^{k_1^2+\dots+k_{n-1}^2}} < \prod_{m=1}^{n-1} \left| 1 - \frac{z}{2^{k_m}} \right|^{k_m} < H \cdot \frac{2^{(k_n+1)(k_1+\dots+k_{n-1})}}{2^{k_1^2+\dots+k_{n-1}^2}}. \quad (5.7)$$

Again, from Lemma 4, it follows that on boundary of  $D_n$  when  $n$  is large,

$$H < \prod_{n=m+1}^{\infty} \left| 1 - \frac{z}{2^{k_m}} \right|^{k_m} < H. \quad (5.8)$$

From (5.6), (5.7) and (5.8) we find that on the boundary of  $D_n$  when  $n$  is large

$$H \cdot 2^{k_n} \left\{ \frac{2^{\frac{1}{k_n}(k_1^2 + \dots + k_{n-1}^2)}}{2^{\left(1 + \frac{1}{k_n}\right)(k_1 + \dots + k_{n-1})}} \right\} < |z - 2^{k_n}| < H 2^{k_n} \left\{ \frac{2^{\frac{1}{k_n}(k_1^2 + \dots + k_{n-1}^2)}}{2^{\left(1 - \frac{1}{k_n}\right)(k_1 + \dots + k_{n-1})}} \right\}.$$

From these inequalities the lemma follows provided the  $k_n$  increase sufficiently rapidly to ensure that

$$k_1^2 + \dots + k_{n-1}^2 = O(k_n) \quad (n \rightarrow \infty). \quad (5.9)$$

**Lemma 6.** *For large  $n$  we have in  $2^{k_{n-1}} \leq |z| \leq 2^{k_{n+1}}$ , but outside  $D_n$ , provided that  $k_n$  increases quickly enough,*

$$\left| \frac{f'(z)}{f(z)} \right| < H \frac{k_n 2^{k_1 + \dots + k_{n-1}}}{|z|}.$$

We have

$$\frac{f'(z)}{f(z)} = \sum_{m=1}^{\infty} \frac{k_m}{z - 2^{k_m}}.$$

If the  $k_n$  increase sufficiently rapidly then, for  $2^{k_{n-1}} \leq |z| \leq 2^{k_{n+1}}$

$$\begin{aligned} \sum_{m=1}^{n-1} \frac{k_m}{z - 2^{k_m}} &\leq \sum_{m=1}^{n-1} \frac{k_m}{2^{k_{n-1}} - 2^{k_m}} \\ &< \frac{2}{2^{k_{n-1}}} \sum_{m=1}^{n-1} k_m \\ &< H \frac{k_n}{2^{k_n}}. \end{aligned} \quad (5.10)$$

Also,

$$\begin{aligned} \sum_{m=n+1}^{\infty} \frac{k_m}{|z - 2^{k_m}|} &\leq \sum_{m=n+1}^{\infty} \frac{k_m}{2^{k_m} - 2^{k_{n+1}}} \\ &< H \sum_{m=n+1}^{\infty} \frac{k_m}{2^{k_m}} \\ &< \frac{H}{2^{k_n}} \sum_{m=n+1}^{\infty} \frac{k_m}{2^{\frac{k_m}{2}}} \\ &< \frac{H}{2^{k_n}}. \end{aligned} \quad (5.11)$$

From Lemma 5 it follows that if the  $k_n$  increase rapidly enough then

$$\frac{k_n}{|z - 2^{k_n}|} < H \frac{k_n 2^{k_1 + \dots + k_{n-1}}}{2^{k_n}}. \quad (5.12)$$

From (5.10), (5.11) and (5.12) the lemma follows.

**Lemma 7.** *If the  $k_n$  increase sufficiently rapidly then for  $2^{k_n+1} \leq |z| \leq 2^{k_{n+1}-1}$  we have*

$$\varrho(f(z)) = O\left(\frac{1}{|z|}\right).$$

If the  $k_n$  increase quickly enough then on  $|z| = 2^{k_n+1}$  we obtain

$$\begin{aligned} \frac{|f'|}{|f|^2} &< H \frac{k_n 2^{k_1 + \dots + k_{n-1}}}{|z|^2} \\ &< \frac{H}{|z|} \end{aligned}$$

by Lemmas 4 and 6. The same inequality is also true for  $|z| = 2^{k_{n+1}-1}$ . Now  $\left|\frac{zf'(z)}{f^2(z)}\right|$  is subharmonic in  $2^{k_n+1} \leq |z| \leq 2^{k_{n+1}-1}$  and since it is bounded by  $H$  on the boundary it is bounded by  $H$  inside the annulus. Therefore in  $2^{k_n+1} \leq |z| \leq 2^{k_{n+1}-1}$ ,

$$\varrho(f(z)) < \frac{|f'(z)|}{|f^2(z)|} = O\left(\frac{1}{|z|}\right).$$

**Lemma 8.** *In  $2^{k_n-1} \leq |z| \leq 2^{k_{n+1}}$  we have*

$$\varrho(f(z)) \leq H \frac{k_n 2^{k_1 + \dots + k_{n-1}}}{|z|}$$

*provided the  $k_n$  increase quickly enough.*

In  $2^{k_n-1} \leq |z| \leq 2^{k_{n+1}}$  but outside  $D_n$  it follows, if the  $k_n$  increase quickly enough, that

$$\left|\frac{zf'(z)}{f^2(z)}\right| < H k_n 2^{k_1 + \dots + k_{n-1}} \quad (5.13)$$

by Lemmas 4 and 6 and the use of subharmonicity as before. Hence the lemma is true in this region.

On the boundary of  $D_n$  we get

$$|zf'(z)| < H k_n 2^{k_1 + \dots + k_{n-1}} \quad (5.14)$$

and so, by the maximum modulus principle, this also holds inside  $D_n$ . From (5.13) and (5.14) the lemma follows.

Given  $\varphi(r)$  as in the theorem choose an increasing sequence of integers  $k_n$  so that the above results hold and also

$$2^{k_1} + \dots + 2^{k_{n-1}} < \varphi(2^{k_n}).$$

Then from Lemmas 7 and 8 we see that

$$\limsup_{r \rightarrow \infty} \frac{r \mu(r, f)}{\varphi(r) \log r} < \infty,$$

since  $\varphi(r)$  is increasing.

This completes the proof of the theorem. It should perhaps be pointed out that given  $\varphi(r)$  where  $\varphi(r) \rightarrow \infty (r \rightarrow \infty)$  it is not difficult to find a  $\psi(r)$  such that  $\psi(r) \rightarrow \infty (r \rightarrow \infty)$ ,  $\varphi(r) \geq \psi(r)$  and  $\psi(r)$  is increasing. Consequently  $\varphi(r)$  was assumed to be increasing in the theorem only for convenience.

## 5.2. Proof of Theorem 7

A number of lemmas are required.

**Lemma 9.** *If  $A > 1$  and  $f(z) = \prod_1^\infty \left(1 + \frac{z}{e^{nA}}\right)^{[A^n]}$  then  $f(z)$  is a function of very regular growth and order  $\frac{\log A}{A}$ .*

For  $e^{nA} \leq |z| \leq e^{(n+1)A}$  we have

$$\begin{aligned} \log M(r, f) &\geq \log |f(e^{nA})| \\ &\geq (A^n - 1) \log 2. \end{aligned} \tag{5.15}$$

Also, in this range,

$$\begin{aligned} \log M(r, f) &\leq \log M(e^{(n+1)A}, f) \\ &\leq \sum_{m=1}^{n+1} A^m \log \{1 + e^{(n+1-m)A}\} + \sum_{m=n+2}^\infty A^m \log \{1 + e^{(n+1-m)A}\} \\ &\leq \sum_{m=1}^{n+1} A^m \{\log 2 + (n+1-m)A\} + \sum_{m=n+2}^\infty A^m e^{-(m-n-1)A} \\ &\leq \frac{A^{n+2} \log 2}{A-1} + A^{n+1} \sum_{\nu=1}^n \frac{\nu}{A^\nu} + A^{n+1} \sum_{\nu=1}^\infty A^\nu e^{-\nu A} \\ &< K(A) A^n. \end{aligned} \tag{5.16}$$

From (5.15) and (5.16) it follows that for  $e^{nA} \leq |z| \leq e^{(n+1)A}$

$$\frac{(A^n - 1) \log 2}{A^{n+1}} < \frac{\log M(r, f)}{r^{(\log A)/A}} < \frac{K(A) \cdot A^n}{A^n},$$

and so the result follows.

**Lemma 10.** *If  $\varphi_n(z) = \left( \sum_1^{n-2} + \sum_{n+1}^{\infty} \right) [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right|$  then for*

$$\frac{e^{nA}}{2} \leq |z| \leq 2e^{nA},$$

$$-\eta A^n \leq \varphi_n(z) \leq \eta A^n$$

where  $\eta = \eta(A) > 0$  and  $\eta \rightarrow 0 (A \rightarrow \infty)$ ;  $\eta$  is not necessarily the same at each occurrence.

We have, in the range of the lemma,

$$\begin{aligned} \sum_1^{n-2} [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right| &\leq \sum_1^{n-2} A^m \log \left( 1 + \frac{2e^{nA}}{e^{mA}} \right) \\ &\leq \sum_1^{n-2} A^m \{ \log 4 + (n-m)A \} \\ &\leq \frac{A^{n-1} \log 4}{A-1} + A^{n-1} \sum_{\nu=0}^{n-3} \frac{\nu+2}{A^\nu} \\ &\leq \eta(A) \cdot A^n. \end{aligned} \tag{5.17}$$

Also, in the above range,

$$\begin{aligned} \sum_{n+1}^{\infty} [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right| &\leq \sum_{n+1}^{\infty} A^m \log \left( 1 + \frac{2e^{nA}}{e^{mA}} \right) \\ &\leq 2 \sum_{n+1}^{\infty} A^m e^{(n-m)A} \\ &= 2A^n \sum_{\nu=1}^{\infty} (A e^{-A})^\nu \\ &\leq \eta(A) A^n. \end{aligned} \tag{5.18}$$

From (5.17) and (5.18) the right hand inequality of the lemma follows.

In the range of the lemma we also have, if  $e^{2A} \geq 4$ ,

$$\begin{aligned} \sum_1^{n-2} [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right| &\geq \sum_1^{n-2} [A^m] \log \left( \frac{e^{nA}}{2e^{mA}} - 1 \right) \\ &\geq 0, \end{aligned} \tag{5.19}$$

and, if  $e^A > 4$ ,



$$\begin{aligned}
\sum_{n+1}^{\infty} [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right| &\geq \sum_{n+1}^{\infty} A^m \log \left( 1 - \frac{2e^{nA}}{e^{mA}} \right) \\
&> -4 \sum_{n+1}^{\infty} A^m e^{(n-m)A} \\
&= -4A^n \sum_{\nu=1}^{\infty} (Ae^{-A})^{\nu} \\
&\geq -\eta(A)A^n,
\end{aligned} \tag{5.20}$$

From (5.19) and (5.20) the left hand inequality of the lemma follows.

**Lemma 11.** For  $|z| = \frac{e^{nA}}{2}$  and  $|z| = 2e^{nA}$ ,

$$\left(\frac{1}{4} - \eta\right)A^n \leq \log |f(z)| \leq (3 + \eta)A^n.$$

If  $|z| = \frac{e^{nA}}{2}$  we have

$$\begin{aligned}
[A^{n-1}] \log \left| 1 + \frac{z}{e^{(n-1)A}} \right| &\leq A^{n-1} \log \left( 1 + \frac{e^A}{2} \right) \\
&\leq A^{n-1} (\log 2 + A) \\
&\leq (1 + \eta)A^n.
\end{aligned} \tag{5.21}$$

Also for  $|z| = \frac{e^{nA}}{2}$ ,

$$\begin{aligned}
[A^n] \log \left| 1 + \frac{z}{e^{nA}} \right| &\leq A^n \log 3/2 \\
&\leq A^n.
\end{aligned} \tag{5.22}$$

From (5.21) and (5.22) and Lemma 10, the right hand inequality of Lemma 11 follows for  $|z| = \frac{e^{nA}}{2}$ .

We have for  $|z| = \frac{e^{nA}}{2}$ , if  $e^A > 4$ ,

$$\begin{aligned}
[A^{n-1}] \log \left| 1 + \frac{z}{e^{(n-1)A}} \right| &\geq [A^{n-1}] \log \left( \frac{e^A}{2} - 1 \right) \\
&\geq (A^{n-1} - 1) (A - \log 4) \\
&\geq (1 - \eta)A^n;
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
[A^n] \log \left| 1 + \frac{z}{e^{nA}} \right| &> -A^n \log 2 \\
&> -\frac{3}{4} A^n.
\end{aligned} \tag{5.24}$$

From (5.23) and (5.24) and Lemma 10, the left hand inequality of the Lemma 11 follows for  $|z| = \frac{e^{nA}}{2}$ .

The result for  $|z| = 2e^{nA}$  follows in a similar manner to the above.

**Lemma 12.** *If  $z$  satisfies  $|z + e^{nA}| \geq \frac{e^{nA}}{4}$  and  $\frac{e^{nA}}{2} \leq |z| \leq 2e^{nA}$  then*

$$\left| \frac{f'(z)}{f(z)} \right| \leq (4 + \eta) \frac{A^n}{e^{nA}}.$$

We have

$$\frac{f'(z)}{f(z)} = \sum_1^{\infty} \frac{[A^m]}{z + e^{mA}}.$$

For  $|z| \geq \frac{e^{nA}}{2}$ , if  $e^A \geq 4$ ,

$$\begin{aligned}
\left| \sum_1^{n-1} \frac{[A^m]}{z + e^{mA}} \right| &\leq \sum_1^{n-1} \frac{A^m}{\frac{e^{nA}}{2} - e^{mA}} \\
&\leq \frac{4}{e^{nA}} \sum_1^{n-1} A^m \\
&< \frac{4A^n}{(A-1)e^{nA}} \\
&\leq \eta \frac{A^n}{e^{nA}};
\end{aligned} \tag{5.25}$$

and for  $|z| \leq 2e^{nA}$ , if  $e^A \geq 4$ ,

$$\begin{aligned}
\left| \sum_{n+1}^{\infty} \frac{[A^m]}{z + e^{mA}} \right| &\leq \sum_{n+1}^{\infty} \frac{A^m}{e^{mA} - 2e^{nA}} \\
&\leq 2 \sum_{n+1}^{\infty} \frac{A^m}{e^{mA}} \\
&= 2 \frac{A^n}{e^{nA}} \sum_{v=1}^{\infty} (Ae^{-A})^v \\
&\leq \eta \frac{A^n}{e^{nA}}.
\end{aligned} \tag{5.26}$$

Finally, if  $|z + e^{nA}| \geq \frac{e^{nA}}{4}$ , then

$$\frac{[A^n]}{|z + e^{nA}|} \leq \frac{4A^n}{e^{nA}}. \quad (5.27)$$

From (5.25), (5.26) and (5.27) the lemma follows.

**Lemma 13.** For  $\frac{e^{nA}}{2} \leq |z| \leq 2e^{nA}$ ,

$$\mu(r, f) \leq K(A) \frac{A^n}{r},$$

provided  $A$  is sufficiently large.

When  $A$  is large enough we see from Lemma 11 that the set  $|f(z)| < 1$  splits into a number of components. Each zero  $e^{nA}$  is contained in a component  $D_n$ , say, and  $D_n$  lies in  $\frac{e^{nA}}{2} \leq |z| \leq 2e^{nA}$ .

First of all we show that when  $A$  is large the disc  $|z + e^{nA}| \leq \frac{e^{nA}}{4}$  is contained in  $D_n$ . From Lemma 10 it follows that in this disc,

$$\begin{aligned} \log |f(z)| &\leq [A^{n-1}] \log \left| 1 + \frac{z}{e^{(n-1)A}} \right| + [A^n] \log \left| 1 + \frac{z}{e^{nA}} \right| + \eta A^n \\ &\leq A^{n-1} \log \left( 1 + \frac{5}{4} e^A \right) - (A^n - 1) \log 4 + \eta A^n \\ &\leq A^{n-1} \left( \log \frac{5}{2} + A \right) - (A^n - 1) \log 4 + \eta A^n \\ &< 0, \end{aligned}$$

provided  $A$  is large enough, independently of  $n$ . Hence we arrive at the desired conclusion.

From Lemma 12 and the above it follows that when  $A$  is large then on the boundary of  $D_n$ ,

$$|f'(z)| = \left| \frac{f'(z)}{f(z)} \right| \leq (4 + \eta) \frac{A^n}{e^{nA}}.$$

Therefore in  $D_n$  and on its boundary,

$$\varrho(f(z)) \leq |f'(z)| \leq (4 + \eta) \frac{A^n}{e^{nA}}. \quad (5.28)$$

In the annulus  $\frac{e^{nA}}{2} \leq |z| \leq 2e^{nA}$  outside  $D_n$  it follows that when  $A$  is large

$$\varrho(f(z)) \leq \left| \frac{f'(z)}{f(z)} \right| \leq (4 + \eta) \frac{A^n}{e^{nA}}, \quad (5.29)$$

by Lemma 12.

Since  $\frac{1}{e^{nA}} \leq \frac{2}{r}$  for  $\frac{e^{nA}}{2} \leq r \leq 2e^{nA}$  the lemma follows from (5.28) and (5.29).

**Lemma 14.** *For large  $A$ , if  $2e^{nA} \leq r \leq \frac{e^{(n+1)A}}{2}$  then*

$$\mu(r, f) < \frac{K(A)}{r}.$$

From Lemmas 11 and 12 it follows that

$$\left| \frac{zf'(z)}{f(z)^2} \right| < K(A)$$

on the boundary of  $2e^{nA} \leq |z| \leq \frac{e^{(n+1)A}}{2}$ . Since the function on the left above is subharmonic in the annulus it follows that the inequality holds throughout the annulus. Hence the lemma follows because  $\varrho(f(z)) \leq \frac{|f'(z)|}{|f(z)|^2}$ .

**5.3.** Before completing the proof of Theorem 7, we observe that the constants  $K(A)$  appearing in Lemmas 13 and 14 remain bounded as  $A \rightarrow \infty$ . From Lemmas 11, 13 and 14 it follows that

$$\limsup_{r \rightarrow \infty} \frac{r\mu(r, f)}{\log M(r, f)} < B,$$

where  $B$  is an absolute constant for all  $f(z)$  for which  $A \geq A_0$ ,  $A_0$  being some fixed value.

We proceed to prove Theorem 7.

If  $0 < \sigma < \frac{\log A_0}{A_0}$  in Theorem 7 we take  $f(z)$  as above with  $A$  given by  $\sigma = \frac{\log A}{A}$ . If  $\sigma > \frac{\log A_0}{A_0}$  we proceed as follows. Let  $A_1 > A_0$  be defined by  $2 \frac{\log A_1}{A_1} = \frac{\log A_0}{A_0}$ . Let  $n$  be the smallest positive integer such that  $\frac{\sigma}{n} \leq \frac{\log A_0}{A_0}$ . Then, since  $n \geq 2$ ,  $\frac{\sigma}{n-1} \geq \frac{\log A_0}{A_0}$  and so  $\frac{\sigma}{n} \geq \frac{n-1}{n} \frac{\log A_0}{A_0} \geq \frac{1}{2} \frac{\log A_0}{A_0}$ . Therefore  $\frac{\sigma}{n} = \frac{\log A}{A}$  where  $A_1 \leq A \leq A_0$ .

We now take, as a function for Theorem 7,  $F(z) = f(z^n)$  where  $f(z)$  is constructed as in Lemma 9 with this value of  $A$ . Then

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{r \mu(r, F)}{\log M(r, F)} &= \limsup_{r \rightarrow \infty} \frac{n r^n \mu(r^n, f)}{\log M(r^n, f)} \\ &\leq n B \\ &= \frac{\log A}{A} \cdot n \cdot \frac{B \cdot A}{\log A} \\ &\leq \frac{2 A_0 B}{\log A_0} \cdot \sigma. \end{aligned}$$

Thus the theorem is proved for  $0 < \sigma < \infty$ .

It can be shown by the same methods as above that if  $K$  is large enough then

$$F(z) = \prod_{n=1}^{\infty} (1 + z e^{-K n^2})^{n^n}$$

is a function of order 0 satisfying the conclusion of the theorem.

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