

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 40 (1965-1966)

Artikel: Bilinear Forms on k -Vektorspaces of Denumerable Dimension in the Case of $\text{char}(k) = 2$.
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DOI: <https://doi.org/10.5169/seals-30638>

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Bilinear Forms on k -Vectorspaces of Denumerable Dimension in the Case of $\text{char } (k) = 2$

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Introduction. The classification, up to metric isomorphism, of finite dimensional k -vector spaces E , supplied with a symmetric bilinear form $\Phi: E \times E \rightarrow k$, is a rather difficult problem; it has been solved for particular fields k , such as the field of rationals, reals, p -adic numbers or function fields in one variable over a finite constant field. KAPLANSKY has shown that for k -vector spaces (E, Φ) of a denumerable (algebraic) dimension, these problems vanish in a large number of cases, E admitting an orthonormal basis for an extensive class of underlying fields ([4]; for an investigation of such fields see [3]). In the denumerable case, an exceptional role is once more played by the fields of characteristic 2. For perfect fields of characteristic 2 KAPLANSKY has proved the following

Theorem. For every \aleph_0 -dimensional k -space (E, Φ) , Φ a non degenerate bilinear form, precisely one of the following four possibilities holds: (1) E possesses an orthonormal basis, (2) E possesses a symplectic basis, (3) E is an orthogonal sum $E = E_0 \oplus L$ where E_0 is spanned by a symplectic basis and L is one-dimensional, (4) E is an orthogonal sum $E = E_0 \oplus L$, where E_0 has a symplectic basis and L is two-dimensional, spanned by an orthogonal basis ([4] p. 15). KAPLANSKY has asked what becomes of this theorem if the assumption that every element in the coefficient field be a square, is dropped.

In the following, we investigate the case of an arbitrary field of characteristic 2. Complete results as regards the classification problem are obtained for all fields k of finite dimension over their subfields k^2 (Theorem 2). As a side-result we obtain an invariant characterization of the k -spaces (E, Φ) of denumerable dimension which admit of orthogonal bases, k an arbitrary field of characteristic 2 (Theorem 3).

I. Notations and Results

Let k be a commutative field. A k -vector space (E, Φ) is a k -vector space E supplied with a symmetric bilinear form $\Phi: E \times E \rightarrow k$. (E, Φ) is called semisimple if $E \cap E^\perp = (0)$. In the following, an isomorphism $(E, \Phi) \cong (G, \psi)$ is a vector space isomorphism $\vartheta: E \rightarrow G$ such that $\psi(\vartheta x, \vartheta y) = \Phi(x, y)$

for all $x, y \in E$. If there is no risk of confusion, we simply talk about E instead of (E, Φ) and, we write (x, y) and $\|x\|$ respectively for $\Phi(x, y)$ and the "length" $\Phi(x, x)$ of $x \in E$. A subspace H of (E, Φ) is always considered as being supplied with the restriction $\Phi|_H$ of Φ to H . The radical of H ($\text{rad } H$) is defined as $H \cap H^\perp$. A subspace $H \subset E$ is said to be closed if $H^{\perp\perp} = H$. If H is a closed subspace of (E, Φ) and F a finite dimensional subspace of (E, Φ) then $H + F$ is closed.

2. The following lemma, proved by KAPLANSKY in [4], will be used in the proof of Lemma 4 below. Lemma: Let (E, Φ) be a semi-simple k -vector space of infinite dimension over an arbitrary field k . Let furthermore F be a finite dimensional subspace of E , spanned by the basis f_1, \dots, f_n , V a subspace of E with $V^\perp = (0)$. Then there exists a vector $x \in E$ with $x \in V$, $x \notin V \cap F$ and $\Phi(x, f_i) = \beta_i$ for arbitrarily prescribed $\beta_i \in k$.

3. Bases being the central object below, the following notations prove convenient. If $\alpha_1, \dots, \alpha_n \in k$ then $\langle \alpha_1, \dots, \alpha_n \rangle$ is an n -dimensional k -space (E, Φ) possessing an orthogonal basis e_1, e_2, \dots, e_n with $\|e_i\| = \alpha_i$. "P" invariably denotes a hyperbolic plane, i.e., a two-dimensional space (E, Φ) having a basis e_1, e_2 with $\|e_1\| = \|e_2\| = 0$ and $(e_1, e_2) = 1$. ΣP is an orthogonal sum of hyperbolic planes (i.e., a space spanned by a symplectic basis). $\Sigma \langle \alpha \rangle$ is a space (E, Φ) spanned by an orthogonal basis (finite or infinite), each basis vector of length α , $\alpha \neq 0$. If $\Sigma \langle \alpha \rangle$ is of denumerable dimension, we denote it by $E_{(\alpha)}$.

4. In the following investigations, k will always be a field of characteristic 2 unless stated otherwise. Every such field is a vector space over its subfield k^2 of squares.

5. If (E, Φ) is a semi-simple k -vector space with $\dim E \leq \aleph_0$ then E is an orthogonal sum $\Sigma P \oplus E_0$, where E_0 is spanned by an orthogonal basis.

6. Let (E, Φ) be a k -vector-space. We have $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E$ as $\text{char } k = 2$. Thus, if H is a subspace of E , then the range of the restriction $\|H\|$ is a subspace of the k^2 -vector space k . This range will be denoted by " $\|H\|$ " throughout. In particular, the set of all isotropic vectors x in E ($\|x\| = 0$) is a vector space. This subspace of E is invariably denoted by E_* . (The subspace of vectors satisfying condition (T) in [1] p. 66.) The subspaces E_* , E_*^\perp , $E_*^{\perp\perp}$, $\text{rad } E_*$ etc. will play an important role since they are invariant subspaces under orthogonal transformations. We notice that $\text{rad}(E_*^\perp) \subset E_*$ by the definition of E_* , hence $\text{rad}(E_*^\perp) \subset \text{rad}(E_*^\perp) \cap E_* = \text{rad } E_*$. Therefore $\text{rad } E_*^\perp = \text{rad } E_*$, the converse inclusion being trivial. This means in particular that $\text{rad } E_* (= \text{rad}(E_*^\perp) = (E_* + E_*^\perp)^\perp)$ is a closed space.

II. Bases

Let us mention a few words about the fields. When describing k -spaces (E, Φ) in terms of orthogonal bases, it is clear that the non-square elements of k play an important role. Let g_k be the multiplicative group of non-zero elements in k modulo square factors. If g_k is finite, then its order is a power of 2 since every element of g_k is of order 2. If $\text{char } k \neq 2$ then one can find, for every natural n , fields with g_k of order 2^n (even among the denumerable fields, [3]). On the other hand, if $\text{char } k = 2$ then k^2 is a subfield of k and the elements of g_k are precisely the straight lines through the origin of the k^2 -vector space k . In other words, the order of g_k is either 1 or equal to $\text{card } (k)$. In particular, since g_k is of order 1 for finite fields, g_k is either of order 1 or infinite. In the following discussion of isomorphisms between \aleph_0 -dimensional k -spaces the fields with finite dimension $[k:k^2]$ over their subfields k^2 are seen to play a special role. Since a simple characterization of all non isomorphic spaces over such fields can be given (Theorem 2), let us mention a few elementary facts about these fields.

Clearly, if $[k:k^2]$ is finite, then $[k:k^2]$ is a power of 2. Furthermore, if \bar{k} is a finite algebraic extension of k , $[k:k^2]$ finite, then $[\bar{k}:\bar{k}^2] = [k:k^2]$ ($[\bar{k}:\bar{k}^2] = [\bar{k}:\bar{k}^2][\bar{k}^2:k^2] = [\bar{k}:k][k:k^2]$ and $[\bar{k}^2:k^2] = [\bar{k}:k]$). From this follows that $[\bar{k}:\bar{k}^2] \leq [k:k^2]$ for an arbitrary algebraic extension \bar{k} of k . ($<$ is witnessed by the transition to the algebraic closure.) On the other hand, if $\bar{k} = k(\xi_1, \dots, \xi_n)$, where ξ_1, \dots, ξ_n are independent transcendentals over k , we have $[\bar{k}:\bar{k}^2] = [k:k^2] \cdot 2^n$ (a basis for \bar{k} over \bar{k}^2 is given by the elements $\alpha_i \xi_1^{\varepsilon_1} \xi_2^{\varepsilon_2} \dots \xi_n^{\varepsilon_n}$, $\varepsilon_j = 0, 1$ and α_i running through a k^2 basis of k). In particular:

If k is a field of characteristic 2 with finite $[k:k^2]$, then $[\bar{k}:\bar{k}^2]$ is finite for an arbitrary over field \bar{k} of k , provided its transcendence degree over k is finite. The fields k with finite $[k:k^2]$ form thus a considerable class.

Let again k be an arbitrary field of characteristic 2. It is well known that WITT's Cancellation Theorem does not hold for bilinear forms in the case of $\text{char } k = 2$. Instead, we have the following orthogonal isomorphisms:

Lemma 1. $\langle \alpha \rangle \oplus \langle \alpha, \alpha \rangle \cong \langle \alpha \rangle \oplus P$ ($0 \neq \alpha \in k$, P a hyperbolic plane and all the sums orthogonal).

Lemma 2. $\langle \alpha, \alpha \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle \cong \langle \bar{\alpha}, \bar{\alpha} \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle$ provided that the elements $\{\alpha, \beta_i\}_{i \in I}$ are independent over k^2 and span the same subspace of k (over k^2) as the elements $\{\bar{\alpha}, \beta_i\}_{i \in I}$ ($\text{card } I$ is finite or infinite; all sums are orthogonal).

Proofs. 1. Let e_1, e_2, e_3 be an orthogonal basis of $\langle \alpha \rangle \oplus \langle \alpha, \alpha \rangle$ with $\|e_i\| = \alpha$. Introduce a new basis $\bar{e}_1, \bar{e}_2, \bar{e}_3$ by $\bar{e}_1 = e_1 + e_2 + e_3$, $\bar{e}_2 = e_1 + e_2$, $\bar{e}_3 = \alpha^{-1}(e_2 + e_3)$.

2. Let $e_{00}, e_0, e_i (i \in I)$ be an orthogonal basis of $\langle \alpha, \alpha \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle$ with $\|e_{00}\| = \|e_0\| = \alpha$, $\|e_i\| = \beta_i$. Since $\{\alpha, \beta_i\}_{i \in I}$ and $\{\bar{\alpha}, \beta_i\}_{i \in I}$ span the same subspace of k we have $\bar{\alpha} = \lambda_0^2 \alpha + \sum_{i=1}^n \lambda_i^2 \beta_i$ for suitable $\lambda_0, \lambda_1, \dots, \lambda_n$. Since the elements $\{\bar{\alpha}, \beta_i\}_{i \in I}$ are independent over k^2 we have $\lambda_0 \neq 0$. For a fixed choice of $\lambda_0, \lambda_1, \dots, \lambda_n$ introduce the following basis

$$\begin{aligned} \bar{e}_{00} &= \frac{\bar{\alpha}}{\lambda_0 \alpha} e_{00} + \left(\lambda_0 + \frac{\bar{\alpha}}{\lambda_0 \alpha} \right) e_0 + \sum_2^n \lambda_i e_i \\ \bar{e}_0 &= \lambda_0 e_0 + \sum_2^n \lambda_i e_i \\ 2 \leq i \leq n : \bar{e}_i &= \frac{\lambda_i \beta_i}{\lambda_0 \alpha} (e_{00} + e_0) + e_i \\ n < i : \bar{e}_i &= e_i. \end{aligned}$$

We shall list a few consequences some of which will be of importance later.

Corollary 1. (i) $\bigoplus_{i \in I} E_{(\alpha_i)} \oplus \Sigma P = \bigoplus E_{(\alpha_i)}$ (all sums orthogonal).

(ii) $\langle \alpha_1 \alpha_1 \alpha_2 \alpha_2 \dots \alpha_m \alpha_m \rangle \cong \langle \bar{\alpha}_1 \bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_2 \dots \bar{\alpha}_m \bar{\alpha}_m \rangle$ provided the elements $\alpha_1, \dots, \alpha_m$ are independent over k^2 and span the same subspace of k (over k^2) as the elements $\bar{\alpha}_1, \dots, \bar{\alpha}_m$.

(iii) $\bigoplus_{j=1}^m \langle \alpha_j \alpha_j \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle \cong \bigoplus_{j=1}^m \langle \bar{\alpha}_j \bar{\alpha}_j \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle$ provided the elements $\{\alpha_1, \dots, \alpha_m, \beta_i\}_{i \in I}$ are independent over k^2 and span the same subspace of k as the elements $\{\bar{\alpha}_1, \dots, \bar{\alpha}_m, \beta_i\}_{i \in I}$ (card I is finite or infinite, m is a natural number, all sums are orthogonal).

We remark that the transformation of Lemma 2 does not lend itself to a generalization of (ii) and (iii) to the case of infinite m . (We have not succeeded in proving or disproving the infinite analogue of (ii) by any other means; cf. Proposition 3.)

Another lemma which we shall use is the following:

Lemma 3. Let (E, Φ) be a k -vector space of denumerable dimension, semi-simple with respect to the bilinear form $\Phi: E \times E \rightarrow k$ and k a field of arbitrary characteristic. Let furthermore R be a closed, totally isotropic subspace of E ($R^{\perp\perp} = R$).

and $R \subset R^\perp$). There exists a basis $(r_i)_{i \in I}$ of R and a subspace R' of E admitting an orthogonal basis $(r'_i)_{i \in I}$ such that $R \oplus R'$ decomposes into an orthogonal sum of semi-simple planes $K_i = k(r_i, r'_i)$,

$$R \oplus R' = \bigoplus_{i \in I} K_i \quad \text{card } I = \dim R = \dim R'$$

and, furthermore, such that $R \oplus R'$ admits of an orthogonal supplement in E : $E = (R \oplus R') \oplus H$, $H \perp R \oplus R'$.

In the case of $\text{char } k \neq 2$, the planes K_i are hyperbolic and $R \oplus R'$ thus possesses a symplectic basis (cf. BOURBAKI, *Formes Sesquilinéaires* p. 78).

Proof. Let S and T be finite dimensional semi-simple subspaces with the following properties:

$$S \perp T, T \subset R^\perp, S = \bigoplus_{i=1}^n K_i, K_i = k(r_i, r'_i) \text{ and } r_i \in R \quad (1)$$

$$(T \oplus S) \cap R = k(r_i)_{1 \leq i \leq n}. \quad (2)$$

Let $(e_m)_{m \geq 1}$ be some fixed basis of the space E and let e_m be the first basis vector not contained in $S \oplus T$. We construct finite dimensional spaces K and L in $(S \oplus T)^\perp$ such that $S' = S \oplus K$ and $T' = T \oplus L$ satisfy the properties (1) and (2) with S' and T' in lieu of S and T and such that $e_m \in S' \oplus T'$. In this fashion we obtain a decomposition of E of the required form:

$$E = \cup S \oplus T = (\cup S) \oplus (\cup T), \quad H = \cup T \text{ and } R \oplus R' = \cup S.$$

Since $S \oplus T$ is semi-simple and finite dimensional, we may decompose e_m : $e_m = e'_m + e''_m$ with $e'_m \in S \oplus T$ and $e''_m \perp S \oplus T$. Thus we may without loss of generality assume that $e_m \perp S \oplus T$.

First case. $e_m \in R$. Therefore $\|e_m\| = 0$ and, since $(S \oplus T)^\perp$ is semi-simple, there exists r' with $(e_m, r') \neq 0$. The space $k(e_m, r')$ is semi-simple and we put $S' = S + k(e_m, r')$ and $T' = T$. We have to determine $(T' + S') \cap R$. Let $r \in (T' \oplus S') \cap R$, $r = t + s + \lambda e_m + \mu r'$ with $t \in T$, $s \in S$ and $r \in R$. Since $T \subset R^\perp$ we obtain $0 = (v, R) = (t, R)$ hence $t = 0$ as T is semi-simple. Therefore, (since $R \subset R^\perp$) we obtain $0 = (r, e_m) = \mu(e_m, r')$. Thus $\mu = 0$ and $v = s + \lambda e_m$. Since $e_m \in R$ in our case therefore $s \in R$ i.e., $s \in S \cap R = k(r_i)_{i \leq n}$ by (2). Thus $(T' \oplus S') \cap R = k(r_1, \dots, r_m, e_m)$ which, upon relabeling e_m as r_{n+1} (and r' as r'_{n+1}), is (2). The remaining conditions are trivially satisfied.

Case 2. $e_m \notin R$ and $e_m \in R^\perp$. We first convince ourselves that $e_m \notin R + (S \oplus T)$; assume that $e_m = r + s + t$ with $r \in R$, $s \in S$ and $t \in T$. Since $e_m \perp S + T$ and $T \subset R^\perp$, we have in particular $0 = (e_m, T) = (t, T)$; hence $t = 0$ as T is semi-simple. Since $e_m \in R^\perp$ in the present case, and $R \subset R^\perp$, we obtain furthermore $0 = (e_m, R \cap S) = (s, R \cap S)$ i. e., $S \perp S \cap R$. From the explicit form of $S = \bigoplus k(r_i, r'_i)$ we see that necessarily $s \in R \cap S$. Thus $e_m = r + s \in R$, a contradiction. Since $(R + S + T)^{\perp\perp} = R + S + T$, we conclude from $e_m \notin R + S + T$ that $(R + S + T)^\perp \not\subset e_m^\perp$. Hence there exists a vector $t \in (R + S + T)^\perp = R^\perp \cap (S + T)^\perp$ with $(e_m, t) \neq 0$. Thus, if $\|e_m\| = 0$ then $k(e_m, t)$ is a semi-simple space and we put $S' = S$, $T' = T + k(e_m, t)$. If, on the other hand, $\|e_m\| \neq 0$, we simply put $S' = S$ and $T' = T + k(e_m)$. We have to determine $(T' \oplus S') \cap R$. Let, in the first case, $r \in T' \oplus S'$ i. e., $r = s + t + \lambda e_m + \mu t$ with $s \in S$, $t \in T$ and $r \in R$. Since $e_m \in R^\perp$ and $\|e_m\| = 0$ we find $0 = (r, e_m) = \mu(t, e_m)$, therefore $\mu = 0$. Since $t \in R^\perp \cap (S \oplus T)^\perp$ we then find $0 = (r, t) = \lambda(e_m, t)$. Hence $\lambda = 0$. This shows that $(T' \oplus S') \cap R = (T \oplus S) \cap R$. In the other case, $\|e_m\| \neq 0$, it is even simpler to verify that $(T' \oplus S') \cap R = (T \oplus S) \cap R$. The remaining conditions (1) are trivially satisfied for S' and T' .

Case 3. $e_m \notin R^\perp$. As in the second case one verifies that $e_m \notin R^\perp + S + T$. Since $(R^\perp + S + T)^{\perp\perp} = R^\perp + S + T$, we conclude from $e_m \notin R^\perp + S + T$ that $(R^\perp + S + T)^\perp \not\subset e_m^\perp$. In other words there exists a vector $r \in (R^\perp + S + T)^\perp = R^{\perp\perp} \cap (S \oplus T)^\perp = R \cap (S \oplus T)^\perp$ with $(e_m, r) \neq 0$. Since $r \in R$ we have $\|r\| = 0$ and the space $k(r, e_m)$ is semi-simple. We put $S' = S \oplus k(r, e_m)$ and $T' = T$. Upon relabeling r as r_{n+1} (and e_m as r'_{n+1}) the conditions (1) and (2) are verified as in case 1. Q. E. D.

Lemma 3 often finds application in the following situation. Suppose that G is a subspace of E such that the radical $R = G \cap G^\perp$ of G happens to be a closed subspace of E . We then have a decomposition $E = (R \oplus R') \oplus H$, $H \perp (R \oplus R')$. Furthermore, one can always find an algebraic complement L of R in G such that $L \subset H$. For, if L_0 is some algebraic complement of R in G then $L_0 \perp R$. Every vector $l_0 \in L_0$ has a decomposition $l_0 = r + r' + h$. Since $l_0 \perp R$ necessarily $r' = 0$. In other words, $L_0 \subset R \oplus H$ which shows that there is a complement L of R in G with $L \subset H$.

We are interested in decompositions of E of the following sort: E is an orthogonal sum $E = \bigoplus E_i$ such that the ranges $\|E_i\|$ of the summands are either 0 or 1-dimensional subspaces of the k^2 -vector space $\|E\|$ and such that the elements spanning the non trivial $\|E_i\|$ are linearly independent over k^2 . In other words,

$$E = \Sigma P \oplus \Sigma \langle \alpha_1 \rangle \oplus \Sigma \langle \alpha_2 \rangle \oplus \dots$$

where the P_i are hyperbolic planes and where the field elements $\alpha_1, \alpha_2, \dots$ are linearly independent over k^2 . In view of Lemma 1 we may assume that the summands $\Sigma \langle \alpha_i \rangle$ are either of infinite dimension or of dimension ≤ 2 . Thus, collecting 1-, 2- and \aleph_0 -dimensional summands we may rewrite the above decomposition as follows:

$$E = \Sigma P \oplus \bigoplus_{i \in I_1} E_{(\beta_i)} \oplus \bigoplus_{i \in I_2} \langle \gamma_i \gamma_i \rangle \oplus \bigoplus_{i \in I_3} \langle \delta_i \rangle \quad (1)$$

where all the field elements $\beta_i, \gamma_j, \delta_l$ together are independent over k^2 .

We shall determine those k -space (E, Φ) which admit of a decomposition of type (1). We first have

Proposition 1. *If E admits of a decomposition (1) then*

$$E_*^\perp \oplus E_*^{\perp\perp} = (\text{rad } E_*)^\perp. \quad (2)$$

Proof. Let for every $i \in I_1$ the space $E_{(\beta_i)}$ be spanned by the vectors $(e_i)_{i \geq 1} \cdot (E_{(\beta_i)})_*$ is spanned by the vectors $(e_{i1} + e_i)_{i \geq 1}$ and, the orthogonal complement of $(E_{(\beta_i)})_*$ in $E_{(\beta_i)}$ is (0) . Let furthermore, for every $i \in I_2$, $\langle \gamma_i \gamma_i \rangle$ be spanned by the vectors f_i, f'_i . Since all the elements $\beta_i, \gamma_j, \delta_e$ together are independent over k^2 (by assumption), we obtain for E_* from (1)

$$E_* = \Sigma P \oplus \bigoplus_{i \in I_1} E_{(\beta_i)*} \oplus \bigoplus_{i \in I_2} k(f_i + f'_i) \oplus (0).$$

Furthermore

$$E_*^\perp = (0) \oplus \bigoplus_{i \in I_1} k(f_i + f'_i) \oplus \bigoplus_{i \in I_3} \langle \delta_i \rangle \quad \text{and} \quad E_*^{\perp\perp} = \Sigma P \bigoplus_{I_1} E_{(\beta_i)*} \oplus \bigoplus_{I_2} k(f_i + f'_i).$$

From this we readily read off that (2) holds.

Condition (2) is not always satisfied. The simplest kind of counter-example is the following. Let E be spanned by the basis vectors $\{e_i\}_{i \geq 1} \cup \{f_i\}_{i \geq 1} \cup \{g_0\}$ and let Φ be defined on the basis as follows: $\|e_i\| = \alpha$ and $(e_i, e_j) = 0$ ($i \neq j, i, j \geq 1$), $\|f_i\| = \beta_i$ and $(f_i, f_j) = 0$ ($i \neq j, i, j \geq 1$), $\|g_0\| = \gamma$ and $(e_i, f_j) = 0$, $(e_i, g_0) = \alpha$, $(f_i, g_0) = \beta_i$, ($i, j \geq 1$) for $\alpha, \gamma, \beta_1, \beta_2, \dots$ independent over k^2 (a field with $[k: k^2] \geq \aleph_0$ is required). Here $\text{rad } E_* = 0$ and $(\text{rad } E_*)^\perp = E$, but $E_*^\perp + E_*^{\perp\perp}$ falls short of E by one dimension. We remark that (2) is equivalent to $E_*^\perp \oplus E_*^{\perp\perp}$ being closed.

We shall prove that the converse of Proposition 1 is true. This is accomplished by reducing the general case to the cases of spaces E with $E_*^\perp = (0)$ or $E_*^\perp = E_*$. We start out with these special cases.

Lemma 4. *Let (E, Φ) be a semi-simple space of denumerable dimension with $E_*^\perp = (0)$. Then for every $\alpha \in \|E\|$ and every orthogonal decomposition $E = H \oplus H^\perp$ with finite dimensional H we have $\alpha \in \|H^\perp\|$.*

Proof. Let $E = H \oplus H^\perp$ be any decomposition with finite dimensional H , furthermore α some arbitrarily fixed element in $\|E\|$. We apply Lemma 1.2 with E_* and H in the roles of V and F respectively. Since $\alpha \in \|E\|$, there exists some vector $x_0 \in E$ with $\|x_0\| = \alpha$. Hence there exists a vector $x \in E_*$ with $(x, f_i) = -(x_0, f_i)$, f_1, \dots, f_n a fixed basis of H . Therefore $(x_0 + x, f_i) = 0$ i.e., $x_0 + x \perp H$. Since $x \in E_*$ we have $\|x_0 + x\| = \|x_0\| = \alpha$.

Proposition 2. *Let (E, Φ) be a semi-simple space of denumerable dimension with $\|E\| \neq 0$. We have an orthogonal decomposition*

$$E = \bigoplus_{i \in I} E_{(\pi_i)}$$

where $\{\pi_i\}_{i \in I}$ is a k^2 -basis for $\|E\|$ if and only if $E_*^\perp = (0)$.

Proof. If E admits such a decomposition it is readily verified that $E_*^\perp = (0)$. Let us then assume that $E_*^\perp = (0)$. We construct a decomposition of E of the required type step by step. Let $F = \Sigma P \oplus \Sigma \langle \pi_1 \rangle \oplus \dots \oplus \Sigma \langle \pi_n \rangle$ be a finite dimensional subspace of E , the P_s hyperbolic planes and the field elements π_1, \dots, π_n linearly independent over k^2 . Let furthermore $(e_i)_{i \geq 1}$ be some fixed basis for the space E and assume that e_m is the first basis vector not contained in F . We shall construct a finite dimensional subspace H in F^\perp such that $e_m \in F \oplus H$ and $F' = F \oplus H$ is of the form $\Sigma P \oplus \Sigma \langle \pi_1 \rangle \oplus \dots \oplus \Sigma \langle \pi_r \rangle$ with π_1, \dots, π_r linearly independent over k^2 .

Since F is finite dimensional and semi-simple, we may decompose $e_m: e_m = e'_m + e''_m$ with $e'_m \in F$ and $e''_m \perp F$. Three cases are possible: $\|e''_m\| = 0$ and e''_m is contained in some hyperbolic plane $P' \subset F^\perp$ or $\|e''_m\| \neq 0$ or $\|e''_m\| = 0$ and $e''_m \in \langle \delta, \delta \rangle \subset F^\perp$ for some $0 \neq \delta \in k$. In the first case we may choose P' for H and we put $F' = F \oplus P'$. In the second case we put $F' = F \oplus k(e''_m)$ provided that $e''_m \notin \|F\|$. If, on the other hand, we should have $e''_m = \sum_{i=1}^n \lambda_i^2 \pi_i$ with, say $\lambda_1 \neq 0$, then we apply Lemma 4 a

finite number of times and find a sequence of mutually orthogonal vectors h_1, h_2, \dots, h_n in $(F + k(e''_m))^\perp$ with $\|h_1\| = \|e''_m\|$, $\|h_i\| = \pi_i$, $2 \leq i \leq n$. By Lemma 2 the space H spanned by $e''_m, h_1, h_2, \dots, h_n$ is isomorphic to $\langle \pi_1 \pi_1 \pi_2 \pi_3, \dots, \pi_n \rangle$ and we put $F' = F \oplus H$. The third case is treated in

the same way, the first two vectors for the construction of H already at hand. Thus, in all three cases we find $F' = F \oplus H$, $e_m \in F'$ where F' again is of the form $\Sigma P \oplus \Sigma \langle \pi_1 \rangle \oplus \dots \oplus \Sigma \langle \pi_r \rangle$, the π_i s linearly independent over k^2 . In this fashion we find an orthogonal decomposition of E as follows, $E = \cup F = \Sigma P \oplus \Sigma \langle \pi_1 \rangle \oplus \Sigma \langle \pi_2 \rangle \oplus \dots$. In view of the independence of the π_i s we have $E_* = \Sigma P \oplus (\Sigma \langle \pi_1 \rangle)_* \oplus \dots$. Not all of the summands $\Sigma \langle \pi_i \rangle$ can be (0) since $\|E\| \neq 0$. Thus, if one of the summands should be finite dimensional we would have $E_*^\perp \neq (0)$, contrary to assumption. Hence all the summands $\Sigma \langle \pi_i \rangle$ are infinite dimensional. Application of Corollary 1 finally yields $E \cong E_{(\pi_1)} \oplus E_{(\pi_2)} \oplus \dots$.

Corollary 2. *If (E, Φ) is a space with $E_*^\perp = (0)$ whose range $\|E\| \neq 0$ is spanned by the elements π_1, \dots, π_m (not necessarily independent over k^2) then E is isomorphic to $E_{(\pi_1)} \oplus \dots \oplus E_{(\pi_m)}$.*

Proof. By Proposition 2 $E \cong E_{(\sigma_1)} \oplus \dots \oplus E_{(\sigma_n)}$ where $\sigma_1 \dots \sigma_n$ is a k^2 -basis for $\|E\|$. Let then π_1, \dots, π_n ($n \leq m$) be a subset of elements independent over k^2 . By Corollary 1 (ii) we have

$$\langle \pi_1 \pi_1 \rangle \oplus \dots \oplus \langle \pi_n \pi_n \rangle \cong \langle \sigma_1 \sigma_1 \rangle \oplus \dots \oplus \langle \sigma_n \sigma_n \rangle.$$

Hence trivially $E_{(\sigma_1)} \oplus \dots \oplus E_{(\sigma_n)} \cong E_{(\pi_1)} \oplus \dots \oplus E_{(\pi_n)}$. Let $\pi_{n+1} = \sum_{i=1}^r \lambda_i^2 \pi_i$. After renumbering $\pi_1 \dots \pi_n$ we may assume that $\lambda_i \neq 0$, $1 \leq r \leq i$. Hence by Corollary 1 (ii) $\langle \pi_{n+1} \pi_{n+1} \pi_2 \dots \pi_r \rangle \cong \langle \pi_1 \pi_1 \pi_2 \dots \pi_n \rangle$. Thus $E_{(\pi_{n+1})} \oplus E_{(\pi_2)} \oplus \dots \oplus E_{(\pi_r)} \cong E_{(\pi_1)} \oplus \dots \oplus E_{(\pi_r)}$ can be arranged in a trivial fashion. In this manner we obtain $E_{(\pi_1)} \oplus \dots \oplus E_{(\pi_m)} \cong E$.

Proposition 3. *Let (E, Φ) be a semi-simple space of at most denumerable dimension. We have an orthogonal decomposition*

$$E = \bigoplus_{i \in I} \langle \pi_i \pi_i \rangle$$

where the π_i form some k^2 -basis for $\|E\|$ if and only if $E_*^\perp = E_*$.

Proof. If E admits such a decomposition we trivially have $E_*^\perp = E_*$. Conversely, let us assume that $E_*^\perp = E_*$. We first remark that E cannot contain a triple of mutually orthogonal vectors of the same length $\neq 0$. For,

assume that z_1, z_2, z_3 were such vectors, $\|z_1\| = \|z_2\| = \|z_3\| \neq 0$. We decompose according to the decomposition $E = E_* \oplus L$: $z_1 = e_1 + l_1$, $z_2 = e_2 + l_2$, $z_3 = e_3 + l_3$. Thus $\|l_1\| = \|l_2\| = \|l_3\|$. Since L contains no isotropic vectors we must necessarily have $l_1 = l_2 = l_3$. Since E_* is totally isotropic in our case, the three orthogonality conditions reduce to $0 = (e_1 + e_2, l_1) + \|l_1\|$, $0 = (e_1 + e_3, l_1) + \|l_1\|$, $0 = (e_2 + e_3, l_1) + \|l_1\|$. Adding the first two of these equations we obtain $(e_2 + e_3, l_1) = 0$ which contradicts the third one as $\|l_1\| \neq 0$. We now construct a decomposition of E step by step as in the proof of Proposition 2. Let $F = \langle \pi_1 \pi_1 \rangle \oplus \langle \pi_2 \pi_2 \rangle \oplus \dots \oplus \langle \pi_n \pi_n \rangle$ be a finite dimensional subspace of E , $\pi_1, \pi_2, \dots, \pi_n$ linearly independent over k^2 . Furthermore, let e_m again be the first basis vector of some fixed basis for E not contained in F . Without loss of generality we may proceed assuming that $e_m \perp F$. We consider first the case that $\|e_m\| \neq 0$. We try to find a vector $l \in F^\perp \cap E_*$ with $(l, e_m) \neq 0$. Suppose that there is no such vector l , in other words $F^\perp \cap E_* \subset e_m^\perp$. Since E_* is closed in our case, we find $(F + E_*^\perp)^\perp = F^\perp \cap E_*^{\perp\perp} = F^\perp \cap E_* \subset e_m^\perp$ therefore $e_m \in (F + E_*^\perp)^{\perp\perp} = F + E_*^\perp$ i.e., $e_m \in F + E_*^\perp = F + E_*$. Thus $e_m = f + f_0$ with $\|e_m\| = \|f\| \neq 0$.

Since $f \in F$ we should therefore have three mutually orthogonal vectors of the same length $\|e_m\| \neq 0$, a contradiction (if F contains one vector of some length $\alpha \neq 0$, then it contains, by virtue of its form, two orthogonal vectors of that length). Thus we must have $F^\perp \cap E_* \not\subset e_m^\perp$ and there exists a vector $l \in F^\perp \cap E_*$ with $(e_m, l) \neq 0$. Hence e_m and $e'_m = e_m + \frac{\|e_m\|}{(l, e_m)} l$ are mutually orthogonal vectors of F^\perp with $\|e_m\| = \|e'_m\|$. We put $F' = F \oplus k(e_m, e'_m)$. There remains the possibility that $\|e_m\| = 0$. Since E_* is totally isotropic, e_m cannot be contained in a hyperbolic plane, therefore $e_m \in \langle \delta, \delta \rangle \subset F^\perp$ for some $0 \neq \delta \in k$ (F^\perp is semi-simple). Since there cannot be more than two orthogonal vectors of the same length $\neq 0$ we must have $\delta \notin \|F\|$ and we put $F' = F \oplus \langle \delta \delta \rangle$ similar to the former case. In this fashion we obtain a decomposition of E of the required form, $E = \cup F = \langle \pi_1 \pi_1 \rangle \oplus \langle \pi_2 \pi_2 \rangle \oplus \dots$ where all the π_i s are linearly independent over k^2 .

We now prove the converse of Proposition 1.

Theorem 1. *Let $\text{char } k = 2$ and (E, Φ) a semi-simple k -space of denumerable dimension and let E_* be the subspace of vectors of length zero. If*

$$E_*^\perp + E_*^{\perp\perp} = (\text{rad } E_*)^\perp$$

then E admits of an orthogonal decomposition

$$E = \bigoplus_{i \in I_1} E_{(\gamma_i)} \oplus \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle \quad (\text{I})$$

or

$$E = \bigoplus_{i \in I_1} P_i \oplus \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle \quad (\text{II})$$

where, in the first case, the elements of the union $\{\gamma_i\}_{i \in I_1} \cup \{\beta_i\}_{i \in I_2} \cup \{\alpha_i\}_{i \in I_3}$ are a k^2 -basis of the range $\|E\|$ over k^2 , in the second case the same for the elements of the union $\{\beta_i\}_{i \in I_2} \cup \{\alpha_i\}_{i \in I_3}$ (the P_i s are hyperbolic planes).

Proof. Let $R = \text{rad}(E_*^{\perp\perp}) = (E_* + E_*^{\perp})^{\perp}$. Since R is totally isotropic and closed, we can apply Lemma 3 and obtain a decomposition

$$E = (R \oplus R') \oplus H, \quad H \perp (R \oplus R')$$

$$R \oplus R' = \bigoplus_{i \in I_2} k(r_i, r'_i), \quad R = \bigoplus_{i \in I_2} k(r_i)_{i \in I_2}. \quad (1)$$

Since $R \perp E_*^{\perp\perp}$, we can find an algebraic complement S of R in $E_*^{\perp\perp}$ with $S \perp R'$ (see the remark following the proof of Lemma 3). Hence $S \perp R \oplus R'$:

$$E_*^{\perp\perp} = R \oplus S, \quad S \subset H. \quad (2)$$

Furthermore S is semi-simple. If T is the orthogonal of S in H , we obtain from (2) $E_*^{\perp} = E_*^{\perp\perp\perp} = R \oplus T$. On the other hand, by the assumption of the theorem $R \oplus H = R^{\perp} = E_*^{\perp} + E_*^{\perp\perp} = R \oplus (S \oplus T)$. Since $S + T \subset H$ therefore $S + T = H$. Furthermore, since S is semi-simple, the sum $S + T$ is direct. Thus E is decomposed into three orthogonal summands:

$$E = (R \oplus R') \oplus S \oplus T \quad (3)$$

and it remains to discuss the spaces $R \oplus R'$, S and T . With regard to S we first remark that

$$E_* = R \oplus S_*. \quad (4)$$

For $R \oplus S_* \subset E_*$ is trivial. Conversely, if $x \in E_* \subset E_*^{\perp\perp} = R \oplus S$ we have $x = r + s$ with $r \in R$ and $s \in S$. Therefore $0 = \|x\| = \|r\| + \|s\| = \|s\|$ and $s \in S_*$. This shows $E_* \subset R + S_*$. Let then $S_*^{\perp_s}$ be the orthogonal of S_* in S . Since $S_*^{\perp_s} \subset S$ and $S \perp R$ we have $S_*^{\perp_s} \subset E_*^{\perp}$ by (4). Also $S_*^{\perp_s} \subset S \subset E_*^{\perp\perp}$, hence $S_*^{\perp_s} \subset E_*^{\perp} \cap E_*^{\perp\perp} = R$. Therefore $S_*^{\perp_s} = (0)$ as

$S_*^\perp \subset S$ and $S \cap R = (0)$. Thus, S is semi-simple and $S_*^\perp = (0)$. Two cases are possible for S : Either $S = S_*$ in which case S is a sum of hyperbolic planes or else $S \neq S_*$ in which case the range $\|S\|$ is different from 0 and Proposition 2 can be quoted: Thus

$$\text{either } S = \bigoplus_{i \in I_1} P_i \text{ or } S = \bigoplus_{i \in I_1} E_{(\gamma_i)} . \quad (5)$$

From (4) we learn that $R' \cap E_* = (0)$. Therefore, taking orthogonals in $R + R'$, we obtain $(R + R')_* = R = R^\perp = (R + R')_*^\perp$ and we may cite Proposition 3:

$$R \oplus R' = \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle . \quad (6)$$

Finally $E_* \cap T = (0)$ by (4), i.e., T contains no isotropic vectors. Hence T possesses an orthogonal basis, $T = \bigoplus_{i \in I_3} \langle \alpha_i \rangle$ where all the α_i s are independent over k^2 . Summarizing the facts about the decomposition (3) we see that E admits of an orthogonal decomposition of the form

$$E = \bigoplus_{i \in I_1} E_{(\gamma_i)} \oplus \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle \text{ or } E = \bigoplus_{i \in I_1} P_i \oplus \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle .$$

A dependence $0 = \sum v_i^2 \gamma_i + \sum \mu_i^2 \beta_i + \sum \kappa_i^2 \alpha_i$ defines an isotropic vector $x = \sum v_i c_i + \sum \mu_i b_i + \sum \kappa_i a_i$, $\sum v_i c_i \in S$, $\sum \mu_i b_i \in R + R'$ and $\sum \kappa_i a_i \in T$. By (4) $x \in E_* = R + S_*$ and thus $\kappa_i = 0$, $\|\sum v_i c_i\| = \sum v_i^2 \gamma_i = 0$ and $\|\sum \mu_i b_i\| = \sum \mu_i^2 \beta_i = 0$. However, the γ_i s are linearly independent over k^2 by Proposition 2. Therefore $v_i = 0$. Proposition 3 guarantees the independence of the β_i s and therefore $\mu_i = 0$. This proves that the elements $\gamma_i, \beta_i, \alpha_i$ together are independent over k^2 and the proof of Theorem 1 is complete.

Theorem 1 can be used to discuss the problem of isomorphism between \aleph_0 -dimensional k -spaces (E, Φ) in a large number of cases. We shall give here a complete discussion of the cases where the underlying field k is of finite dimension over its subfield k^2 . Thus, let k be a field with $[k : k^2]$ finite. For a space (E, Φ) we have $\text{codim } E_* \leq [k : k^2]$ or else an algebraic complement of E_* in E should contain an isotropic vector which is impossible. Since $\dim E_*^\perp \leq \text{codim } E_*$, the space E_*^\perp is finite dimensional and $E_*^{\perp\perp} + E_*^\perp$ is therefore closed. Hence every space of denumerable dimension over such a field admits of a basis as described by Theorem 1. (The following discussion also includes that of spaces (E, Φ) with $\|E\|$ finite dimensional over k^2 , k an arbitrary field.)

Theorem 2. *Let k be a field of characteristic 2 of finite dimension n over its subfield k^2 ($n = [k : k^2]$), (E, Φ) an n_0 -dimensional semi-simple space over k . Then (i) E is of the form:*

$$E = E_{(\gamma_1)} \oplus \dots \oplus E_{(\gamma_r)} \oplus \langle \beta_1 \beta_1 \beta_2 \beta_2 \dots \beta_s \beta_s \rangle \oplus \langle \alpha_1 \alpha_2 \dots \alpha_t \rangle \quad r \geq 1 \quad (\text{I})$$

or

$$E = \sum_{\infty} P \oplus \langle \beta_1 \beta_1 \beta_2 \beta_2 \dots \beta_p \beta_p \rangle \oplus \langle \alpha_1 \alpha_2 \dots \alpha_q \rangle, \quad (\text{II})$$

where all the sums are orthogonal and, in the first case, the elements $\gamma_1, \dots, \gamma_r, \beta_1, \dots, \beta_s, \alpha_1, \dots, \alpha_t$ are independent over k^2 and the same for $\beta_1, \dots, \beta_p, \alpha_1, \dots, \alpha_q$ in the second case (thus $r + s + t \leq n, p + q \leq n$).

(ii) E is uniquely determined, up to orthogonal isomorphism, by its range $\|E\|$, the range $\|E_*^{\perp\perp}\|$ and by the space E_*^{\perp} . (In particular, the numbers r, s and t , respectively p and q are orthogonal invariants of the space E .)

(iii) In terms of the above bases: If $\|E_*^{\perp\perp}\| \neq 0$ (i.e., E_* not closed) then E is of type (I), if $\|E_*^{\perp\perp}\| = 0$ (i.e., E_* closed) then E is of type (II). (Thus (I) and (II) represent non isomorphic spaces.) A space of type (I) is uniquely determined, up to orthogonal isomorphism, by $\|E\|$, the subspace of k (over k^2) spanned by the elements $\gamma_1, \dots, \gamma_r$ and by the space $\langle \alpha_1, \dots, \alpha_t \rangle$. A space of type (II) is uniquely determined, up to isomorphism, by $\|E\|$ and by the space $\langle \alpha_1, \dots, \alpha_q \rangle$.

Proof. It only remains to discuss the question of isomorphisms. For a space of type (I) let $E_{(\gamma_i)}$ be spanned by a basis $\{e_{ij}\}_{j \geq 1}$. $E_{(\gamma_i)}^*$ is then spanned by the vectors $e_{i1} + e_{ij}$ ($j \geq 1$) and the orthogonal of $E_{(\gamma_i)}^*$ in $E_{(\gamma_i)}$ is 0. Let $\langle \beta_1 \beta_1, \dots, \beta_s \beta_s \rangle$ be spanned by a basis $\{e_i, e'_i\}_{1 \leq i \leq s}$ and let R be the totally isotropic space $k(e_i + e'_i)_{1 \leq i \leq s}$. We then have, by virtue of the independence of the elements $\gamma_1, \dots, \beta_1, \dots, \alpha_1, \dots$

$$E_* = E_{(\gamma_1)^*} \oplus \dots \oplus E_{(\gamma_r)^*} \oplus R, \quad E_*^{\perp} = R \oplus \langle \alpha_1, \dots, \alpha_t \rangle,$$

$$E_*^{\perp\perp} = E_{(\gamma_1)} \oplus \dots \oplus E_{(\gamma_r)} \oplus R.$$

Let \bar{E} be another space falling into category (I), $\bar{E} = E_{(\bar{\gamma}_1)} \oplus \dots \oplus E_{(\bar{\gamma}_{\bar{r}})} \oplus \langle \bar{\beta}_1 \bar{\beta}_1, \dots, \bar{\beta}_{\bar{s}} \bar{\beta}_{\bar{s}} \rangle \oplus \langle \bar{\alpha}_1, \dots, \bar{\alpha}_{\bar{t}} \rangle$ such that $\|E\| = \|\bar{E}\|, \|E_*^{\perp\perp}\| = \|\bar{E}_*^{\perp\perp}\|$ and $E_*^{\perp} \cong \bar{E}_*^{\perp}$. We have to prove that $E \cong \bar{E}$. Since $\gamma_1, \dots, \gamma_r$ and $\bar{\gamma}_1, \dots, \bar{\gamma}_{\bar{r}}$ are independent over k^2 we first have $r = \bar{r}$ (since $\|E_*^{\perp\perp}\| = \|\bar{E}_*^{\perp\perp}\|$). By Corollary 2 we see that $E_*^{\perp\perp} \cong \bar{E}_*^{\perp\perp}$. Hence we may intro-

duce a new basis in $\bar{E}_*^{\perp\perp}$ such that $\bar{\gamma}_i = \gamma_i$, $1 \leq i \leq r$. From the isomorphism $R \oplus \langle \alpha_1, \dots, \alpha_t \rangle \cong \bar{R} \oplus \langle \bar{\alpha}_1, \dots, \bar{\alpha}_t \rangle$ we conclude that $\langle \alpha_1, \dots, \alpha_t \rangle \cong \langle \bar{\alpha}_1, \dots, \bar{\alpha}_t \rangle$ since R and \bar{R} are totally isotropic orthogonal summands and since both $\langle \alpha_1, \dots, \alpha_t \rangle$ and $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_t \rangle$ are semi-simple (even non-isotropic by the independence of the α s). Thus $t = \bar{t}$ and we may introduce a new basis in $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_t \rangle$ such that $\bar{\alpha}_i = \alpha_i$, $1 \leq i \leq t$. Finally, since $\|E\| = \|\bar{E}\|$ and since $\gamma_1, \dots, \beta_1, \dots, \alpha_1, \dots$ and $\bar{\gamma}_1, \dots, \bar{\beta}_1, \dots, \bar{\alpha}_1, \dots$ are independent over k^2 we have $r + s + t = \bar{r} + \bar{s} + \bar{t}$; therefore $s = \bar{s}$ as $r = \bar{r}$ and $t = \bar{t}$. Furthermore, having introduced the new bases in $\bar{E}_*^{\perp\perp}$ and $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_t \rangle$ we may cite Corollary 1 (ii), $\langle \gamma_1, \dots, \gamma_r \rangle \oplus \langle \beta_1\beta_1, \dots, \beta_s\beta_s \rangle \oplus \langle \alpha_1, \dots, \alpha_t \rangle \cong \langle \bar{\gamma}_1, \dots, \bar{\gamma}_r \rangle \oplus \langle \bar{\beta}_1\bar{\beta}_1, \dots, \bar{\beta}_s\bar{\beta}_s \rangle \oplus \langle \bar{\alpha}_1, \dots, \bar{\alpha}_t \rangle$. A fortiori $E_{(\gamma_1)} \oplus \dots \oplus E_{(\gamma_r)} \oplus \langle \beta_1\beta_1, \dots, \beta_s\beta_s \rangle \oplus \langle \alpha_1, \dots, \alpha_t \rangle \cong E_{(\bar{\gamma}_1)} \oplus \dots \oplus E_{(\bar{\gamma}_r)} \oplus \langle \bar{\beta}_1\bar{\beta}_1, \dots, \bar{\beta}_s\bar{\beta}_s \rangle \oplus \langle \bar{\alpha}_1, \dots, \bar{\alpha}_t \rangle$ and thus $E \cong \bar{E}$. The simpler case of spaces falling into category (II) is treated in the same way. This proves Theorem 2.

Theorem 2 may also be expressed in the following way: If $[k : k^2]$ is finite and (E, Φ) an \aleph_0 -dimensional, semi-simple k -space with E_* not closed, then there exist three finite dimensional k -spaces F, G and H such that $F \oplus G \oplus H$ contains no isotropic vectors and E is isomorphic to the (external) orthogonal sum $(\sum^\infty F) \oplus G \oplus G \oplus H$. E is uniquely determined by the ranges $\|F + G + H\|$, $\|F\|$ and by the space H ; on the other hand, if E_* is closed, then there exist two finite dimensional k -spaces G and H such that $G \oplus H$ contains no isotropic vector and E is isomorphic to the (external) orthogonal sum $(\sum^\infty P) \oplus G \oplus G \oplus H$. In this case E is uniquely determined by the ranges $\|G + H\|$ and by the space H .

We should like to mention that Theorem 2 alone can be obtained more directly by proving Theorem 1 only for spaces E with $\|E\|$ of finite dimension over k^2 . This is done by an induction on $\dim_{k^2} \|E\|$. For $\dim_{k^2} \|E\| = 0$ we have $E = \Sigma P$. After induction assumption two cases arise which have to be treated differently: First case, there exists some decomposition $E = H \oplus H^\perp$ with finite dimensional H such that $\dim_{k^2} \|H^\perp\| < \dim_{k^2} \|E\|$. Hence there is a basis of the required sort for H^\perp by the induction assumption. The required basis for E is then found easily by applications of Corollary 1. Second case, there is no such decomposition of E . In that case, one proves directly that $E = E_{(\pi_1)} \oplus \dots \oplus E_{(\pi_n)}$ where π_1, \dots, π_n span $\|E\|$. This is accomplished along the line of the proof of Proposition 2, where now the assumption of our case replaces the function of Lemma 4.

Thus, for fields k with finite $[k:k^2]$ a complete list of non isomorphic k -spaces (E, Φ) of denumerable dimension can easily be given on the basis of Theorem 2, provided one knows the *finite* dimensional, non-isotropic k -spaces $(\langle \alpha_1, \dots, \alpha_t \rangle!)$. It is advantageous to first subdivide the spaces according to the dimensions of E/E_* , E_*^\perp and $\text{rad}(E_*)$. In the notations of Theorem 2: $p + q, r + s + t = \dim(E/E_*)$; $p + q, s + t = \dim(E_*^\perp)$; $p, s = \dim(\text{rad } E_*)$ $p + q, r + s + t \leq [k:k^2]$. We may use uniformly the notations r, s, t by interpreting a triple (r, s, t) with $r = 0$ as belonging to a space of type (II). There are $\frac{(n+1)(n+2)(n+3)}{6}$ ordered triples (r, s, t) with $0 \leq r + s + t \leq n$; they yield a subdivision of all semi-simple \aleph_0 -dimensional k -spaces (E, Φ) according to their dimensions of E/E_* , E_*^\perp and $\text{rad } E_*$ into $\frac{(n+1)(n+2)(n+3)}{6}$ classes ($n = [k:k^2]$). The particular choices for $\gamma_1, \dots, \gamma_r, \beta_1, \dots, \beta_s, \alpha_1, \dots, \alpha_t$ are then taken. For the sake of illustration, we give a complete list for an underlying field k with $[k:k^2] = 2$:

$\dim E/E_*$ $r + s + t$	$\dim E_*^\perp$ $s + t$	\dim $(\text{rad } E_*)$ s	
0	0	0	$\sum^\infty P$
1	0	0	$E_{(\nu)}$
1	1	0	$\sum^\infty P \oplus \langle \nu \rangle$
1	1	1	$\sum^\infty P \oplus \langle \nu, \nu \rangle$
2	0	0	$E_{(\alpha)} \oplus E_{(\beta)}$
2	1	0	$E_{(\nu)} \oplus \langle \mu \rangle \quad \nu \neq \mu$
2	1	1	$E_{(\alpha)} \oplus \langle \beta, \beta \rangle, E_{(\nu)} \oplus \langle \alpha, \alpha \rangle \quad \nu \neq \alpha$
2	2	0	$\sum^\infty P \oplus \langle \alpha, \nu \rangle \quad \nu \neq \alpha$
2	2	1	$\sum^\infty P \oplus \langle \beta, \beta \rangle \oplus \langle \alpha \rangle, \sum^\infty P \oplus \langle \alpha, \alpha \rangle \oplus \langle \nu \rangle \quad \nu \neq \alpha$
2	2	2	$\sum^\infty P \oplus \langle \alpha, \alpha \rangle \oplus \langle \beta, \beta \rangle$

All the sums are orthogonal, $\{\alpha, \beta\}$ is some fixed basis of k over k^2 ; ν and μ run independently through a fixed set of representatives of g_k (the multi-

plicative group of k modulo square factors), subject only to conditions listed in the table. All the spaces thus obtained are mutually non isomorphic and they are, up to orthogonal isomorphisms, all semi-simple k -spaces (E, Φ) of denumerable dimension.

III. Orthogonal bases

Let k be an arbitrary field of characteristic 2. If the semi-simple k -space (E, Φ) is finite dimensional, then either $E = \Sigma P$ or E possesses an orthogonal basis (Lemma 1). Let (E, Φ) be a space of denumerable dimension. E is an orthogonal sum $\Sigma P \oplus E_0$ where E_0 possesses an orthogonal basis. If $\dim_k(E/E_*)$ is infinite (i.e., $\dim_k ||E||$ is infinite), then $\dim E_0$ is infinite and E has an orthogonal basis by virtue of Lemma 1. Thus, if E does not admit of an orthogonal basis, then E/E_* is of finite dimension and there exists a decomposition of E as described in Theorem 2 (necessarily of type (II)): $E = \Sigma P \oplus E_0$, where E_0 is finite dimensional and spanned by an orthogonal basis. Conversely, a space of this form does not admit of an orthogonal basis for, $\Sigma P \oplus E_0 \subset \bigoplus_{i=1}^{\infty} k(e_i)$ gives $E_0 \subset \bigoplus_{i=1}^N k(e_i)$ for a suitable N and thus, for the respective orthogonals, we obtain $\bigoplus_{i=N+1}^{\infty} k(e_i) \subset \Sigma P$. This is a contradiction as $||e_i|| \neq 0$ for an orthogonal basis of a semi-simple space. Thus, a space (E, Φ) of denumerable dimension admits of no orthogonal basis if and only if E_* is closed and E/E_* finite dimensional. These conditions may be formulated in various ways. Here is a selection:

Theorem 3. *Let k be an arbitrary field of characteristic 2, (E, Φ) a semi-simple k -space of denumerable dimension. The following statements are equivalent:*

- (j) E possesses no orthogonal basis;
- (jj) E/E_* is finite dimensional and E_* is closed;
- (jjj) E_*^\perp is finite dimensional and $\dim E/E_* = \dim E_*^\perp$;
- (jv) E/E_* is finite dimensional and $\dim(\text{rad } E_*) = \dim E/(E_* + E_*^\perp)$.

IV. Automorphisms

We shall add here a few remarks about the group $\mathfrak{O}(E, \Phi)$ of all metric automorphisms of a space (E, Φ) , i.e., the group of all vector space auto-

morphisms $T: E \rightarrow E$ which satisfy $\Phi(Tx, Ty) = \Phi(x, y)$ for all $x, y \in E$. The underlying field k is of characteristic 2 and $\dim E = \aleph_0$. The structure of the group $\mathfrak{O}(E, \Phi)$ is unknown in the general case. If (E, Φ) satisfies the conditions

$$E_*^\perp + E_*^{\perp\perp} \text{ is closed, } \dim(\text{rad } E_*) < \aleph_0 \quad (1)^1$$

– which always takes place when the underlying field is of finite dimension $[k: k^2]$ over k^2 – then the study of $\mathfrak{O}(E, \Phi)$ can be reduced to the study of simpler groups. They are the (symplectic) group $\mathfrak{O}(E, \Phi)$, where the \aleph_0 -dimensional space (E, Φ) is an orthogonal sum of hyperbolic planes, and the group $\mathfrak{O}(E, \Phi)$, where (E, Φ) is an orthogonal sum $E_{(\alpha_1)} \oplus E_{(\alpha_2)} \oplus \dots$ and the elements $\alpha_1, \alpha_2, \dots$ independent over k^2 (cf. 1.3 for notations). This reduction, possible for the spaces subject to (1), shall be carried out here.

For a space satisfying (1) there is decomposition (Theorem 1):

$$E = E_0 \oplus (R + R') \oplus E_1, \quad (2)$$

where $E_0, R \oplus R'$ and E_1 are orthogonal summands such that

$$R = \text{rad } E_*, \quad E_* = E_{0*} \oplus R, \quad E_*^\perp = R \oplus E_1, \quad E_*^{\perp\perp} = E_0 \oplus R \quad (3)$$

and, furthermore, $R \oplus R'$ is an orthogonal sum of planes $k(r_i, r'_i)$, $i \in I$ for $\{r_i\}_{i \in I}$ and $\{r'_i\}_{i \in I}$ a basis of R and R' respectively. For every $T \in \mathfrak{O}(E, \Phi)$ we have $T(E_*) = E_*$, $T(R) = R$, $T(E_*^\perp) = E_*^\perp$ and $T(E_*^{\perp\perp}) = E_*^{\perp\perp}$. When $x \in R' \oplus E_1$ we write $Tx = x + Lx$. Hence $\|Lx\| = 0$ and $Lx \in E_* \subset E_*^{\perp\perp}$,

$$Lx \in E_0 \oplus R \text{ for } x \in R' \oplus E_1. \quad (4)$$

In particular, if $x \in R$ and $y \in R'$ then $(x, y) = (Tx, Ty) = (Tx, y + Ly) = (Tx, y)$ since $Tx \in R \perp E_0 \oplus R$. Therefore $(x - Tx, y) = 0$ for all $y \in R'$ or $x - Tx \in R'^\perp$, $R'^\perp \cap R = 0$; hence $x - Tx = 0$ since $x - Tx$ also belongs to R . Thus the restriction T/R of T to R leaves the vectors of R fixed,

$$T|_R = I_R. \quad (5)$$

¹⁾ We recall an earlier example where the second condition is satisfied but not the first. See the remark at the end of this section.

Let then $x \in E_1$ and $y \in R'$. Since $E_1 \subset E_*^\perp$ and $T(E_*^\perp) = E_*^\perp$ we have $Lx \in R$; hence $(x, y) = (Tx, Ty) = (x + Lx, y + Ly) = (x, y) + (Lx, y)$. Thus $(Lx, y) = 0$ for all $y \in R'$ i.e., $Lx \in R'^\perp$, $R'^\perp \cap R = 0$ and therefore $Lx = 0$ as $Lx \in R$. In other words,

$$T|_{E_1} = I_{E_1}. \quad (6)$$

Thus, every automorphism of E leaves E_*^\perp pointwise fixed. Therefore we have for every $x \in R'$ and $y \in E_*^\perp$ that $(x, y) = (Tx, Ty) = (Tx, y)$ hence $x - Tx \in E_*^{\perp\perp} = E_0 + R$ for every $x \in R'$. Therefore, and in view of (5) and (6) we can decompose the image Tx for every $x \in (R \oplus R') \oplus E_1$ as follows, $Tx = x + L_0x + L_1x$ with $L_0x \in E_0$ and $L_1x \in R$. Computing $\|Tx\|$ shows furthermore that even $L_0x \in E_{0*}$. We therefore have $(x \in R \oplus R' \oplus E_1)$

$$Tx = x + L_0x + L_1x \quad (7)$$

where the projections L_0 and L_1 are linear maps

$$L_0: R \oplus R' \oplus E_1 \rightarrow E_{0*}, \quad L_0(R \oplus E_1) = (0);$$

$$L_1: R \oplus R' \oplus E_1 \rightarrow R, \quad L_1(R \oplus E_1) = (0).$$

On the other hand, for $x \in E_0 \subset E_*^{\perp\perp} = E_0 \oplus R$ we have

$$(x \in E_0) \quad Tx = L_2x + L_3x \quad L_2x \in E_0, \quad L_3x \in R. \quad (8)$$

Since R is totally isotropic and orthogonal to E_0 , $L_2: E_0 \rightarrow E_0$ is a metric automorphism of E_0 ; L_3 is some linear map $E_0 \rightarrow R$. If we express Tx for an arbitrary $x \in E$ by using (7) and (8), then the condition that $(x, y) = (Tx, Ty)$ for all $x, y \in E$, $T \in \mathfrak{O}(E, \Phi)$ is equivalent with the conditions

$$(x, L_3y) + (L_0x, L_2y) = 0 \quad \text{for all } x \in R', y \in E_0 \quad (9)$$

$$(x, L_1y) + (L_1x, y) + (L_0x, L_0y) = 0 \quad \text{for all } x, y \in R' \quad (10)$$

(9) and (10) permits a discussion of $\mathfrak{O}(E, \Phi)$ as in the finite dimensional case

([2]). First, the system (9) and (10) admits of solutions L_0 and L_1 for arbitrarily prescribed L_2 and L_3 , L_2 an automorphism of E_0 and $L_3: E_0 \rightarrow R$ a linear map. Indeed. For given L_2 and L_3 (9) defines a linear map $L_0: R' \rightarrow E_{0*}$ in a unique manner. We then extend it to $L_0: R \oplus R' \oplus E_1 \rightarrow E_{0*}$ by defining $L_0(R \oplus E_1) = (0)$. Appealing to the basis of $R \oplus R' = \bigoplus_I k(r_i, r'_i)$ we put $L_1 r'_i = \sum \alpha_{ij} r_j$. Condition (10) is satisfied with the previously found L_0 provided that $\alpha_{ij} + \alpha_{ji} = (L_0 r'_i, L_0 r'_j)$. Since $(L_0 r'_i, L_0 r'_i) = \|L_0 r'_i\| = 0$ as $L_0 r'_i \in E_{0*}$, there are always solutions for the unknowns α_{ij} ; (this is the only place where use is made of the assumption (1) that $\dim R < \aleph_0$). This proves our assertion. Thus, if T runs through $\mathfrak{D}(E, \Phi)$ then the restriction $T|_{E_0 \oplus R}$ (it leaves $E_0 \oplus R = E_*^{\perp \perp}$ invariant!) runs through the group \mathfrak{G} of all automorphisms of the space $E_0 \oplus R$ that leave R pointwise fixed (as we have just proved, every element of \mathfrak{G} can be extended to an automorphism of E). $T \rightarrow T|_{E_0 \oplus R}$ defines an epimorphism

$$\varphi: \mathfrak{D}(E, \Phi) \rightarrow \mathfrak{G}. \quad (11)$$

The kernel $\mathfrak{C} = \ker \varphi$ can easily be described. $T \in \mathfrak{C}$ means that $T|_{E_0 \oplus R}$ is the identical transformation of $E_0 \oplus R$. For such a T and every $x \in E_0 \oplus R \oplus E_1$, $y \in R'$ we obtain from $(x, y) = (Tx, Ty) = (x, Ty)$ that $y - Ty \in (E_0 \oplus R \oplus E_1)^\perp = R$. Thus

$$Tx = x + L_4 x, \quad L_4 x \in R, \quad x \in E, \quad L_4(E_0 \oplus R \oplus E_1) = (0) \quad (12)$$

$(x, y) = (Tx, Ty)$ yields

$$(y, L_4 x) + (L_4 y, x) = (0). \quad (13)$$

Conversely, every linear map $L_4: R' \rightarrow R$ meeting (13) defines an element $T \in \mathfrak{C}$ by means of (12). \mathfrak{C} is thus seen to be isomorphic to the additive group of linear maps $L: R \rightarrow R'$ satisfying (13). Thus, as $s = \dim R$ is finite, $\mathfrak{C} \cong k^{\frac{s(s+1)}{2}}$. Let us turn to the group \mathfrak{G} . It contains the subgroup \mathfrak{G}_0 of automorphisms $T': E_0 \oplus R \rightarrow E_0 \oplus R$ of the form $T': x \rightarrow x + L_5 x$ where L_5 is an arbitrary linear map $L_5: E_0 \oplus R \rightarrow R$ with $L_5(R) = (0)$. \mathfrak{G}_0 is an invariant subgroup of \mathfrak{G} and $\mathfrak{G}/\mathfrak{G}_0 \cong \mathfrak{D}(E_0, \Phi|_{E_0})$. \mathfrak{G}_0 is isomorphic to the additive group of all linear maps $L: E_0 \rightarrow R$, and $\mathfrak{G}_0 \cong k^\omega$ or $\mathfrak{G}_0 \cong (1)$.

Thus, if we put $\mathfrak{C}_0 = \varphi^{-1} \mathfrak{G}_0$, we have the series of invariant subgroups

$$\mathfrak{C} \subset \mathfrak{C}_0 \subset \mathfrak{D}(E, \Phi)$$

with $\mathfrak{C} \cong k^{\frac{s(s+1)}{2}}$, $\mathfrak{C}_0/\mathfrak{C} \cong \mathfrak{G}_0$, $\mathfrak{D}(E, \Phi)/\mathfrak{C}_0 \cong \mathfrak{D}(E_0, \Phi|_{E_0})$, $s = \dim(\text{rad } E_*)$. E_0 is an algebraic complement of $\text{rad } E_*$ in $E_*^{\perp\perp}$; it is either an orthogonal sum of hyperbolic planes or an orthogonal sum $E_{(\alpha_1)} \oplus \dots \oplus E_{(\alpha_n)}$, the elements $\alpha_1, \alpha_2, \dots, \alpha_n$ independent over k^2 .

Remark (added in proof). The condition in (1) that $\dim R = \dim(\text{rad } E_*) < \aleph_0$ is quite unnecessary for the discussion that followed. Setting $L_1 r'_i = \sum \alpha_{ij} r_j$, the matrix equation $\alpha_{ij} + \alpha_{ji} = (L_0 r'_i, L_0 r'_j)$ admits row-finite solutions (which actually define a map L_1); for example $\alpha_{ij} = 0$ ($j \geq i$), $\alpha_{ij} = (L_0 r'_i, L_0 r'_j)$ for $j < i$. For the normal series of groups obtained we have in the case $\dim R = \aleph_0$: $G_0 \cong k^\omega$ and $C \cong k^\omega$.

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This paper was in part supported by NSF grant GP-4205.

(Received September 13, 1965)