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Autor: Gross, H. / Engle, Robert D.

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Bilinear Forms on k-Vector spaces of Denumerable Dimension in the Case of char (k) = 2

by Herbert Gross and Robert D. Engle, Bozeman (Mont.)

Introduction. The classification, up to metric isomorphism, of finite dimensional k-vector spaces E, supplied with a symmetric bilinear form $\Phi: E \times E \to k$, is a rather difficult problem; it has been solved for particular fields k, such as the field of rationals, reals, p-adic numbers or function fields in one variable over a finite constant field. Kaplansky has shown that for k-vector spaces (E, Φ) of a denumerable (algebraic) dimension, these problems vanish in a large number of cases, E admitting an orthonormal basis for an extensive class of underlying fields ([4]; for an investigation of such fields see [3]). In the denumerable case, an exceptional role is once more played by the fields of characteristic 2. For perfect fields of characteristic 2 Kaplansky has proved the following

Theorem. For every \aleph_0 -dimensional k-space (E, Φ) , Φ a non degenerate bilinear form, precisely one of the following four possibilities holds: (1) E possesses an orthonormal basis, (2) E possesses a symplectic basis, (3) E is an orthogonal sum $E = E_0 \oplus L$ where E_0 is spanned by a symplectic basis and L is one-dimensional, (4) E is an orthogonal sum $E_0 \oplus L$, where E_0 has a symplectic basis and L is two-dimensional, spanned by an orthogonal basis ([4] p. 15). Kaplansky has asked what becomes of this theorem if the assumption that every element in the coefficient field be a square, is dropped.

In the following, we investigate the case of an arbitrary field of characteristic 2. Complete results as regards the classification problem are obtained for all fields k of finite dimension over their subfields k^2 (Theorem 2). As a sideresult we obtain an invariant characterization of the k-spaces (E, Φ) of denumerable dimension which admit of orthogonal bases, k an arbitrary field of characteristic 2 (Theorem 3).

I. Notations and Results

Let k be a commutative field. A k-vector space (E, Φ) is a k-vector space E supplied with a symmetric bilinear form $\Phi: E \times E \to k$. (E, Φ) is called semisimple if $E \cap E^{\perp} = (0)$. In the following, an isomorphism $(E, \Phi) \cong (G, \psi)$ is a vector space isomorphism $\vartheta: E \to G$ such that $\psi(\vartheta x, \vartheta y) = \Phi(x, y)$

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- for all $x, y \in E$. If there is no risk of confusion, we simply talk about E instead of (E, Φ) and, we write (x, y) and ||x|| respectively for $\Phi(x, y)$ and the "length" $\Phi(x, x)$ of $x \in E$. A subspace H of (E, Φ) is always considered as being supplied with the restriction Φ/H of Φ to H. The radical of H (rad H) is defined as $H \cap H^{\perp}$. A subspace $H \subset E$ is said to be closed if $H^{\perp \perp} = H$. If H is a closed subspace of (E, Φ) and F a finite dimensional subspace of (E, Φ) then H + F is closed.
- 2. The following lemma, proved by Kaplansky in [4], will be used in the proof of Lemma 4 below. Lemma: Let (E, Φ) be a semi-simple k-vector space of infinite dimension over an arbitrary field k. Let furthermore F be a finite dimensional subspace of E, spanned by the basis f_1, \ldots, f_n , V a subspace of E with $V^{\perp} = (0)$. Then there exists a vector $x \in E$ with $x \in V$, $x \notin V \cap F$ and $\Phi(x, f_i) = \beta_i$ for arbitrarily prescribed $\beta_i \in k$.
- 3. Bases being the central object below, the following notations prove convenient. If $\alpha_1, \ldots, \alpha_n \in k$ then $\langle \alpha_1, \ldots, \alpha_n \rangle$ is an *n*-dimensional *k*-space (E, Φ) possessing an orthogonal basis e_1, e_2, \ldots, e_n with $||e_i|| = \alpha_i$. "P" invariably denotes a hyperbolic plane, i.e., a two-dimensional space (E, Φ) having a basis e_1, e_2 with $||e_1|| = ||e_2|| = 0$ and $(e_1, e_2) = 1$. ΣP is an orthogonal sum of hyperbolic planes (i.e., a space spanned by a symplectic basis). $\Sigma \langle \alpha \rangle$ is a space (E, Φ) spanned by an orthogonal basis (finite or infinite), each basis vector of length $\alpha, \alpha \neq 0$. If $\Sigma \langle \alpha \rangle$ is of denumerable dimension, we denote it by $E_{(\alpha)}$.
- 4. In the following investigations, k will always be a field of characteristic 2 unless stated otherwise. Every such field is a vector space over its subfield k^2 of squares.
- 5. If (E, Φ) is a semi-simple k-vector space with dim $E \leq \aleph_0$ then E is an orthogonal sum $\Sigma P \oplus E_0$, where E_0 is spanned by an orthogonal basis.
- 6. Let (E, Φ) be a k-vector-space. We have ||x+y|| = ||x|| + ||y|| for all $x, y \in E$ as char k = 2. Thus, if H is a subspace of E, then the range of the restriction ||H|| is a subspace of the k^2 -vector space k. This range will be denoted by "||H||" throughout. In particular, the set of all isotropic vectors x in E (||x|| = 0) is a vector space. This subspace of E is invariably denoted by E_* . (The subspace of vectors satisfying condition (T) in [1] p. 66.) The subspaces E_* , E_*^{\perp} , $E_*^{\perp \perp}$, rad E_* etc. will play an important role since they are invariant subspaces under orthogonal transformations. We notice that rad $(E_*^{\perp}) \subset E_*$ by the definition of E_* , hence rad $(E_*^{\perp}) \subset \operatorname{rad}(E_*^{\perp}) \cap E_* = \operatorname{rad}E_*$. Therefore rad $E_*^{\perp} = \operatorname{rad}E_*$, the converse inclusion being trivial. This means in particular that rad E_* (= rad $(E_*^{\perp}) = (E_* + E_*^{\perp})^{\perp}$) is a closed space.

II. Bases

Let us mention a few words about the fields. When describing k-spaces (E, Φ) in terms of orthogonal bases, it is clear that the non-square elements of k play an important role. Let g_k be the multiplicative group of non-zero elements in k modulo square factors. If g_k is finite, then its order is a power of 2 since every element of g_k is of order 2. If char $k \neq 2$ then one can find, for every natural n, fields with g_k of order 2^n (even among the denumerable fields, [3]). On the other hand, if char k=2 then k^2 is a subfield of k and the elements of g_k are precisely the straight lines through the origin of the k^2 -vector space k. In other words, the order of g_k is either 1 or equal to card (k). In particular, since g_k is of order 1 for finite fields, g_k is either of order 1 or infinite. In the following discussion of isomorphisms between \aleph_0 -dimensional k-spaces the fields with finite dimension $[k:k^2]$ over their subfields k^2 are seen to play a special role. Since a simple characterization of all non isomorphic spaces over such fields can be given (Theorem 2), let us mention a few elementary facts about these fields.

Clearly, if $[k:k^2]$ is finite, then $[k:k^2]$ is a power of 2. Furthermore, if \bar{k} is a finite algebraic extension of k, $[k:k^2]$ finite, then $[\bar{k}:\bar{k}^2]=[k:k^2]$ ($[\bar{k}:k^2]=[k:k^2]$ ($[\bar{k}:k^2]=[k:k]$). From this follows that $[\bar{k}:\bar{k}^2] \leq [k:k]$ for an arbitrary algebraic extension \bar{k} of k. (< is witnessed by the transition to the algebraic closure.) On the other hand, if $\bar{k}=k(\xi_1,\ldots,\xi_n)$, where ξ_1,\ldots,ξ_n are independent transcendentals over k, we have $[\bar{k}:\bar{k}^2]=[k:k^2]\cdot 2^n$ (a basis for \bar{k} over \bar{k}^2 is given by the elements $\alpha_i \; \xi_1^{s_1} \; \xi_2^{s_2} \; \ldots \; \xi_n^{s_n}, \; \varepsilon_j=0$, 1 and α_i running through a k^2 basis of k). In particular:

If k is a field of characteristic 2 with finite $[k:k^2]$, then $[\bar{k}:\bar{k}^2]$ is finite for an arbitrary over field \bar{k} of k, provided its transcendence degree over k is finite. The fields k with finite $[k:k^2]$ form thus a considerable class.

Let again k be an arbitrary field of characteristic 2. It is well known that Witt's Cancellation Theorem does not hold for bilinear forms in the case of char k=2. Instead, we have the following orthogonal isomorphisms:

Lemma 1. $\langle \alpha \rangle \oplus \langle \alpha, \alpha \rangle \cong \langle \alpha \rangle \oplus P$ (0 $\neq \alpha \in k$, P a hyperbolic plane and all the sums orthogonal).

Lemma 2. $\langle \alpha, \alpha \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle \cong \langle \overline{\alpha}, \overline{\alpha} \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle$ provided that the elements $\{\alpha, \beta_i\}_{i \in I}$ are independent over k^2 and span the same subspace of k (over k^2) as the elements $\{\overline{\alpha}, \beta_i\}_{i \in I}$ (card I is finite or infinite; all sums are orthogonal).

Proofs. 1. Let e_1 , e_2 , e_3 be an orthogonal basis of $\langle \alpha \rangle \oplus \langle \alpha, \alpha \rangle$ with $||e_i|| = \alpha$. Introduce a new basis \overline{e}_1 , \overline{e}_2 , \overline{e}_3 by $\overline{e}_1 = e_1 + e_2 + e_3$, $\overline{e}_2 = e_1 + e_2$, $\overline{e}_3 = \alpha^{-1}$ $(e_2 + e_3)$.

2. Let e_{00} , e_{0} , e_{i} ($i \in I$) be an orthogonal basis of $\langle \alpha, \alpha \rangle \oplus \bigoplus_{i \in I} \langle \beta_{i} \rangle$ with $||e_{00}|| = ||e_{0}|| = \alpha$, $||e_{i}|| = \beta_{i}$. Since $\{\alpha, \beta_{i}\}_{i \in I}$ and $\{\overline{\alpha}, \beta_{i}\}_{i \in I}$ span the same subspace of k we have $\overline{\alpha} = \lambda_{0}^{2} \alpha + \sum_{i=1}^{n} \lambda_{i}^{2} \beta_{i}$ for suitable λ_{0} , λ_{1} , ..., λ_{n} . Since the elements $\{\overline{\alpha}, \beta_{i}\}_{i \in I}$ are independent over k^{2} we have $\lambda_{0} \neq 0$. For a fixed choice of λ_{0} , λ_{1} , ..., λ_{n} introduce the following basis

$$egin{aligned} \overline{e}_{00} &= rac{\overline{lpha}}{\lambda_0 \, lpha} \, e_{00} + \left(\lambda_0 + rac{\overline{lpha}}{\lambda_0 \, lpha}
ight) e_0 + rac{\Sigma}{2} \, \lambda_i \, e_i \ & ar{e}_0 &= & \lambda_0 \, e_0 + rac{\Sigma}{2} \, \lambda_i \, e_i \ & 2 \leq i \leq n : \overline{e}_i = rac{\lambda_i \, eta_i}{\lambda_0 \, lpha} \, (e_{00} + e_0) & + e_i \ & n < i : \overline{e}_i = e_i \, . \end{aligned}$$

We shall list a few consequences some of which will be of importance later.

Corollary 1. (i) $\bigoplus_{i \in I} E_{(\alpha_i)} \oplus \Sigma P = \bigoplus_{(\alpha_i)}$ (all sums orthogonal).

- (ii) $\langle \alpha_1 \alpha_1 \alpha_2 \alpha_2 \dots \alpha_m \alpha_m \rangle \cong \langle \overline{\alpha}_1 \overline{\alpha}_1 \overline{\alpha}_2 \overline{\alpha}_2 \dots \overline{\alpha}_m \overline{\alpha}_m \rangle$ provided the elements $\alpha_1, \dots, \alpha_m$ are independent over k^2 and span the same subspace of k (over k^2) as the elements $\overline{\alpha}_1, \dots, \overline{\alpha}_m$.
- (iii) $\bigoplus_{j=1}^{m} \langle \alpha_j \alpha_j \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle \cong \bigoplus_{j=1}^{m} \langle \overline{\alpha}_j \overline{\alpha}_j \rangle \oplus \bigoplus_{i \in I} \langle \beta_i \rangle$ provided the elements $\{\alpha_1, \ldots, \alpha_m, \beta_i\}_{i \in I}$ are independent over k^2 and span the same subspace of k as the elements $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_m, \beta_i\}_{i \in I}$ (card I is finite or infinite, m is a natural number, all sums are orthogonal).

We remark that the transformation of Lemma 2 does not lend itself to a generalization of (ii) and (iii) to the case of infinite m. (We have not succeeded in proving or disproving the infinite analogue of (ii) by any other means; cf. Proposition 3.)

Another lemma which we shall use is the following:

Lemma 3. Let (E, Φ) be a k-vector space of denumerable dimension, semi-simple with respect to the bilinear form $\Phi: E \times E \to k$ and k a field of arbitrary characteristic. Let furthermore R be a closed, totally isotropic subspace of $E(R^{\perp \perp} = R)$

and $R \subset R^{\perp}$). There exists a basis $(r_i)_{i \in I}$ of R and a subspace R' of E admitting an orthogonal basis $(r'_i)_{i \in I}$ such that $R \oplus R'$ decomposes into an orthogonal sum of semi-simple planes $K_i = k(r_i, r'_i)$,

$$R \oplus R' = \bigoplus_{i \in I} K_i \text{ card } I = \dim R = \dim R'$$

and, furthermore, such that $R \oplus R'$ admits of an orthogonal supplement in $E: E = (R \oplus R') \oplus H$, $H \perp R \oplus R'$.

In the case of char $k \neq 2$, the planes K_i are hyperbolic and $R \oplus R'$ thus possesses a sympletic basis (cf. Bourbaki, Formes Sesquilinéaires p. 78).

Proof. Let S and T be finite dimensional semi-simple subspaces with the following properties:

$$S \perp T$$
, $T \subset R^{\perp}$, $S = \bigoplus_{i=1}^{n} K_i$, $K_i = k(r_i, r'_i)$ and $r_i \in R$ (1)

$$(T \oplus S) \cap R = k(r_i)_{1 < i < n}. \tag{2}$$

Let $(e_m)_{m\geq 1}$ be some fixed basis of the space E and let e_m be the first basis vector not contained in $S\oplus T$. We construct finite dimensional spaces K and L in $(S\oplus T)^{\perp}$ such that $S'=S\oplus K$ and $T'=T\oplus L$ satisfy the properties (1) and (2) with S' and T' in lieu of S and T and such that $e_m \in S' \oplus T'$. In this fashion we obtain a decomposition of E of the required form:

$$E = \cup S \oplus T = (\cup S) \oplus (\cup T)$$
 , $H = \cup T$ and $R \oplus R' = \cup S$.

Since $S \oplus T$ is semi-simple and finite dimensional, we may decompose $e_m : e_m = e'_m + e''_m$ with $e'_m \in S \oplus T$ and $e''_m \perp S \oplus T$. Thus we may without loss of generality assume that $e_m \perp S \oplus T$.

First case. $e_m \in R$. Therefore $||e_m|| = 0$ and, since $(S \oplus T)^{\perp}$ is semisimple, there exists r' with $(e_m, r') \neq 0$. The space $k(e_m, r')$ is semi-simple and we put $S' = S + k(e_m, r')$ and T' = T. We have to determine $(T' + S') \cap R$. Let $r \in (T' \oplus S') \cap R$, $r = t + s + \lambda e_m + \mu r'$ with $t \in T$, $s \in S$ and $r \in R$. Since $T \subset R^{\perp}$ we obtain 0 = (v, R) = (t, R) hence t = 0 as T is semi-simple. Therefore, (since $R \subset R^{\perp}$) we obtain $0 = (r, e_m) = \mu(e_m, r')$. Thus $\mu = 0$ and $v = s + \lambda e_m$. Since $e_m \in R$ in our case therefore $s \in R$ i.e., $s \in S \cap R = k(r_i)_{i \leq n}$ by (2). Thus $(T' \oplus S') \cap R = k(r_1, \ldots, r_m, e_m)$ which, upon relabeling e_m as r_{n+1} (and r' as r'_{n+1}), is (2). The remaining conditions are trivially satisfied.

Case 2. $e_m \notin R$ and $e_m \in R^{\perp}$. We first convince ourselves that $e_m \notin R$ + $+ (S \oplus T)$; assume that $e_m = r + s + t$ with $r \in R$, $s \in S$ and $t \in T$. Since $e_m \perp S + T$ and $T \subset R^{\perp}$, we have in particular $0 = (e_m, T) = (t, T)$; hence t=0 as T is semi-simple. Since $e_m \in R^{\perp}$ in the present case, and $R \subset R^{\perp}$, we obtain furthermore $0 = (e_m, R \cap S) = (s, R \cap S)$ i.e., $S \perp S \cap R$. From the explicit form of $S = \bigoplus k(r_i, r'_i)$ we see that necessarily $s \in R \cap S$. Thus $e_m = r + s \in R$, a contradiction. Since $(R + S + T)^{\perp \perp} =$ =R+S+T, we conclude from $e_m \in R+S+T$ that $(R+S+T)^\perp \subset e_m^\perp$. Hence there exists a vector $t \in (R + S + T)^{\perp} = R^{\perp} \cap (S + T)^{\perp}$ with $(e_m, t) \neq 0$. Thus, if $||e_m|| = 0$ then $k(e_m, t)$ is a semi-simple space and we put S' = S, $T' = T + k(e_m, t)$. If, on the other hand, $||e_m|| \neq 0$, we simply put and S' = S and $T' = T + k(e_m)$. We have to determine $(T' \oplus S') \cap R$. Let, in the first case, $r \in T' \oplus S'$ i.e., $r = s + t + \lambda e_m + \mu t$ with $s \in S$, $t \in T$ and $r \in R$. Since $e_m \in R^{\perp}$ and $||e_m|| = 0$ we find $0 = (r, e_m) = \mu(t, e_m)$, therefore $\mu = 0$. Since $t \in \mathbb{R}^{\perp} \cap (S \oplus T)^{\perp}$ we then find $0 = (r, t) = \lambda(e_m, t)$. Hence $\lambda = 0$. This shows that $(T' \oplus S') \cap R = (T \oplus S) \cap R$. In the other case, $||e_m|| \neq 0$, it is even simpler to verify that $(T' \oplus S') \cap R = (T \oplus S) \cap R$. The remaining conditions (1) are trivially satisfied for S' and T'.

Case 3. $e_m \notin R^{\perp}$. As in the second case one verifies that $e_m \notin R^{\perp} + S + T$. Since $(R^{\perp} + S + T)^{\perp \perp} = R^{\perp} + S + T$, we conclude from $e_m \notin R^{\perp} + S + T$ that $(R^{\perp} + S + T)^{\perp} \not\subset e_m^{\perp}$. In other words there exists a vector $r \in (R^{\perp} + S + T)^{\perp} = R^{\perp \perp} \cap (S \oplus T)^{\perp} = R \cap (S \oplus T)^{\perp}$ with $(e_m, r) \neq 0$. Since $r \in R$ we have ||r|| = 0 and the space $k(r, e_m)$ is semi-simple. We put $S' = S \oplus k(r, e_m)$ and T' = T. Upon relabeling r as r_{n+1} (and e_m as r'_{n+1}) the conditions (1) and (2) are verified as in case 1. Q.E.D.

Lemma 3 often finds application in the following situation. Suppose that G is a subspace of E such that the radical $R = G \cap G^{\perp}$ of G happens to be a closed subspace of E. We then have a decomposition $E = (R \oplus R') \oplus H$, $H \perp (R \oplus R')$. Furthermore, one can always find an algebraic complement L of R in G such that $L \subset H$. For, if L_0 is some algebraic complement of R in G then $L_0 \perp R$. Every vector $l_0 \in L_0$ has a decomposition $l_0 = r + r' + h$. Since $l_0 \perp R$ necessarily r' = 0. In other words, $L_0 \subset R \oplus H$ which shows that there is a complement L of R in G with $L \subset H$.

We are interested in decompositions of E of the following sort: E is an orthogonal sum $E=\oplus E_i$ such that the ranges $||E_i||$ of the summands are either 0 or 1-dimensional subspaces of the k^2 -vector space ||E|| and such that the elements spanning the non trivial $||E_i||$ are linearly independent over k^2 . In other words,

$$E = \Sigma P \oplus \Sigma \langle \alpha_1 \rangle \oplus \Sigma \langle \alpha_2 \rangle \oplus \dots$$

where the P_s are hyperbolic planes and where the field elements $\alpha_1, \alpha_2, \ldots$ are linearly independent over k^2 . In view of Lemma 1 we may assume that the summands $\Sigma \langle \alpha_i \rangle$ are either of infinite dimension or of dimension ≤ 2 . Thus, collecting 1-, 2- and \aleph_0 -dimensional summands we may rewrite the above decomposition as follows:

$$E = \sum P \oplus_{i \in I_1} E_{(\beta_i)} \oplus_{i \in I_2} \langle \gamma_i \gamma_i \rangle \oplus_{i \in I_3} \langle \delta_i \rangle$$
 (1)

where all the field elements β_i , γ_i , δ_i together are independent over k^2 .

We shall determine those k-space (E, Φ) which admit of a decomposition of type (1). We first have

Proposition 1. If E admits of a decomposition (1) then

$$E_{\star}^{\perp} \oplus E_{\star}^{\perp \perp} = (\text{rad } E_{\star})^{\perp}. \tag{2}$$

Proof. Let for every $i \in I_1$ the space $E_{(\beta i)}$ be spanned by the vectors $(e_{ii})_{i\geq 1} \cdot (E_{(\beta_i)})_*$ is spanned by the vectors $(e_{i1} + e_{ii})_{i\geq 1}$ and, the orthogonal complement of $(E_{(\beta_i)})_*$ in $E_{(\beta_i)}$ is (0). Let furthermore, for every $i \in I_2$, $\langle \gamma_i \gamma_i \rangle$ be spanned by the vectors f_i, f'_j . Since all the elements $\beta_i, \gamma_j, \delta_e$ together are independent over k^2 (by assumption), we obtain for E_* from (1)

$$\boldsymbol{E}_{*} = \boldsymbol{\Sigma} \boldsymbol{P} \oplus \boldsymbol{\oplus} \boldsymbol{E}_{(\boldsymbol{\beta_{i}})*} \oplus \boldsymbol{\oplus}_{i \in I_{2}} \boldsymbol{k} \left(f_{i} + f_{i}' \right) \boldsymbol{\oplus} \left(0 \right).$$

Furthermore

$$E_{\,igstar}^{\,\perp} = (0) \oplus \oplus k \, (f_i + f_i') \oplus \oplus \langle \delta_i
angle \ \ ext{and} \ \ E_{\,igstar}^{\,\perp\,\perp} = \varSigma P \oplus E_{(eta_i)\,igstar} \, E_{(eta_i)\,igstar} \oplus \oplus k \, (f_i + f_i').$$

From this we readily read off that (2) holds.

Condition (2) is not always satisfied. The simplest kind of counter-example is the following. Let E be spanned by the basis vectors $\{e_i\}_{i\geq 1} \cup \{f_i\}_{i\geq 1} \cup \{g_0\}$ and let Φ be defined on the basis as follows: $||e_i|| = \alpha$ and $(e_i, e_j) =$ $=0 (i \neq j, i, j \geq 1), ||f_i|| = \beta_i \text{ and } (f_i, f_j) = 0 (i \neq j, i, j \geq 1), ||g_0|| = \gamma$ and $(e_i, f_j) = 0$, $(e_i, g_0) = \alpha$, $(f_i, g_0) = \beta_i$, $(i, j \ge 1)$ for $\alpha, \gamma, \beta_1, \beta_2, \ldots$ independent over k^2 (a field with $[k:k^2] \geq \aleph_0$ is required). Here rad $E_* = 0$ and $(\operatorname{rad} E_*)^{\perp} = E$, but $E_*^{\perp} + E_{*-}^{\perp \perp}$ falls short of E by one dimension. We remark that (2) is equivalent to $E_*^{\perp} \oplus E_*^{\perp \perp}$ being closed.

We shall prove that the converse of Proposition 1 is true. This is accomplished by reducing the general case to the cases of spaces E with $E_*^{\perp} = (0)$ or $E_*^{\perp} = E_*$. We start out with these special cases.

Lemma 4. Let (E, Φ) be a semi-simple space of denumerable dimension with $E_*^{\perp} = (0)$. Then for every $\alpha \in ||E||$ and every orthogonal decomposition $E = H \oplus H^{\perp}$ with finite dimensional H we have $\alpha \in ||H^{\perp}||$.

Proof. Let $E = H \oplus H^{\perp}$ be any decomposition with finite dimensional H, furthermore α some arbitrarily fixed element in ||E||. We apply Lemma 1.2 with E_* and H in the roles of V and F respectively. Since $\alpha \in ||E||$, there exists some vector $x_0 \in E$ with $||x_0|| = \alpha$. Hence there exists a vector $x \in E_*$ with $(x, f_i) = -(x_0, f_i), f_1, \ldots, f_n$ a fixed basis of H. Therefore $(x_0 + x, f_i) = 0$ i.e., $x_0 + x \perp H$. Since $x \in E_*$ we have $||x_0 + x|| = ||x_0|| = \alpha$.

Proposition 2. Let (E, Φ) be a semi-simple space of denumerable dimension with $||E|| \neq 0$. We have an orthogonal decomposition

$$E=\mathop{\oplus}\limits_{i\,\epsilon\, I}E_{(\pi_i)}$$

where $\{\pi_i\}_{i \in I}$ is a k²-basis for ||E|| if and only if $E_*^{\perp} = (0)$.

Proof. If E admits such a decomposition it is readily verified that $E_*^{\perp}=(0)$. Let us then assume that $E_*^{\perp}=(0)$. We construct a decomposition of E of the required type step by step. Let $F=\Sigma P\oplus \Sigma\langle\pi_1\rangle\oplus\ldots\oplus\Sigma\langle\pi_n\rangle$ be a finite dimensional subspace of E, the P_s hyperbolic planes and the field elements π_1,\ldots,π_n linearly independent over k^2 . Let furthermore $(e_i)_{i\geq 1}$ be some fixed basis for the space E and assume that e_m is the first basis vector not contained in F. We shall construct a finite dimensional subspace E in E such that E is and E is an E in E is of the form E is an E in E

Since F is finite dimensional and semi-simple, we may decompose $e_m: e_m = e'_m + e''_m$ with $e'_m \in F$ and $e''_m \perp F$. Three cases are possible: $||e''_m|| = 0$ and e''_m is contained in some hyperbolic plane $P' \subset F^\perp$ or $||e''_m|| \neq 0$ or $||e''_m|| = 0$ and $e''_m \in \langle \delta, \delta \rangle \subset F^\perp$ for some $0 \neq \delta \in k$. In the first case we may choose P' for H and we put $F' = F \oplus P'$. In the second case we put $F' = F \oplus k(e''_m)$ provided that $e''_m \notin ||F||$. If, on the other hand, we should have $e''_m = \sum_{i=1}^{n} \lambda_i^2 \pi_i$ with, say $\lambda_1 \neq 0$, then we apply Lemma 4 a finite number of times and find a sequence of mutually orthogonal vectors h_1, h_2, \ldots, h_n in $(F + k(e''_m))^\perp$ with $||h_1|| = ||e''_m||, ||h_i|| = \pi_i, 2 \leq i \leq n$. By Lemma 2 the space H spanned by $e''_m, h_1, h_2, \ldots, h_n$ is isomorphic to $\langle \pi_1 \pi_1 \pi_2 \pi_3, \ldots, \pi_n \rangle$ and we put $F' = F \oplus H$. The third case is treated in

the same way, the first two vectors for the construction of H already at hand. Thus, in all three cases we find $F' = F \oplus H$, $e_m \in F'$ where F' again is of the form $\Sigma P \oplus \Sigma \langle \pi_1 \rangle \oplus \ldots \oplus \Sigma \langle \pi_r \rangle$, the $\pi_i s$ linearly independent over k^2 . In this fashion we find an orthogonal decomposition of E as follows, $E=\cup F=\varSigma P\oplus \varSigma \langle \pi_1 \rangle \oplus \varSigma \langle \pi_2 \rangle \oplus \ldots$. In view of the independence of the $\pi_i s$ we have $E_* = \Sigma P \oplus (\Sigma \langle \pi_1 \rangle)_* \oplus \ldots$ Not all of the summands $\Sigma\langle\pi_i\rangle$ can be (0) since $||E||\neq 0$. Thus, if one of the summands should be finite dimensional we would have $E_*^{\perp} \neq (0)$, contrary to assumption. Hence all the summands $\Sigma \langle \pi_i \rangle$ are infinite dimensional. Application of Corollary 1 finally yields $E \cong E_{(\pi_1)} \oplus E_{(\pi_2)} \oplus \dots$

Corollary 2. If (E, Φ) is a space with $E_*^{\perp} = (0)$ whose range $||E|| \neq 0$ is spanned by the elements π_1, \ldots, π_m (not necessarily independent over k^2) then E is isomorphic to $E_{(\pi_n)} \oplus \ldots \oplus E_{(\pi_m)}$.

Proof. By Proposition 2 $E \cong E_{(\sigma_n)} \oplus \ldots \oplus E_{(\sigma_1)}$ where $\sigma_1 \ldots \sigma_n$ is a k^2 -basis for ||E||. Let then π_1, \ldots, π_n $(n \leq m)$ be a subset of elements independent over k^2 . By Corollary 1 (ii) we have

$$\langle \pi_1 \pi_1 \rangle \oplus \ldots \oplus \langle \pi_n \pi_n \rangle \cong \langle \sigma_1 \sigma_1 \rangle \oplus \ldots \oplus \langle \sigma_n \sigma_n \rangle.$$

Hence trivially $E_{(\sigma_1)} \oplus \ldots \oplus E_{(\sigma_n)} \cong E_{(\pi_1)} \oplus \ldots \oplus E_{(\pi_n)}$. Let $\pi_{n+1} = \sum_{i=1}^{r} \lambda_i^2 \pi_i$. After renumbering $\pi_1 \ldots \pi_n$ we may assume that $\lambda_i \neq 0$, $1 \leq r \leq i$. Hence by Corollary 1 (ii) $\langle \pi_{n+1} \pi_{n+1} \pi_2 \dots \pi_r \rangle \cong \langle \pi_1 \pi_1 \pi_2 \dots \pi_n \rangle$. Thus $E_{(\pi_{n+1})} \oplus E_{(\pi_2)} \oplus \ldots \oplus E_{(\pi_r)} \cong E_{(\pi_1)} \oplus \ldots \oplus E_{(\pi_r)}$ can be arranged in a trivial fashion. In this manner we obtain $E_{(\pi_1)} \oplus \ldots \oplus E_{(\pi_m)} \cong E$.

Proposition 3. Let (E, Φ) be a semi-simple space of at most denumerable dimension. We have an orthogonal decomposition

$$E = \bigoplus_{i \in I} \langle \pi_i \pi_i \rangle$$

where the π_i form some k^2 -basis for ||E|| if and only if $E_*^{\perp} = E_*$.

Proof. If E admits such a decomposition we trivially have $E_*^{\perp} = E_*$. Conversely, let us assume that $E_*^{\perp} = E_*$. We first remark that E cannot contain a triple of mutually orthogonal vectors of the same length $\neq 0$. For,

assume that z_1, z_2, z_3 were such vectors, $||z_1|| = ||z_2|| = ||z_3|| \neq 0$. We decompose according to the decomposition $E = E_* \oplus L : z_1 = e_1 + l_1$, $z_2 = e_2 + l_2$, $z_3 = e_3 + l_3$. Thus $||l_1|| = ||l_2|| = ||l_3||$. Since L contains no isotropic vectors we must necessarily have $l_1 = l_2 = l_3$. Since E_* is totally isotropic in our case, the three orthogonality conditions reduce to $0 = (e_1 + e_2, l_1) + || l_1 ||, \ 0 = (e_1 + e_3, l_1) + || l_1 ||, \ 0 = (e_2 + e_3, l_1) + || l_1 ||.$ Adding the first two of these equations we obtain $(e_2 + e_3, l_1) = 0$ which contradicts the third one as $||l_1|| \neq 0$. We now construct a decomposition of E step by step as in the proof of Proposition 2. Let $F=\langle \pi_1\,\pi_1\rangle\oplus\langle \pi_2\,\pi_2\rangle\oplus$ $\oplus \ldots \oplus \langle \pi_n \pi_n \rangle$ be a finite dimensional subspace of E, $\pi_1, \pi_2, \ldots, \pi_n$ linearly independent over k^2 . Furthermore, let e_m again be the first basis vector of some fixed basis for E not contained in F. Without loss of generality we may proceed assuming that $e_m \perp F$. We consider first the case that $||e_m|| \neq 0$. We try to find a vector $l \in F^{\perp} \cap E_*$ with $(l, e_m) \neq 0$. Suppose that there is no such vector l, in other words $F^{\perp} \cap E_* \subset e_m^{\perp}$. Since E_* is closed in our case, we find $(F + E_*^{\perp})^{\perp} = F^{\perp} \wedge E_*^{\perp} = F^{\perp} \wedge E_* \subset e_m^{\perp}$ therefore $e_m \in (F + E_*^{\perp})^{\perp \perp} = F + E_*^{\perp}$ i.e., $e_m \in F + E_*^{\perp} = F + E_*$. Thus $e_m = f + f_0$ with $||e_m|| = ||f|| \neq 0$.

Since $f \in F$ we should therefore have three mutually orthogonal vectors of the same length $||e_m|| \neq 0$, a contradiction (if F contains one vector of some length $\alpha \neq 0$, then it contains, by virtue of its form, two orthogonal vectors of that length). Thus we must have $F^{\perp} \cap E_* \subset e_m^{\perp}$ and there exists a vector $l \in F^{\perp} \cap E_*$ with $(e_m, l) \neq 0$. Hence e_m and $e'_m = e_m + \frac{||e_m||}{(l, e_m)} l$ are mutually orthogonal vectors of F^{\perp} with $||e_m|| = ||e'_m||$. We put $F' = F \oplus k (e_m, e'_m)$. There remains the possibility that $||e_m|| = 0$. Since E_* is totally isotropic, e_m cannot be contained in a hyperbolic plane, therefore $e_m \in \langle \delta, \delta \rangle \subset F^{\perp}$ for some $0 \neq \delta \in k$ (F^{\perp} is semi-simple). Since there cannot be more than two orthogonal vectors of the same length $\neq 0$ we must have $\delta \notin ||F||$ and we put $F' = F \oplus \langle \delta \delta \rangle$ similar to the former case. In this fashion we obtain a decomposition of E of the required form, $E = \cup F = \langle n_1 n_1 \rangle \oplus \langle n_2 n_2 \rangle \oplus \ldots$ where all the $\pi_i s$ are linearly independent over k^2 . We now prove the converse of Proposition 1.

Theorem 1. Let char k=2 and (E,Φ) a semi-simple k-space of denumerable dimension and let E_* be the subspace of vectors of length zero. If

$$E_{*}^{\scriptscriptstyle \perp} + E_{*}^{\scriptscriptstyle \perp \perp} = (\operatorname{rad} E_{*})^{\scriptscriptstyle \perp}$$

then E admits of an orthogonal decomposition

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$$E = \bigoplus_{i \in I_1} E_{(r_i)} \oplus \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle \oplus \bigoplus_{i \in I_2} \langle \alpha_i \rangle \tag{I}$$

or

$$E = \bigoplus_{i \in I_1} P_i \oplus \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle$$
 (II)

where, in the first case, the elements of the union $\{\gamma_i\}_{i \in I_1} \cup \{\beta_i\}_{i \in I_2} \cup \{\alpha_i\}_{i \in I_3}$ are a k^2 -basis of the range ||E|| over k^2 , in the second case the same for the elements of the union $\{\beta_i\}_{i \in I_2} \cup \{\alpha_i\}_{i \in I_3}$ (the P_i s are hyperbolic planes).

Proof. Let $R = \text{rad}(E_*^{\perp \perp}) = (E_* + E_*^{\perp})^{\perp}$. Since R is totally isotropic and closed, we can apply Lemma 3 and obtain a decomposition

$$E = (R \oplus R') \oplus H$$
, $H \perp (R \oplus R')$
 $R \oplus R' = \bigoplus_{i \in I_2} k (r_i, r'_i)$, $R = \bigoplus k (r_i)_{i \in I_2}$. (1)

Since $R \perp E_*^{\perp \perp}$, we can find an algebraic complement S of R in $E_*^{\perp \perp}$ with $S \perp R'$ (see the remark following the proof of Lemma 3). Hence $S \perp R \oplus R'$:

$$E_*^{\perp \perp} = R \oplus S$$
, $S \subset H$. (2)

Furthermore S is semi-simple. If T is the orthogonal of S in H, we obtain from (2) $E_*^{\perp} = E_*^{\perp \perp \perp} = R \oplus T$. On the other hand, by the assumption of the theorem $R \oplus H = R^{\perp} = E_*^{\perp} + E_*^{\perp \perp} = R \oplus (S \oplus T)$. Since $S + T \subset H$ therefore S + T = H. Furthermore, since S is semi-simple, the sum S + T is direct. Thus E is decomposed into three orthogonal summands:

$$E = (R \oplus R') \oplus S \oplus T \tag{3}$$

and it remains to discuss the spaces $R \oplus R'$, S and T. With regard to S we first remark that

$$E_* = R \oplus S_* . \tag{4}$$

For $R \oplus S_* \subset E_*$ is trivial. Conversely, if $x \in E_* \subset E_*^{\perp \perp} = R \oplus S$ we have x = r + s with $r \in R$ and $s \in S$. Therefore 0 = ||x|| = ||r|| + ||s|| = ||s|| and $s \in S_*$. This shows $E_* \subset R + S_*$. Let then $S_*^{\perp s}$ be the orthogonal of S_* in S. Since $S_*^{\perp s} \subset S$ and $S \perp R$ we have $S_*^{\perp s} \subset E_*^{\perp}$ by (4). Also $S_*^{\perp s} \subset S \subset E_*^{\perp \perp}$, hence $S_*^{\perp s} \subset E_*^{\perp} \cap E_*^{\perp \perp} = R$. Therefore $S_*^{\perp s} = (0)$ as

 $S_*^{\perp_s} \subset S$ and $S \cap R = (0)$. Thus, S is semi-simple and $S_*^{\perp_s} = (0)$. Two cases are possible for S: Either $S = S_*$ in which case S is a sum of hyperbolic planes or else $S \neq S_*$ in which case the range ||S|| is different from 0 and Proposition 2 can be quoted: Thus

either
$$S = \bigoplus_{i \in I_1} P_i$$
 or $S = \bigoplus_{i \in I_1} E_{(\gamma_i)}$. (5)

From (4) we learn that $R' \cap E_* = (0)$. Therefore, taking orthogonals in R + R', we obtain $(R + R')_* = R = R^{\perp} = (R + R')_*$ and we may cite Proposition 3:

$$R \oplus R' = \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle . \tag{6}$$

Finally $E_* \cap T = (0)$ by (4), i.e., T contains no isotropic vectors. Hence T possesses an orthogonal basis, $T = \bigoplus_{i \in I_s} \langle \alpha_i \rangle$ where all the $\alpha_i s$ are independent over k^2 . Summarizing the facts about the decomposition (3) we see that E admits of an orthogonal decomposition of the form

$$E = \underset{i \in I_1}{\oplus} E_{(?_i)} \oplus \underset{i \in I_2}{\oplus} \langle \beta_i \beta_i \rangle \oplus \underset{i \in I_3}{\oplus} \langle \alpha_i \rangle \quad \text{or} \quad E = \underset{i \in I_1}{\oplus} P_i \oplus \underset{i \in I_2}{\oplus} \langle \beta_i \beta_i \rangle \oplus \underset{i \in I_3}{\oplus} \langle \alpha_i \rangle.$$

A dependence $0 = \sum v_i^2 \gamma_i + \sum \mu_i^2 \beta_i + \sum \kappa_i^2 \alpha_i$ defines an isotropic vector $x = \sum v_i c_i + \sum \mu_i b_i + \sum \kappa_i a_i$, $\sum v_i c_i \in S$, $\sum \mu_i b_i \in R + R'$ and $\sum \kappa_i a_i \in T$. By (4) $x \in E_* = R + S_*$ and thus $\kappa_i = 0$, $||\sum v_i c_i|| = \sum v_i^2 \gamma_i = 0$ and $||\sum \mu_i b_i|| = \sum \mu_i^2 \beta_i = 0$. However, the $\gamma_i s$ are linearly independent over k^2 by Proposition 2. Therefore $v_i = 0$. Proposition 3 guarantees the independence of the $\beta_i s$ and therefore $\mu_i = 0$. This proves that the elements $\gamma_i, \beta_j, \alpha_e$ together are independent over k^2 and the proof of Theorem 1 is complete.

Theorem 1 can be used to discuss the problem of isomorphism between \aleph_0 -dimensional k-spaces (E, Φ) in a large number of cases. We shall give here a complete discussion of the cases where the underlying field k is of finite dimension over its subfield k^2 . Thus, let k be a field with $[k:k^2]$ finite. For a space (E, Φ) we have codim $E_* \leq [k:k^2]$ or else an algebraic complement of E_* in E should contain an isotropic vector which is impossible. Since dim $E_*^{\perp} \leq \operatorname{codim} E_*$, the space E_*^{\perp} is finite dimensional and $E_*^{\perp \perp} + E_*^{\perp}$ is therefore closed. Hence every space of denumerable dimension over such a field admits of a basis as described by Theorem 1. (The following discussion also includes that of spaces (E, Φ) with ||E|| finite dimensional over k^2 , k an arbitrary field.)

Theorem 2. Let k be a field of characteristic 2 of finite dimension n over its subfield k^2 $(n = [k : k^2])$, (E, Φ) an \aleph_0 -dimensional semi-simple space over k. Then (i) E is of the form:

$$E = E_{(\gamma_1)} \oplus \ldots \oplus E_{(\gamma_r)} \oplus \langle \beta_1 \beta_1 \beta_2 \beta_2 \ldots \beta_s \beta_s \rangle \oplus \langle \alpha_1 \alpha_2 \ldots \alpha_t \rangle \quad r \geq 1 \quad (I)$$

or

$$E = \overset{\boldsymbol{\circ}}{\Sigma} P \oplus \langle \beta_1 \beta_1 \beta_2 \beta_2 \dots \beta_p \beta_p \rangle \oplus \langle \alpha_1 \alpha_2 \dots \alpha_q \rangle, \tag{II}$$

where all the sums are orthogonal and, in the first case, the elements $\gamma_1, \ldots, \gamma_r, \beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_t$ are independent over k^2 and the same for $\beta_1, \ldots, \beta_p, \alpha_1, \ldots, \alpha_q$ in the second case (thus $r + s + t \leq n, p + q \leq n$).

- (ii) E is uniquely determined, up to orthogonal isomorphism, by its range ||E||, the range $||E_{*}^{\perp\perp}||$ and by the space E_{*}^{\perp} . (In particular, the numbers r, s and t, respectively p and q are orthogonal invariants of the space E.)
- (iii) In terms of the above bases: If $||E_{*}^{\perp}|| \neq 0$ (i.e., E_{*} not closed) then E is of type (I), if $||E_*^{\perp\perp}|| = 0$ (i.e., E_* closed) then E is of type (II). (Thus (I) and (II) represent non isomorphic spaces.) A space of type (I) is uniquely determined, up to orthogonal isomorphism, by ||E||, the subspace of k (over k^2) spanned by the elements $\gamma_1, \ldots, \gamma_r$ and by the space $\langle \alpha_1, \ldots, \alpha_t \rangle$. A space of type (II) is uniquely determined, up to isomorphism, by ||E|| and by the space $\langle \alpha_1, \ldots, \alpha_q \rangle$.

Proof. It only remains to discuss the question of isomorphisms. For a space of type (I) let $E_{(\gamma_i)}$ be spanned by a basis $\{e_{ij}\}_{j\geq 1} \cdot E_{(\gamma_i)_*}$ is then spanned by the vectors $e_{i1} + e_{ij}$ $(j \ge 1)$ and the orthogonal of $E_{(\gamma_i)}$ in $E_{(\gamma_i)}$ is 0. Let $\langle \beta_1 \beta_1, \ldots, \beta_s \beta_s \rangle$ be spanned by a basis $\{e_i, e_i'\}_{1 \le i \le s}$ and let R be the totally isotropic space $k(e_i + e'_i)_{1 \leq i \leq s}$. We then have, by virtue of the independence of the lements $\gamma_1, \ldots, \beta_1, \ldots, \alpha_1, \ldots$

$$egin{aligned} E_* &= E_{(\gamma_1)\,*} \oplus \ldots \oplus E_{(\gamma_r)\,*} \oplus R, \ E_*^\perp &= R \oplus \langle lpha_1, \, \ldots, lpha_t
angle \, , \end{aligned}$$
 $E_*^{\perp\perp} &= E_{(\gamma_1)} \oplus \ldots \oplus E_{(\gamma_r)} \oplus R \, .$

Let \overline{E} be another space falling into category (I), $\overline{E} = E_{(\gamma_1)} \oplus \ldots \oplus E_{(\gamma_{\overline{r}})} \oplus$ $\oplus \langle \overline{\beta_1} \overline{\beta_1}, \ldots, \overline{\beta_{\bar{s}}} \overline{\beta_{\bar{s}}} \rangle \oplus \langle \overline{\alpha_1}, \ldots, \overline{\alpha_t} \rangle$ such that $||E|| = ||\overline{E}||, ||E_{*}^{\perp \perp}|| = ||\overline{E}_{*}^{\perp \perp}||$ and $E_*^{\perp} \cong \overline{E}_*^{\perp}$. We have to prove that $E \cong \overline{E}$. Since $\gamma_1, \ldots, \gamma_r$ and $\overline{\gamma}_1, \ldots, \overline{\gamma}_{\overline{r}}$ are independent over k^2 we first have $r = \overline{r}$ (since $||E_*^{\perp \perp}|| =$ $= ||\overline{E}_*^{\perp \perp}||$). By Corollary 2 we see that $|E_*^{\perp \perp} \cong \overline{E}_*^{\perp \perp}$. Hence we may introduce a new basis in $\overline{E}_{*}^{\perp \perp}$ such that $\overline{\gamma}_{i}$, $= \gamma_{i}$, $1 \leq i \leq r$. From the isomorphism $R \oplus \langle \alpha_{1}, \ldots, \alpha_{t} \rangle \cong \overline{R} \oplus \langle \overline{\alpha}_{1}, \ldots, \overline{\alpha}_{\overline{t}} \rangle$ we conclude that $\langle \alpha_{1}, \ldots, \alpha_{t} \rangle \cong \overline{\alpha}_{1}, \ldots, \overline{\alpha}_{\overline{t}} \rangle$ since R and \overline{R} are totally isotropic orthogonal summands and since both $\langle \alpha_{1}, \ldots, \alpha_{t} \rangle$ and $\langle \overline{\alpha}_{1}, \ldots, \overline{\alpha}_{\overline{t}} \rangle$ are semi-simple (even non-isotropic by the independence of the αs). Thus t = t and we may introduce a new basis in $\langle \overline{\alpha}_{1}, \ldots, \overline{\alpha}_{t} \rangle$ such that $\overline{\alpha}_{i} = \alpha_{i}$, $1 \leq i \leq t$. Finally, since $||E|| = ||\overline{E}||$ and since $\gamma_{1}, \ldots, \beta_{1}, \ldots, \alpha_{1}, \ldots$ and $\overline{\gamma}_{1}, \ldots, \overline{\beta}_{1}, \ldots, \overline{\alpha}_{1}, \ldots$ are independent over k^{2} we have $r + s + t = \overline{r} + \overline{s} + \overline{t}$; therefore $s = \overline{s}$ as $r = \overline{r}$ and $t = \overline{t}$. Furthermore, having introduced the new bases in $\overline{E}_{*}^{\perp \perp}$ and $\langle \overline{\alpha}_{1}, \ldots, \overline{\alpha}_{t} \rangle$ we may cite Corollary 1 (ii), $\langle \gamma_{1}, \ldots, \gamma_{r} \rangle \oplus \langle \beta_{1}\beta_{1}, \ldots, \beta_{s}\beta_{s} \rangle \oplus \langle \alpha_{1}, \ldots, \alpha_{t} \rangle \cong \langle \overline{\gamma}_{1}, \ldots, \overline{\gamma}_{r} \rangle \oplus \langle \overline{\beta}_{1}\overline{\beta}_{1}, \ldots, \overline{\beta}_{s}\overline{\beta}_{s} \rangle \oplus \langle \overline{\alpha}_{1}, \ldots, \overline{\alpha}_{t} \rangle$. A fortiori $E_{(\gamma_{1})} \oplus \ldots \oplus E_{(\gamma_{r})} \oplus \langle \beta_{1}\beta_{1}, \ldots, \beta_{s}\beta_{s} \rangle \oplus \langle \alpha_{1}, \ldots, \alpha_{t} \rangle \cong E_{(\overline{\gamma}_{1})} \oplus \ldots \oplus E_{(\overline{\gamma}_{r})} \oplus \langle \overline{\beta}_{1}\overline{\beta}_{1}, \ldots, \overline{\beta}_{s}\overline{\beta}_{s} \rangle \oplus \langle \overline{\alpha}_{1}, \ldots, \overline{\alpha}_{t} \rangle$ and thus $E \cong \overline{E}$. The simpler case of spaces falling into category (II) is treated in the same way. This proves Theorem 2.

Theorem 2 may also be expressed in the following way: If $[k:k^2]$ is finite and (E,Φ) an \aleph_0 -dimensional, semi-simple k-space with E_* not closed, then there exist three finite dimensional k-spaces F,G and H such that $F\oplus G\oplus H$ contains no isotropic vectors and E is isomorphic to the (external) orthogonal sum $(\Sigma F) \oplus G \oplus G \oplus H$. E is uniquely determined by the ranges ||F+G+H||, ||F|| and by the space H; on the other hand, if E_* is closed, then there exist two finite dimensional k-spaces G and H such that $G \oplus H$ contains no isotropic vector and E is isomorphic to the (external) orthogonal sum $(\Sigma P) \oplus G \oplus G \oplus H$. In this case E is uniquely determined by the ranges ||G+H|| and by the space H.

We should like to mention that Theorem 2 alone can be obtained more directly by proving Theorem 1 only for spaces E with ||E|| of finite dimension over k^2 . This is done by an induction on $\dim_{k^2}||E||$. For $\dim_{k^2}||E||=0$ we have $E=\Sigma P$. After induction assumption two cases arise which have to be treated differently: First case, there exists some decomposition $E=H\oplus H^\perp$ with finite dimensional H such that $\dim_{k^2}||H^\perp||<\dim_{k^2}||E||$. Hence there is a basis of the required sort for H^\perp by the induction assumption. The required basis for E is then found easily by applications of Corollary 1. Second case, there is no such decomposition of E. In that case, one proves directly that $E=E_{(\pi_1)}\oplus\ldots\oplus E_{(\pi_n)}$ where π_1,\ldots,π_n span ||E||. This is accomplished along the line of the proof of Proposition 2, where now the assumption of our case replaces the function of Lemma 4.

Thus, for fields k with finite $[k:k^2]$ a complete list of non isomorphic k-spaces (E, Φ) of denumerable dimension can easily be given on the basis of Theorem 2, provided one knows the finite dimensional, non-isotropic k-spaces $(\langle \alpha_1, \ldots, \alpha_t \rangle!)$. It is advantageous to first subdivide the spaces according to the dimensions of E/E_* , E_*^\perp and $\operatorname{rad}(E_*)$. In the notations of Theorem 2: p+q, $r+s+t=\dim(E/E_*)$; p+q, $s+t=\dim(E_*^\perp)$; $p,s=\dim(\operatorname{rad}E_*)$ p+q, $r+s+t\leq [k:k^2]$. We may use uniformly the notations r,s,t by interpreting a triple (r,s,t) with r=0 as belonging to a space of type (II). There are $\frac{(n+1)(n+2)(n+3)}{6}$ ordered triples (r,s,t) with $0\leq r+s+t\leq n$; they yield a subdivision of all semi-simple (r,s,t) with (r,s,t) and (r,s,t) are then taken of (r,s,t) into (r,s,t) and (r,s,t) are then taken. For the sake of illustration, we give a complete list for an underlying field k with $[k:k^2]=2$:

$\dim E/E*$ $r+s+t$	$\dim E_{*}^{\perp}$ $s+t$	$\dim \atop (\operatorname{rad} E_*)$	
0	0	0	$\overset{\infty}{\Sigma}P$
1	0	0	$E_{(u)}$
1	1	0	$\overset{igstyle \infty}{\Sigma}P\oplus\langle v angle$
1	1	1	$\overset{f \infty}{\Sigma}P\oplus\langle v,v angle$
2	0	0	$E_{(\alpha)} \oplus E_{(\beta)}$
2	1	0	$E_{(\nu)} \oplus \langle \mu \rangle \ \nu \neq \mu$
2	1	1	$E_{(\alpha)} \oplus \langle \beta, \beta \rangle, \ E_{(\nu)} \oplus \langle \alpha, \alpha \rangle \ \nu \neq \alpha$
2	2	0	$\overset{\mathbf{\sigma}}{\Sigma}P\oplus\langlelpha, u angle\ \ u eqlpha$
2	2	1	$\overset{\infty}{\Sigma}P\oplus\langleeta,eta angle\oplus\langlelpha angle,\;\overset{\infty}{\Sigma}P\oplus\langlelpha,lpha angle\oplus\langle u angle\;\; u eqlpha$
2	2	2	$\overset{oldsymbol{lpha}}{\mathcal{\Sigma}P} \oplus \langle lpha, lpha angle \oplus \langle eta, eta angle$

All the sums are orthogonal, $\{\alpha, \beta\}$ is some fixed basis of k over k^2 ; ν and μ run independently through a fixed set of representatives of g_k (the multi-

plicative group of k modulo square factors), subject only to conditions listed in the table. All the spaces thus obtained are mutually non isomorphic and they are, up to orthogonal isomorphisms, all semi-simple k-spaces (E, Φ) of denumerable dimension.

III. Orthogonal bases

Let k be an arbitrary field of characteristic 2. If the semi-simple k-space (E, Φ) is finite dimensional, then either $E = \Sigma P$ or E possesses an orthogonal basis (Lemma 1). Let (E, Φ) be a space of denumerable dimension. E is an orthogonal sum $\Sigma P \oplus E_0$ where E_0 possesses an orthogonal basis. If $\dim_k(E/E_*)$ is infinite (i.e., $\dim_{k^2}||E||$ is infinite), then dim E_0 is infinite and E has an orthogonal basis by virtue of Lemma 1. Thus, if E does not admit of an orthogonal basis, then E/E_* is of finite dimension and there exists a decomposition of E as described in Theorem 2 (necessarily of type (II)): $E = \Sigma P \oplus E_0$, where E_0 is finite dimensional and spanned by an orthogonal basis. Conversely, a space of this form does not admit of an orthogonal basis for, $\Sigma P \oplus E_0 \subset \bigoplus_{i=1}^{\infty} k(e_i)$ gives $E_0 \subset \bigoplus_{i=1}^{N} k(e_i)$ for a suitable N and thus, for the respective orthogonals, we obtain $\bigoplus_{N+1} k\left(e_i\right) \subset \Sigma P$. This is a contradiction as $||e_i|| \neq 0$ for an orthogonal basis of a semi-simple space. Thus, a space (E, Φ) of denumerable dimension admits of no orthogonal basis if and only if E_* is closed and E/E_* finite dimensional. These conditions may be formulated in various ways. Here is a selection:

Theorem 3. Let k be an arbitrary field of characteristic 2, (E, Φ) a semi-simple k-space of denumerable dimension. The following statements are equivalent:

- (j) E possesses no orthogonal basis;
- (jj) E/E_* is finite dimensional and E_* is closed;
- (jjj) E_*^{\perp} is finite dimensional and dim $E/E_* = \dim E_*^{\perp}$;
- (jv) E/E_* is finite dimensional and dim (rad E_*) = dim $E/(E_*+E_*^{\perp})$.

IV. Automorphisms

We shall add here a few remarks about the group $\mathfrak{D}(E, \Phi)$ of all metric automorphisms of a space (E, Φ) , i.e., the group of all vector space auto-

morphisms $T: E \to E$ which satisfy $\Phi(Tx, Ty) = \Phi(x, y)$ for all $x, y \in E$. The underlying field k is of characteristic 2 and dim $E = \aleph_0$. The structure of the group $\mathfrak{D}(E, \Phi)$ is unknown in the general case. If (E, Φ) satisfies the conditions

$$E_*^{\perp} + E_*^{\perp\perp}$$
 is closed, dim (rad E_*) $< \aleph_0$ (1)1)

- which always takes place when the underlying field is of finite dimension $[k:k^2]$ over k^2 - then the study of $\mathfrak{D}(E,\Phi)$ can be reduced to the study of simpler groups. They are the (sympletic) group $\mathfrak{D}(E,\Phi)$, where the \mathfrak{R}_0 -dimensional space (E,Φ) is an orthogonal sum of hyperbolic planes, and the group $\mathfrak{D}(E,\Phi)$, where (E,Φ) is an orthogonal sum $E_{(\alpha_1)} \oplus E_{(\alpha_2)} \oplus \ldots$ and the elements α_1,α_2,\ldots independent over k^2 (cf. 1.3 for notations). This reduction, possible for the spaces subject to (1), shall be carried out here.

For a space satisfying (1) there is decomposition (Theorem 1):

$$E = E_0 \oplus (R + R') \oplus E_1 , \qquad (2)$$

where E_0 , $R \oplus R'$ and E_1 are orthogonal summands such that

$$R = \operatorname{rad} E_{*}, \ E_{*} = E_{0*} \oplus R, \ E_{*}^{\perp} = R \oplus E_{1}, \ E_{*}^{\perp \perp} = E_{0} \oplus R$$
 (3)

and, furthermore, $R \oplus R'$ is an orthogonal sum of planes $k(r_i, r_i')$, $i \in I$ for $\{r_i\}_{i \in I}$ and $\{r_i'\}_{i \in I}$ a basis of R and R' respectively. For every $T \in \mathfrak{D}(E, \Phi)$ we have $T(E_*) = E_*$, T(R) = R, $T(E_*^{\perp}) = E_*^{\perp}$ and $T(E_*^{\perp \perp}) = E_*^{\perp}$. When $x \in R' \oplus E_1$ we write Tx = x + Lx. Hence ||Lx|| = 0 and $Lx \in E_* \subset E_*^{\perp \perp}$,

$$Lx \in E_0 \oplus R \text{ for } x \in R' \oplus E_1.$$
 (4)

In particular, if $x \in R$ and $y \in R'$ then (x, y) = (Tx, Ty) = (Tx, y + Ly) = (Tx, y) since $Tx \in R \perp E_0 \oplus R$. Therefore (x - Tx, y) = 0 for all $y \in R'$ or $x - Tx \in R'^{\perp}$, $R'^{\perp} \cap R = 0$; hence x - Tx = 0 since x - Tx also belongs to R. Thus the restriction T/R of T to R leaves the vectors of R fixed,

$$T|_{R} = \mathbf{1}_{R}. \tag{5}$$

¹⁾ We recall an earlier example where the second condition is satisfied but not the first. See the remark at the end of this section.

Let then $x \in E_1$ and $y \in R'$. Since $E_1 \subset E_*^{\perp}$ and $T(E_*^{\perp}) = E_*^{\perp}$ we have $Lx \in R$; hence (x, y) = (Tx, Ty) = (x + Lx, y + Ly) = (x, y) + (Lx, y). Thus (Lx, y) = 0 for all $y \in R'$ i.e., $Lx \in R'^{\perp}$, $R'^{\perp} \cap R = 0$ and therefore Lx = 0 as $Lx \in R$. In other words,

$$T|_{E_1} = \mathbf{1}_{E_1}. \tag{6}$$

Thus, every automorphism of E leaves E_*^{\perp} pointwise fixed. Therefore we have for every $x \in R'$ and $y \in E_*^{\perp}$ that (x,y) = (Tx,Ty) = (Tx,y) hence $x - Tx \in E_*^{\perp \perp} = E_0 + R$ for every $x \in R'$. Therefore, and in view of (5) and (6) we can decompose the image Tx for every $x \in (R \oplus R') + E_1$ as follows, $Tx = x + L_0x + L_1x$ with $L_0x \in E_0$ and $L_1x \in R$. Computing ||Tx|| shows furthermore that even $L_0x \in E_{0*}$. We therefore have $(x \in R \oplus R' \oplus E_1)$

$$Tx = x + L_0 x + L_1 x \tag{7}$$

where the projections L_0 and L_1 are linear maps

$$L_0: R \oplus R' \oplus E_1 \rightarrow E_{0*}, L_0(R \oplus E_1) = (0);$$

$$L_1: R \oplus R' \oplus E_1 \rightarrow R, \ L_1(R \oplus E_1) = (0).$$

On the other hand, for $x \in E_0 \subset E_*^{\perp \perp} = E_0 \oplus R$ we have

$$(x \in E_0)$$
 $Tx = L_2x + L_3x$ $L_2x \in E_0$, $L_3x \in R$. (8)

Since R is totally isotropic and orthogonal to E_0 , $L_2: E_0 \to E_0$ is a metric automorphism of E_0 ; L_3 is some linear map $E_0 \to R$. If we express Tx for an arbitrary $x \in E$ by using (7) and (8), then the condition that (x, y) = (Tx, Ty) for all $x, y \in E$, $T \in \mathfrak{D}(E, \Phi)$ is equivalent with the conditions

$$(x, L_3 y) + (L_0 x, L_2 y) = 0$$
 for all $x \in R', y \in E_0$ (9)

$$(x, L_1 y) + (L_1 x, y) + (L_0 x, L_0 y) = 0$$
 for all $x, y \in R'$ (10)

(9) and (10) permits a discussion of $\mathfrak{D}(E,\Phi)$ as in the finite dimensional case

([2]). First, the system (9) and (10) admits of solutions L_0 and L_1 for arbitrarily prescribed L_2 and L_3 , L_2 an automorphism of E_0 and L_3 : $E_0 \to R$ a linear map. Indeed. For given L_2 and L_3 (9) defines a linear map L_0 : $R' \to E_{0*}$ in a unique manner. We then extend it to L_0 : $R \oplus R' \oplus E_1 \to E_{0*}$ by defining $L_0(R \oplus E_1) = (0)$. Appealing to the basis of $R \oplus R' = \bigoplus k (r_i, r_i')$ we put $L_1 r_i' = \sum \alpha_{ij} r_j$. Condition (10) is satisfied with the previously found L_0 provided that $\alpha_{ij} + \alpha_{ji} = (L_0 r_i', L_0 r_j')$. Since $(L_0 r_i', L_0 r_i') = ||L_0 r_i'|| = 0$ as $L_0 r_i' \in E_{0*}$, there are always solutions for the unknowns α_{ij} ; (this is the only place where use is made of the assumption (1) that dim $R < \aleph_0$. This proves our assertion. Thus, if T runs through $\mathfrak{D}(E, \Phi)$ then the restriction $T|_{E_0 \oplus R}$ (it leaves $E_0 \oplus R = E_+^{\perp}$ invariant!) runs through the group \mathfrak{G} of all automorphisms of the space $E_0 \oplus R$ that leave R pointwise fixed (as we have just proved, every element of \mathfrak{G} can be extended to an automorphism of E). $T \to T|_{E_0 \oplus R}$ defines an epimorphism

$$\varphi: \mathfrak{D}(E, \Phi) \to \mathfrak{G}. \tag{11}$$

The kernel $\mathfrak{C}=\ker\varphi$ can easily be described. $T\in\mathfrak{C}$ means that $T|_{E_0\oplus R}$ is the identical transformation of $E_0\oplus R$. For such a T and every $x\in E_0\oplus R\oplus E_1,\ y\in R'$ we obtain from (x,y)=(Tx,Ty)=(x,Ty) that $y-Ty\in(E_0+R+E_1)^\perp=R$. Thus

$$Tx = x + L_4x$$
, $L_4x \in R$, $x \in E$, $L_4(E_0 + R + E_1) = (0)$ (12)

(x, y) = (Tx, Ty) yields

$$(y, L_4 x) + (L_4 y, x) = (0).$$
 (13)

Conversely, every linear map $L_4: R' \to R$ meeting (13) defines an element $T \in \mathbb{C}$ by means of (12). \mathbb{C} is thus seen to be isomorphic to the additive group of linear maps $L: R \to R'$ satisfying (13). Thus, as $s = \dim R$ is finite, $\mathbb{C} \cong k^{\frac{s(s+1)}{2}}$. Let us turn to the group \mathfrak{G} . It contains the subgroup \mathfrak{G}_0 of automorphisms $T': E_0 \oplus R \to E_0 \oplus R$ of the form $T': x \to x + L_5 x$ where L_5 is an arbitrary linear map $L_5: E_0 \oplus R \to R$ with $L_5(R) = (0)$. \mathfrak{G}_0 is an invariant subgroup of \mathfrak{G} and $\mathfrak{G}/\mathfrak{G}_0 \cong \mathfrak{D}(E_0, \Phi|_{E_0})$. \mathfrak{G}_0 is isomorphic to the additive group of all linear maps $L: E_0 \to R$, and $\mathfrak{G}_0 \cong k^{\omega}$ or $\mathfrak{G}_0 \cong (1)$.

Thus, if we put $\mathfrak{C}_0 = \varphi^{-1}\mathfrak{G}_0$, we have the series of invariant subgroups

$$\mathfrak{C} \subset \mathfrak{C}_0 \subset \mathfrak{D}(E, \Phi)$$

with $\mathfrak{C} \cong k^{\frac{s(s+1)}{2}}$, $\mathfrak{C}_0/\mathfrak{C} \cong \mathfrak{G}_0$, $\mathfrak{D}(E, \Phi)/\mathfrak{C}_0 \cong \mathfrak{D}(E_0, \Phi|_{E_0})$, $s = \dim (\operatorname{rad} E_*)$. E_0 is an algebraic complement of $\operatorname{rad} E_*$ in $E_*^{\perp \perp}$; it is either an orthogonal sum of hyperbolic planes or an orthogonal sum $E_{(\alpha_1)} \oplus \ldots \oplus E_{(\alpha_n)}$, the elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ independent over k^2 .

Remark (added in proof). The condition in (1) that dim $R = \dim (\operatorname{rad} E_*) <_{\aleph_0}$ is quite unnecessary for the discussion that followed. Setting $L_1 r_i' = \sum \alpha_{ij} r_j$ the matrix equation $\alpha_{ij} + \alpha_{ji} = (L_0 r_i', L_0 r_j')$ admits row-finite solutions (which actually define a map L_1); for example $\alpha_{ij} = 0$ $(j \ge i)$, $\alpha_{ij} = (L_0 r_i', L_0 r_j')$ for j < i. For the normal series of groups obtained we have in the case dim $R = \aleph_0$: $G_0 \cong k^{\omega}$ and $C \cong k^{\omega}$.

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