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The Spectral Sequence of a Postnikov System[1])

by Donald W. Kahn

The notion of a spectral sequence first came into prominence in connection with the problem of relating the homology of a fibre space with that of the fibre and base ([12], [20]). Subsequently, a similar situation was observed for extensions of groups [9]. In all such cases, the function of the spectral sequence was to show how the homology (or cohomology) of two objects influenced that of a third. The spectral sequence of Adams [2] is of a different nature; it takes the cohomology of a space as a module over the Steenrod Algebra and constructs, from this module, the stable homotopy of the space. The Adams spectral sequence thus exhibits some of the inner forces which determine the homotopy of a space.

On the other hand, the Postnikov system ([17], [19], [21]) is a geometric construction which in a formal way completely determines the homotopy structure of a space. That is to say, the ingredients of a Postnikov system, the homotopy groups and the k-invariants, completely determine the homotopy-type of a space. Nevertheless, there has not been given a full description of how the homotopy groups and k-invariants fit together to determine the homology of a space. (Homology has usually been studied by applying the Serre spectral sequence [20] to individual terms.) The general purpose of this paper is to fill this gap. I will study here a spectral sequence which begins with the homology of $K(\pi_n, n)$ spaces, where $\pi_n = \pi_n(X)$, and which converges to $H_*(X)$. There are similar sequences with coefficients or in cohomology. In a heuristic sense and in the stable range, this sequence is dual to the Adams sequence which converges to homotopy. The most important differences are the following; 1. Our sequence is not restricted to the stable range, although outside the stable range we do not have a full description of the first term. 2. The differentials in this sequence may be directly related to the k-invariants. 3. The Hurewicz map occurs naturally in our sequence.

The principal results of this paper, according to section, are as follows: Section 1. contains preliminaries. 2. and 3. are devoted to the construction of the spectral sequence, with the more technical properties in the latter. Section 4. is devoted to the Hurewicz map $H_n : \pi_n(X) \to H_n(X)$, while 5. concerns the order of elements in ker (H_n). In section 6., we introduce a filtration on ker (H_n). 7. deals with stable homotopy groups of spheres, in which case we have a

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filtration by ideals. In 8. we relate the spectral sequence to compositions, and we derive some algebraic properties which a non-nilpotent element in the stable homotopy ring would necessarily have. In section 9. we interpret the WHITEHEAD product in the spectral sequence and obtain a result about the non-vanishing of this product.

It is pleasure to acknowledge two sources of inspiration for this work. For the special case of B_U, the classifying space for the infinite unitary group, and within the stable range, a similar sequence was used in [3]. The general universal coefficient theorem of [7] suggested the duality with the sequence of [2], as mentioned above.

1. Preliminaries

Throughout this paper, we consider spaces which have the homotopy-type of a 1-connected, countable CW-complex. All spaces are understood to have base points, which are respected by maps. However, it is convenient to omit the base points from the notation. I shall use singular homology, denoted by $H_n(X, G)$ or $H_*(X, G) = \sum_n \oplus H_n(X, G)$, or singular cohomology denoted $H^n(X, G)$, etc. In case $G = Z$, the integers, we omit the coefficient groups from the notation.

By a fibre space, I shall always mean a fibration which satisfies the absolute covering homotopy property (the $ACHP$), that is to say the covering homotopy property with respect to maps of all spaces. It is known that up to fibre homotopy equivalence, these fibre spaces are the same as the fibre spaces in the sense of SERRE [20]. By a result of MILNOR [16], our constructions of such fibre spaces do not take us outside our basic category. We shall also use the well-known construction which converts any map into a fibre map (see [10]) and which is functorial.

If X is a space, we shall use the notation $\{X_n, p_n, \pi_n\}$ for a POSTNIKOV system for X (see [10] for details). Here, $p_n : X \to X_n$, and $\pi_n : X_n \to X_{n-1}$ is a principal fibre space [18] with fibre $K(\pi_n(X), n)$. If $i^n \in H^n(\pi_n(X), n; \pi_n(X))$ is the fundamental class and τ_n is the transgression in the fibre space $\pi_n : X_n \to X_{n-1}$, we shall use the definition

$$k^{n+1} = \tau_n i^n \in H^{n+1}(X_{n-1}; \pi_n(X))$$

for the k-invariants. We note that in the semi-simplicial case, as in [17], the construction of the POSTNIKOV system is functorial and the maps $p_n: X \to X_n$ are fibre maps. However, the notion of k-invariant does not fit naturally into that theory. Hence, we work in the geometric case, where instead of functoriality, we have the weaker notion of "induced map" [10]. It can be shown by example (due to A. Dold) that in the geometric case, one cannot have a functorial construction of Postnikov systems (in which $\pi_n : X_n \to X_{n-1}$ is always a principal fibration).

2. The Spectral Sequence

We shall first construct arbitrarily large finite portions of the spectral sequence; then we shall show how these fit together to form the entire spectral sequence.

Let X be a given space in our category and let $\{X_n, p_n, \pi_n\}$ be a Postnikov system for X. Assume that X is $(p-1)$-connected, where $p > 1$. Choose $m > p$ and convert the map

$$p_m : X \to X_m$$

into a fibre map. For simplicity, we keep the same notation for this new equivalent map. We then have a tower of fibre spaces

$$X \xrightarrow{p_m} X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} \ldots \longrightarrow X_p = K(\pi_p(X), p).$$

We note that the map

$$\pi_K \circ \ldots \circ \pi_m \circ p_m$$

is homotopically equivalent to the map p_{K-1}, and thus we shall use p_{K-1} to refer to the composition. Because our fibre spaces satisfy the $ACHP$, the composition of any finite number of fibre maps is a fibre map. In particular, each P_{K-1}, for $K-1 \leq m$, is a fibre map. In the fibre space

$$p_i : X \to X_i, \; i \leq m$$

let $F_{i+1} = p_i^{-1}$ (base point) denote the fibre.

Lemma 2.1.

a) F_{i+1} is i-connected.
b) F_{i+1} is fibred over $K(\pi_{i+1}(X), i+1)$, with fibre F_{i+2}.
c) $F_{m+1} \subset F_m \subset \ldots \subset F_p = \ldots = X$ is a finite increasing filtration of the space.

Proof: a) is trivial. For b), it is sufficient to note that π_{i+1} projects F_{i+1} onto $K(\pi_{i+1}(X), i+1)$. But $K(\pi_{i+1}(X), i+1)$ is a subspace of X_{i+1} which includes the basepoint. Part c) follows by taking successive inclusions of fibres.

We shall now define the spectral sequence of length m. We set F_n equal to the base point, if $n > m+1$.

Def. 2.1. The homology couple of length m, of the POSTNIKOV system $\{X_n, p_n, \pi_n\}$ for X, is the exact couple [14] in homology of the filtration in Lemma 2.1 c). Specifically, we put

$$D_{r,s} = H_{r+s}(F_r); \quad E_{r,s} = H_{r+s}(F_r, F_{r+1}).$$

The couple maps i, j, and k are the usual maps from the exact sequence of a pair. They have respective bidegrees $(-1, 1)$, $(0, 0)$, and $(1, -2)$.

The cohomology couple is

$$D^{r,s} = H^{r+s}(F_r); \quad E^{r,s} = H^{r+s}(F_r, F_{r+1})$$

for which we have bidegrees $(1, -1)$, $(-1, 2)$ and $(0, 0)$.

We denote these couples by

$$\langle D_{r,s}; E_{r,s}\rangle_m \quad \text{and} \quad \langle D^{r,s}; E^{r,s}\rangle_m.$$

We refer to the spectral sequences associated to these couples as the spectral sequences, of length m, of the POSTNIKOV system $\{X_n, p_n, \pi_n\}$.

Remark. a) Our numbering conventions differ slightly from the usual ones. For example, in this spectral sequence, the first significant term will actually be E^1.

b) One may define such sequences with coefficients in the obvious way.

Our first task is that of showing how to obtain a spectral sequence for $\{X_n, p_n, \pi_n\}$, without regard to m.

Prop. 2.1. Suppose $m_1 > m$. Suppose also that we have constructed, as described above, two finite POSTNIKOV systems

$$X \xrightarrow{p_m} X_m \xrightarrow{\pi_m} X_{m-1} \to \cdots$$

and

$$X^1 \xrightarrow{p_{m_1}} X_{m_1} \xrightarrow{\pi_{m_1}} X_{m_1-1} \to \cdots$$

Then, X and X^1 have the same homotopy-type, and furthermore, there is a homotopy equivalence

$$f: X \to X^1$$

and a family of homotopy equivalences

$$f_i: X_i \to X_i\,, \quad i \leq m$$

such that

$$\pi_i f_i = f_{i-1} \cdot \pi_i \quad \text{and} \quad f_i p_i = p_i f.$$

We also have

$$f(F_i) \subset F_i^1\,, \quad \text{for} \quad i \leq m+1\,,$$

with the map f/F_i being a homotopy equivalence.

We thus conclude that f defines a couple map.

Proof. Since we have only altered the space X by converting a map into a fibre map, X and X^1 clearly have the same homotopy-type. Select a homotopy equivalence $\bar{f}: X \to X^1$. According to [10], we have a family of induced maps $\bar{f}_i: X_i \to X_i$ such that $\pi_i \cdot \bar{f}_i = \bar{f}_{i-1} \cdot \pi_i$ and $p_i \cdot \bar{f} \simeq \bar{f}_i \cdot p_i$. In particular, $p_m \bar{f} \simeq \bar{f}_m p_m$, so that by the $ACHP$, there is a map f, $f \simeq \bar{f}$, with $p_m f = \bar{f}_m p_m$. Let $f_i = \bar{f}_i$, $i \leq m$. Then the first two conditions are clearly satisfied.

It is clear that $f(F_i) \subset F_i^1$. By the five-lemma and WHITEHEAD's theorem, we see that f/F_i is a homotopy equivalence.

It follows from this proposition that the terms $E_{r,s}^1$, $r < m$, are the same regardless of which POSTNIKOV system is used in constructing the couple. Hence, in a range of dimensions which goes to infinity with m, the two spectral sequences will be the same.

Def. 2.2. The limits, in m, of the two spectral sequences defined in Def. 2.1 are denoted

$$\{E_{r,s}^n;\, d_n\} \text{ (homology) and } \{E_n^{r,s};\, d_n\} \text{ (cohomology)}.$$

Our interest is in these sequences which are defined without regard to m. We will next study the convergence of these spectral sequences. Let $\langle D_{r,s};\, E_{r,s}\rangle$ denote the limit couple whose sequence is $\{E_{r,s}^n;\, d_n\}$.

Prop. 2.2.

a) If $n > s$, $D_{r,s}^n = 0$.
b) If $n > \max.\,(r, s-1)$, then $E_{r,s}^n = E_{r,s}^{n+1} = \ldots$.

Proof. Elements of $D^n_{r,s}$ are representend by classes in $H_{r+s}(F_r)$ which may be pulled back to $H_{r+s}(F_{r+n})$. Since $s < n$ and F_{r+n} is $(r + n - 1)$-connected, the result follows. Part b) is similar.

Prop. 2.3.

$$E^\infty_{r,s} = \frac{Im\,(D_{r,s} \to H_{r+s}(X))}{Im\,(D_{r+1,s-1} \to H_{r+s}(X))}$$

with the similar result for cohomology.

Proof. Easy.

In order to apply the spectral sequence to some specific problem, we must have some information about the first term. To know this term in general seems to be difficult. However, the following proposition determines it in the "stable range".

Prop. 2.4.

$$\left.\begin{aligned} E^1_{r,s} &= H_{r+s}(\pi_r(X), r) \\ E_1^{r,s} &= H^{r+s}(\pi_r(X), r)\,. \end{aligned}\right. \qquad 0 \leq s \leq r$$

Similar statements hold for other coefficients, for example a principal ideal ring.

Proof. F_r is fibred over $K(\pi_r(X), r)$ with fibre F_{r+1}. $K(\pi_r(X), r)$ is $(r - 1)$-connected, while F_{r+1} is r-connected. The assertion follows from [20] (see Cor. 1, p. 469), and similarly in cohomology.

Remark. If one pictures the term E^1 as lying in the first quadrant in the plane, then this stable range is the region $\{(r, s) \mid 0 \leq s \leq r\}$.

3. The Spectral Sequence (con't.)

In this section, I shall consider the following special properties of the spectral sequence: a) the effect of a map, b) the relation between the differentials and the k-invariants, c) cohomology operations in the spectral sequence for cohomology, d) the duality between the spectral sequences for homology and cohomology, when the coefficients are a field, and e) the effect of the loop space functor Ω. We begin with some remarks on naturality.

The notion of POSTNIKOV system in the category of spaces and maps is not a functor. In fact, the requirement that our fibrations be principal prevents there from being a functorial construction. Furthermore, our construction of the spectral sequence from the POSTNIKOV system depended on choices (covering homotopies, etc.). However, in the case of POSTNIKOV systems, one does have the notion of induced maps [10] which are associated with a map of spaces. Here we have a similar situation, as follows:

Prop. 3.1. Let $f\colon X \to X^1$ be a map. Suppose X and X^1 have POSTNIKOV systems $\{X_n, p_n, \pi_n\}$ and $\{X_n^1, p_n^1, \pi_n^1\}$, and couples $\langle D_{r,s}; E_{r,s}\rangle$ and $\langle D_{r,s}^1; E_{r,s}^1\rangle$ (as in section 2). Then there is a couple map $\langle f_1, f_2\rangle$,

$$f_1\colon D_{r,s} \to D_{r,s}^1$$
$$f_2\colon E_{r,s} \to E_{r,s}^1$$

such that $\langle f_1, f_2\rangle$ is compatible with the induced maps

$$f_*\ \pi_i(X) \to \pi_i(X^1)$$
$$f_*\ H_i(X) \to H_i(X^1).$$

Furthermore, if $g \simeq f\colon X \to X'$, then $\langle f_1, f_2\rangle$ is also compatible with g.

Proof. By our construction, it is sufficient to consider finite POSTNIKOV systems of length m, m large. Altering f, if necessary, by a homotopy, we may form a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X_m & \longrightarrow & X_{m-1} & \longrightarrow & \cdots \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f_m}\big\downarrow & & {\scriptstyle f_{m-1}}\big\downarrow & & \\ X^1 & \longrightarrow & X_m^1 & \longrightarrow & X_{m-1}^1 & \longrightarrow & \cdots \end{array}$$

As all maps are base-point preserving, we have

$$f(F_r) \subset F_r^1, \quad r \leq m+1.$$

We may then form the following commutative diagram, in which the horizontal maps are inclusions:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & F_{r+1} & \longrightarrow & F_r & \longrightarrow & \cdots & \cdots & \longrightarrow & X \\ & {\scriptstyle f|F_{r+1}}\big\downarrow & & {\scriptstyle f|F_r}\big\downarrow & & & & & {\scriptstyle f}\big\downarrow \\ \cdots \longrightarrow & F_{r+1}^1 & \longrightarrow & F_r^1 & \longrightarrow & \cdots & \cdots & \longrightarrow & X^1\,. \end{array}$$

It is now easy to construct the desired couple maps. If $g \simeq f$, then $g_m \simeq f_m$, so that the last assertion follows easily.

It is clear that the k-invariants of the POSTNIKOV system determine the fibre spaces $(F_r, F_{r+1}, K(\pi_r(X), r))$, and thus, they determine all the differentials in the spectral sequence, at least up to identifications. The stable portion of E_1 consists of the cohomology of the fibres in the fibre spaces $(X_n, K(\pi_n(X), n), X_{n-1}; \pi_n)$. If i^* is induced by the inclusion $i : K(\pi_n(X), n) \to X_n$, then $i^* k^{n+2} \in H^{n+2}(\pi_n(X), n; \pi_{n+1}(X))$ is an element of E_1, when the coefficients are taken in $\pi_{n+1}(X)$. The following proposition identifies this class.

Prop. 3.2. Suppose X is $(p-1)$-connected, $n > p > 2$. Let

$$\iota^{n+1} \in H^{n+1}(\pi_{n+1}(X), n+1; \pi_{n+1}(X))$$

be the fundamental class, $i : K(\pi_n(X), n) \to X_n$ the inclusion of the fibre. Then in the spectral sequence for a POSTNIKOV system for X, we have

$$d_1 \, \iota^{n+1} = i^* k^{n+2} .$$

Proof. We shall use the identifications of Prop. 2.4. Consider the following commutative diagram, in which the horizontal maps describe fibre space maps:

$$\begin{tikzcd}
F_{n+1} \arrow[r, "="] \arrow[d] & F_{n+1} \arrow[r, "p_{n+1} | F_{n+1}"] \arrow[d] & K(\pi_{n+1}(X), n+1) \arrow[d] \\
F_n \arrow[r] \arrow[d, "p"] & X \arrow[r, "p_{n+1}"] \arrow[d] & X_{n+1} \arrow[d] \\
K(\pi_n(X), n) \arrow[r] & X_n \arrow[r, "="] & X_n .
\end{tikzcd}$$

Let τ and $\bar{\tau}$ denote the transgressions in the right- and lefthand fibre spaces (resp.). Let $\bar{\iota}^{n+1}$ be the fundamental class in F_{n+1}. Since the transgression is natural, we have

$$i^* k^{n+2} = i^* \tau \, \iota^{n+1} = \bar{\tau} \, \bar{\iota}^{n+1} .$$

Identifying the cohomology, in positive dimensions, of $(A, \text{pt.})$ with that of A, for any space A, we have

$$i^* k^{n+2} = p^{*-1} \delta^* \bar{i}^{n+1} = p^{*-1} \delta^* k^* \bar{p}^* i^{n+1},$$

where $k: F_{n+1} \to (F_{n+1}, F_{n+2})$ and $\bar{p}: (F_{n+1}, F_{n+2}) \to (K(\pi_{n+1}(X), n+1), \text{pt.})$. Since p^* and $\bar{p}^*$ are just identifications, and $\delta^* k^*$ gives the differential d_1, we have the formula

$$i^* k^{n+2} = d_1 i^{n+1}$$

as desired.

Remark. One may prove a similar result about the higher differentials. Roughly speaking if $d_j\{i^{n+1}\} = 0$ for $1 \le j < m$, then τi^{n+1} is in the image of $(\pi_{n-m+2} \circ \ldots \circ \pi_{n-1} \circ \pi_n)^*$. If a is an element which this map sends to τi^{n+1}, then $i^* a$ equals $d_m\{i^{n+1}\}$, modulo an appropriate subgroup. Here, as before, i is an inclusion of a fibre. Details are left to the reader.

We now study the relationship between cohomology operations and differentials in the spectral sequence in cohomology. It is convenient to use the following terminology:

Def. 3.1. A family of cohomology operations, Φ_n, defined as additive, natural transformations of relative cohomology which augment degree by $d(n)$, with fixed coefficients G, is called *ordinary*, if the Φ_n commute with the coboundary for all pairs, e. g.

$$\begin{array}{ccc} H^m(X; G) & \xrightarrow{\Phi_n} & H^{m+d(n)}(X; G) \\ \downarrow \delta^* & & \downarrow \delta^* \\ H^{m+1}(X, A; G) & \xrightarrow{\Phi_n} & H^{m+1+d(n)}(X, A; G) \end{array}$$

commutes for all pairs in our category.

Prop. 3.3. Let $\{E_r^{p,q}; d_r\}$ be the spectral sequence for a Postnikov system for a space X in our category. Let the coefficients be G, and suppose Φ_n is an ordinary family of cohomology operations with coefficients in G. Suppose $x \in E_1^{p,q}$ represents $\{x\} \in E_r^{p,q}$. Then $\Phi_n(x)$ represents a class in $E_r^{p,q+d(n)}$, denoted $\{\Phi_n(x)\}$, and

$$d_r\{\Phi_n(x)\} = \{\Phi_n(u)\}$$

where u represents $\mathrm{dr}\{x\}$.

Proof. Since the Φ_n commute with coboundaries and induced maps, clearly $d_j\{x\} = 0$, $j < r$, implies $d_j\{\Phi_n(x)\} = 0$, $j < r$. The second assertion follows similarly.

Remark. The differentials thus commute with the STEENROD operations, which are, of course, the most important example of a family which is ordinary. Similarly, with appropriate signs, the differentials commute with some families of BOCKSTEIN homomorphisms. However, the differentials do not commute with all natural cohomology operations. The following proposition gives an example of a BOCKSTEIN which does not commute with differentials.

Prop. 3.4. Let $X = S^n$, n large. Let $u \in H^{n+3}(Z_8, n+3; Z_2)$ be the non-zero element, and let $\Phi \in H^{n+4}(Z_8, n+3; Z_2)$ be the non-zero element. Then, we have in the spectral sequence for cohomology with coefficients Z_2,

$$d_1(u) = 0 \quad \text{and} \quad d_2(\{u\}) = 0$$

but

$$d_1(\Phi(u)) = d_1(\Phi) \neq 0.$$

Proof. The first assertion follows from the fact that there must be some element in the spectral sequence to kill $S_q^4 i \in H^{n+4}(Z_1 n; Z_2)$. u is the only possible element. Thus, $d_3(\{u\}) = \{S_q^4 i\}$. But, there is no element in $E_3^{n,5}$, and $\{\Phi\} \in E_r^{n+4,1}$ is never an image under any dr. Thus, $\{\Phi\}$ does not remain until E_3. In fact, it is easy to show that $d_1(\Phi) \neq 0$.

Remark. Further details on the spectral sequence for a sphere, in low dimensions, may be found in section 8. The only information needed for the computations of Prop. 3.4 is Prop. 3.3 and the ADEM relations among the STEENROD squares.

Next, we consider the spectral sequences in homology and in cohomology with coefficients in a field. In the case of coefficients in a field, the groups $H_i(X, A)$ and $H^i(X, A)$ are actually dual vector spaces, and the induced maps f_* and f^* are dual homomorphisms. A similar duality applies on the chain level, and this shows that the connecting homomorphisms are dual homomorphisms. We thus have a duality between the couples, and hence the

Prop. 3.5. When the coefficients are taken in a field, the two spectral sequences $\{E^r_{p,q}; \text{dr}\}$ and $\{E_r^{p,q}; \text{dr}\}$ are dual spectral sequences of vector spaces.

The last special property of this spectral sequence, which we are going to discuss, is the relationship with the loop functor. In order to do this, we need some special properties of the suspension homomorphism, with respect to pairs of spaces. Although these do not seem to be available in the literature, they are not difficult and are probably well-known. I will only sketch the theory, leaving details to the reader.

We require a transgression homomorphism for pairs that is a map

$$\tau : H^i(\Omega X, \Omega A) \to H^{i+1}(X, A)$$

which is to be defined in a stable range, and which is to be compatible with respect to the ordinary transgression and the exact sequence of a pair.

To begin, let $\chi_0 \in A \subset X$ be the base point, and define

$$P_{\chi_0, A} = \{\alpha \in PX \mid \alpha(1) \in A\}, \text{ where } PX = \text{paths on } X.$$

We define the relative transgression to be the correspondence

$$H^i(P_{\chi_0, A}) \xrightarrow{\delta^*} H^{i+1}(PX, P_{\chi_0, A}) \xleftarrow{p^*} H^{i+1}(X, A).$$

We shall show how this can be modified to give a transgression for pairs as required. We operate primarily in the stable range, i.e. our dimensions will be less than twice the connectivity of the space. For simplicity, we omit any questions of sign.

There is a commutative diagram

$$\begin{tikzcd}
H^{i-1}(\Omega A) \arrow[r] \arrow[d] & H^i(P_{\chi_0, A}) \arrow[r, "i^*"] \arrow[d] & H^i(\Omega X) \arrow[d] \\
H^i(A) \arrow[r, "\delta^*"] & H^{i+1}(X, A) \arrow[r, "j^*"] & H^{i+1}(X) .
\end{tikzcd}$$

Consider the case where X is fibred over Y with fibre A. Let $\pi : X \to Y$ be the projection map. We then have a commutative diagram

$$\begin{tikzcd}
\Omega A \arrow[r, "\Omega(i)"] \arrow[d] & \Omega X \arrow[r, "\Omega(\pi)"] \arrow[d] & \Omega Y \arrow[d] \\
PA \arrow[r, "P(i)"] \arrow[d] & PX \arrow[r, "P(\pi)"] \arrow[d] & PY \arrow[d, "p"] \\
A \arrow[r, "i"] & X \arrow[r, "\pi"] & Y
\end{tikzcd}$$

in which each column or row is a fibration. $p \cdot P(\pi)$ has fibre $P_{\chi_0, A}$. In our stable range, there are isomorphisms

$$\tau_1 : H^{i-1}(P_{\chi_0, A}) \xrightarrow{\approx} H^i(Y)$$

$$\tau_2 : H^{i-1}(\Omega Y) \xrightarrow{\approx} H^i(Y)$$

$$\Omega(\pi)^* : H^{i-1}(\Omega Y) \xrightarrow{\approx} H^{i-1}(\Omega X, \Omega A) .$$

If we define

$$f = \Omega(\pi)^* \circ \tau_2^{-1} \circ \tau_1 : H^{i-1}(P_{\chi_0, A}) \xrightarrow{\approx} H^{i-1}(\Omega X, \Omega A),$$

then there is a commutative diagram

$$\begin{tikzcd}[column sep=small]
 & & & H^i(P_{\chi_0, A}) \arrow[dd, "f"', "||"] \arrow[dr, "i^A"] & & \\
\cdots \arrow[r] & H^{i-1}(\Omega X) \arrow[r] & H^{i-1}(\Omega A) \arrow[ur, "p^* \cdot \tau"] \arrow[dr, "\delta^*"'] & & H^i(\Omega X) \arrow[r] & \cdots \\
 & & & H^i(\Omega X, \Omega A) \arrow[ur, "s^*"'] & &
\end{tikzcd}$$

In other words, under our assumptions, if we identify $H^i(P_{\chi_0, A})$ and $H^i(\Omega X, \Omega A)$ via f, then there is a transgression for pairs

$$\tau : H^i(\Omega X, \Omega A) \to H^{i+1}(X, A)$$

which, along with the usual transgression, commutes with induced maps and coboundaries.

Now, it follows formally that in the stable range, there is a transgression map

$$\tau : E_r^{p-1,q}(\Omega X) \to E_r^{p,q}(X)$$

which commutes with differentials, and which, in this range, is an isomorphism of spectral sequences. In the usual way, one may convert this to a map of spectral sequences

$$\Sigma : E_r^{p,q}(X) \to E_r^{p+1,q}(SX).$$

Using these maps, one may pass to the limit, and one thus obtains a stable spectral sequence associated with a space. In homology, the groups $E_{p,0}^1$ in this limit spectral sequence are just the stable homotopy groups of the space. In sections 7. and 8., we shall use this spectral sequence in studying the stable homotopy groups (ring) of spheres.

4. The HUREWICZ Homomorphism

If we consider the spectral sequence for a POSTNIKOV system for X in integral homology, we note that $\pi_*(X)$ occurs naturally in E^1, while E^∞ is a graded group associated with $H_*(X)$. In this section, we identify the edge homomorphism

$$\pi_*(X) \to H_*(X)$$

with the HUREWICZ homomorphism. We conclude with conditions for the HUREWICZ map to be a monomorphism or an epimorphism, and with a new, short proof of the fact that the HUREWICZ map is always onto in the second non-trivial dimension.

Theorem 4.1. Let X be a space with a given POSTNIKOV system and spectral sequence $\{E^r_{p,q};\,\mathrm{dr}\}$ in integral homology. Then, the (essentially finite) composition

$$\pi_r(X) \xrightarrow{\approx} E^1_{r,0} \to E^2_{r,0} \to \dots \to E^\infty_{r,0} \subset H_r(X)$$

for which the first map is the HUREWICZ isomorphism in dimension r in the space $K(\pi_r(X), r)$, the last map is the obvious inclusion, and the other maps are the natural projections, is the HUREWICZ homomorphism

$$H_r : \pi_r(X) \to H_r(X)$$

for the space X.

Proof. Consider our spectral sequence as defined by a finite POSTNIKOV system of m terms, $m >> r$, Then, we have a commutative diagram

$$\begin{array}{ccccc} \pi_r(F_r, F_{r+1}) & \xleftarrow{\approx} & \pi_r(F_r) & \xrightarrow{\approx} & \pi_r(X) \\ \downarrow \| & & \downarrow \| & & \downarrow \\ H_r(F_r, F_{r+1}) & \xleftarrow{\approx} & H_r(F_r) & \longrightarrow & H_r(X) \end{array}$$

in which the vertical arrows represent HUREWICZ homomorphisms. We are interested in the map on the right, so that it is clear that we must study, up to these identifications, the map

$$H_r(F_r) \xrightarrow{i_*} H_r(X)$$

in relation to the spectral sequence.

Consider the epimorphism

$$E^1_{r,0} \to E^2_{r,0}.$$

This map consists in projecting $E^1_{r,0}$ onto its quotient by the image of d_1. Identifying $H_r(F_r)$ with $H_r(F_r, F_{r+1})$, we see that an element in $E^1_{r,0}$ belongs to $I_m(d_1)$, if and only if it is an image under

$$\delta_* : H_{r+1}(F_{r-1}, F_r) \to H_r(F_r).$$

By exactness, an element in $E^1_{r,0}$ is in $Im(d_1)$, if and only if it is annihilated by $H_r(F_r) \to H_r(F_{r-1})$.

More generally, an element of $E^i_{r,0}$ is in $Im(d_i)$, if and only if its class is annihilated by $H_r(F_r) \to H_r(F_{r-i})$.

Therefore, the composition

$$H_r(F_r) = H_r(F_r, F_{r+1}) = E^1_{r,0} \to E^2_{r,0} \to \dots \to E^\infty_{r,0}$$

is just the projection onto the quotient of $H_r(F_r)$ by the kernel of

$$H_r(F_r) \to H_r(F_1) = H_r(X).$$

It is easy to see that the inclusion $E^\infty_{r,0} \subset H_r(X)$ is the same as the imbedding of $Im(H_r(F_r) \to H_r(X))$ in $H_r(X)$, proving the theorem.

Cor. 4.1. If X is $(p-1)$-connected, and if $E^\infty_{r-i,i} = 0$, for $0 < i \leq r-p$, then the HUREWICZ map H_r, for dimension r, is an epimorphism.

Proof. In this case, $E^\infty_{r,0} = H_r(X)$.

Cor. 4.2. If X is $(p-1)$-connected, and $E^i_{r-i,i+1} = 0$, for $0 < i \leq r-p$, then H_r is a monomorphism.

Proof. In this case, each $E^n_{r,0} \to E^{n+1}_{r,0}$ is a monomorphism.

As an elementary application, we have the following:

Prop. 4.1 (EILENBERG-MACLANE). Let r be a positive integer, π an abelian group. Then $H_{r+1}(\pi, r; Z) = 0$.

Proof. For $r = 1$, this is a well-known algebraic remark, and we shall consider the case $r > 1$. Let X be a connected space such that

$$H_r(X) = \pi$$

$$H_s(X) = 0, \; s \neq r, \; s > 0.$$

(It is easy to see that such spaces always exist.) Consider the homology spectral sequence for a POSTNIKOV system for X. Then

$$E^1_{r,1} = H_{r+1}(\pi, r; Z).$$

But if $E^1_{r,1} \neq 0$, $E^\infty_{r,1}$ would be non-zero, since no element in $E^n_{r,1}$ is an image under any differential. But then we would have $H_{r+1}(X) \neq 0$, which is a contradiction.

Prop. 4.2 (G. WHITEHEAD). Let X be $(p-1)$-connected, $p > 1$. Then

$$H_{p+1} : \pi_{p+1}(X) \to H_{p+1}(X)$$

is onto.

Proof. By the above, $E^\infty_{p,1} = E^1_{p,1} = 0$, so that this is a special case of Cor. 4.1.

5. The Kernel of the HUREWICZ Homomorphism

In the previous section, we have characterized the HUREWICZ homomorphism as an edge homomorphism in the spectral sequence of a POSTNIKOV system. Here we propose to use these methods to study the kernel of the HUREWICZ homomorphism. It is convenient to make the following definition:

Def. 5.1. The HUREWICZ homomorphism

$$H_n : \pi_n(X) \to H_n(X)$$

will be called *stable*, if X is $(m-1)$-connected, and $n < 2m - 1$.

Our work here will be restricted to a study of stable HUREWICZ homomorphisms, because in this range of dimensions, we have complete information about the E^1-term in the spectral sequence for a POSTNIKOV system. The following theorem relates the order of elements in the kernel of the HUREWICZ homomorphism to the density of non-zero homotopy groups of the space.

Theorem 5.1. Let X be an $(m-1)$-connected space in our category, which is assumed to have finitely-generated integral homology groups. Let

$$H_n : \pi_n(X) \to H_n(X)$$

be a stable HUREWICZ homomorphism, e.g. $n < 2m - 1$. Let $k_p(n)$ denote the number of dimensions i, $0 < i < n$, for which $\pi_i(X)$ contains an element whose order is either a power of p or is infinite. Let $o_p(\alpha)$ denote the smallest natural number c such that $p^c \alpha = 0$.

If $\alpha \in \operatorname{Ker}(H_a)$ is a non-zero element whose order is a power of p, a prime, then

$$o_p(\alpha) < k_p(n) + 1.$$

Proof. If π is any finitely-generated abelian group, then it has been shown by H. CARTAN [6] (see also [8]) that p-torsion every element of $H_i(\pi, n; Z)$, $n < i < 2n$, is order p. Clearly, such a group has elements of order p, if and only if, π has elements of order p or ∞.

Therefore, $E^1_{r,s}$ $0 < r < n$, $s < r$, has elements of order p, exactly when $\pi_r(X)$ contains elements of order p. There are exactly $k_p(n)$ such groups. Since α is in $\operatorname{Ker}(H_n)$, its class is annihilated by some sequence of differentials d_i. The domain of each such differential is a group all of whose elements have prime order. The desired inequality follows immediately.

Cor. 5.1. With the same notations as Theorem 5.1, let $o_p(G)$ denote the maximum of $o_p(\alpha)$, when α ranges over G. Then, when all the numbers involved are finite, we have

$$o_p(\pi_n(X)) - k_p(n) - 1 \leq o_p(I_m(H_n)) \leq o_p(H_n(X)).$$

Cor. 5.2. Let $\alpha \in \pi_j(S^n)$, $n < j < 2n - 1$. Denote by $l_p(j)$ the number of i, $n < i < j$, so that $\pi_i(S^n)$ has an element of order p. Then,

$$o_p(\alpha) \leq l_p(j) + 1.$$

Remark. 1. ADAMS [1] and LIULEVICIUS[2]) have obtained absolute bounds for the order of elements in the stable homotopy groups of spheres. On the other hand, KERVAIRE and MILNOR (11) construct elements in the stable homotopy groups of spheres or arbitrary high order. This fact shows that in some limit sense, Cor. 5.2 is best possible. However, we shall show, in the next section, that the estimate of Cor. 5.2 may be improved by 1.

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[2]) ADAMS's results first appeared as Lecture Notes from the Univ. of California. LIULEVICIUS's paper is in Proc. Amer. Math. Soc. *14* (1963).

2. One also has some information about elements $\alpha \in \pi_n(X)$, $\alpha \in \operatorname{Ker}(H_n)$, which lie outside the stable range, provided one assumes that $\{\alpha\} = 0 \in E^{[n/2]}_{n,0}$. In this case, one may also bound order in terms of the density of elements in the range of dimensions $[n/2] < i < n$. Clearly, such elements lie in a "metastable" range, in that they share some properties with the stable elements. Unfortunately, I know no condition on α which would insure that it was metastable in this sense.

6. The Filtration of Ker (H_n)

We shall now define a filtration on the elements of the homotopy groups of a space and then examine some of its special properties.

Def. 6.1. Let $\alpha \in \pi_n(X)$, $n > 1$. We identify α with an element of $E^1_{n,0}$, via the HUREWICZ isomorphism $\pi_n(X) \approx E^1_{n,0}$. Suppose that $0 \neq \{\alpha\} \in E^i_{n,0}$, $1 \leq i \leq m - 1$, but that $0 = \{\alpha\} \in E^m_{n,0}$. Then we say that α has filtration m and write $f(\alpha) = m$. If $\alpha = 0$, put $f(\alpha) = 0$; if $\alpha \notin \operatorname{Ker}(H_n)$, put $f(\alpha) = \infty$.

Def. 6.2. Define

$$J^k_n(X) = \{\alpha \in \pi_n(X) \mid f(\alpha) \leq k\}.$$

Remark. It is clear that if $\alpha \in \pi_n(X)$, $f(\alpha) = k$, then if a is an integer, $f(a\alpha) \leq k$. Clearly the set $J^k_n(X)$ is a subgroup of $\pi_n(X)$. We put

$$J^k(X) = \sum_n \oplus J^k_n(X).$$

The subgroups $J^k(X)$ are then a filtration of

$$\pi_*(X) = \sum_n \oplus \pi_n(X).$$

Remark. In we consider the spectral sequence with coefficients in a field, then the set of elements such that $f(\alpha) = k$ also forms a subgroup, and in fact, one can define a filtration by vector subspaces.

These subgroups and filtrations will play an important role in subsequent sections. In the remainder of this section, I propose to obtain bounds on the filtration of elements. We shall also indicate why elements of arbitrarily high filtration occur in the stable spectral sequence for a sphere.

Prop. 6.1. Let X be $(m-1)$-connected, $m \geq 2(p-1)$. Let $\alpha \in \pi_n(X)$ be an element whose order is a power of the prime p. Then, we have

$$f(\alpha) \geq 2(p-1)-1.$$

Proof. We begin by noting that the homology of an EILENBERG–MAC LANE space, with coefficients in the field of p elements, Z_p, vanishes in the range of dimensions $m+1 < i < 2(p-1)$. By the universal coefficient theorem, the p-torsion of the integral homology must also vanish in this range. Therefore, the first differential whose image can possibly be $\{\alpha\}$ is $d_{2(p-1)-1}$.

Prop. 6.2. Let X have a single non-vanishing free homology group π in dimension m. Suppose $\alpha \in \pi_n(X)$, $m < n < 2m-1$.

1. If α has order a power of p, p an odd prime, then

$$f(\alpha) \leq \max. [n-m-(2(p-1)-1), 2(p-1)-1].$$

2. If the order of α is a power of 2, then either the above holds with $p=2$, or else $n-m = 1, 3$, or 7.

Proof. We have $\{\alpha\} = d_j\{\beta\}$, for some j and β. In case $\beta \in E^1_{m,2(p-1)}$, $f(\alpha) = 2(p-1)-1$. In this case, α lies in the first non-vanishing homotopy group of X, whose dimension is greater then m, and which has p-torsion.

We must show that for $n > m+2(p-1)-1$, $f(\alpha) \leq n-m-2(p-1)+1$. Consider the coefficient homomorphism $Z \to Z_p$ into the spectral sequence with coefficients Z_p. In that portion of the spectral sequence where β lies, all elements have prime order, so that this coefficient homomorphism is actually an isomorphism on p-torsion. It is then clearly sufficient to show that in the dual cohomology spectral sequence, with Z_p coefficients, if

$$d_j\{\bar{\alpha}\} = \{\bar{\beta}\}$$

then $j \leq n-m-2(p-1)+1$.

But suppose that $d_j\{\bar{\alpha}\} = \{\bar{\beta}\}$, $\bar{\beta} \in E_1^{m,n-m+1}$. Then, we must have $E^j_{m,i} = 0$, $i < n-m+1$.

Then there is an element P in the mod p STEENROD algebra such that $\bar{\beta} = Pi$. P is not P^1. If p is odd, then according to LIULEVICIUS [13] and others, there can be no space where this operation is non-trivial, and there is no cohomology in dimensions between those of i and Pi. This shows that $\bar{\beta} \in E_1^{m,n-m+1}$ was impossible.

In case $p = 2$, the argument is similar, except for the fact that Sq^2, Sq^4, and Sq^8 do not factor (see [4]).

Now, referring to Theorem 5.1, we see that the presence of elements of high order in the stable p-component of the homotopy groups of a space implies the existence of elements of high filtration in the spectral sequence. Using the work of KERVAIRE and MILNOR [11], we have the following observation:

Prop. 6.3. There exist spheres which contain elements in their stable homotopy groups of arbitrarily high filtration.

Remark. The results on the WHITEHEAD product, in the last section of this paper, will show that there also exist unstable elements, in the homotopy groups of spheres, which have arbitrarily high filtration.

7. The Filtration for Stable Homotopy Groups of Spheres

We now examine the filtration in the case of stable homotopy groups of spheres. We use the notation

$$G_k = \lim_{n \to \infty} \pi_{n+k}(S^n)\,;\; G = \sum_k \oplus\, G_k .$$

In section 3, we sketched a theory of relative suspension and transgression which enabled us to pass to the limit and define a spectral sequence which contains the stable homotopy groups of spheres. In the case of homology with integer coefficients, we denote this stable spectral sequence, which is the limit of the sequences for the spheres S^n, by

$$\{{}_SE^r_{p,q}\,;\; d_r\} .$$

Exactly as in the previous section, we make the following

Def. 7.1. $\alpha \in G_n$, $n > 0$, has filtration m, written $f(\alpha) = m$, if $0 \neq \{\alpha\} \in {}_SE^r_{n,0}$, for $r < m$, but $0 = \{\alpha\} \in {}_SE^m_{n,0}$.

We define

$$_{S}J_n^k = \{\alpha \in G_n \mid f(\alpha) \leq k\}$$

and

$$_{S}J^k = \sum_n \oplus \, _{S}J_n^k.$$

Remarks. The $_{S}J^k$ clearly define a filtration of the stable homotopy groups of spheres by subgroups. It is easy to see that the bounds, given by Prop. 6.2, on the filtration of an element, may be carried over to estimate the filtration of elements in G.

It is well-known (see [5]) that G possesses the additional structure of an anti-commutative ring. The multiplication is defined via the composition of mappings. We shall study the effect of this multiplication on the stable spectral sequence and on the filtration of elements.

First we note that an element $\alpha \in G_k$ is represented by a map

$$\alpha : S^{n+k} \to S^n, \ n \text{ large,}$$

and hence, α defines, by means of proposition 3.1, a homomorphism $\bar{\alpha}$ of $\{_{S}E_{p,q}^r; d_r\}$ of bidegree $(k, 0)$. If $i \in G_0$ is a specified generator, we have that $\bar{\alpha}(i) = \alpha$. The following proposition is immediate:

Prop. 7.1. Suppose $\alpha \in G_k$, $\beta \in G_{k^1}$. Let $\bar{\alpha}$ denote the homomorphism of $\{_{S}E_{p,q}^r; d_r\}$ defined by α. Then, under the identification of $_{S}E_{k+k^1,0}^1$ with G_{k+k^1}, we have

$$\bar{\alpha}(\beta) = \alpha \circ \beta.$$

Concerning the relation of multiplication to filtration, we have

Prop. 7.2. Suppose $\alpha \in G_k$, $\beta \in G_{k^1}$, $f(\alpha) = m$, $f(\beta) = m^1$. Then

$$f(\alpha \circ \beta) = f(\beta \circ \alpha) \leq \min. (m, m^1).$$

Proof. From anti-commutativity, we see that $f(\alpha \circ \beta) = f(\beta \circ \alpha)$. Since $f(\beta) = m^1$, there is a class u such that

$$\{\beta\} = d_m \cdot u.$$

Then

$$\{\alpha \circ \beta\} = d_{m^1} (\bar{\alpha}(u)).$$

Thus

$$f(\alpha \circ \beta) \leq m^1.$$

Similarly

$$f(\alpha \circ \beta) \leq m.$$

Remark. The case where $f(\alpha \circ \beta) < \min. (m, m^1)$ does occur. For example, the generator of G_3, which is of filtration 3, has a non-zero power which is divisible by the generator of G_1. The latter has filtration 1. Hence, an element of filtration 3 can have a non-zero power which has filtration 1.

In the sequel, it will be convenient to refer to those products, for which $f(\alpha \circ \beta) = \min. (m, m^1)$, as *non-degenerate.* A non-zero product with $f(\alpha \circ \beta) < < \min. (m, m^1)$ will be called *degenerate.*

Cor. 7.1. a) ${}_SJ^k$ is an ideal in the ring G.

b) If $k < k^1$, ${}_SJ^k$ is an ideal in ${}_SJ^{k^1}$.

Proof. Let $\alpha \in {}_SJ^k$, $\beta \in G$. Then, $f(\alpha \circ \beta) \leq k$, so that $\alpha \circ \beta \in {}_SJ^k$. Case b) is similar.

Remark. I do not know the exact relationship of this filtration to the other filtrations which have been defined on the stable homotopy groups.

8. The Composition Product for Homotopy Groups

It has been said that cohomological methods for studying homotopy fail to shed light on compositions (and higher order composition products). In this section we shall show how this is not the case. Using the spectral sequence, we shall give conditions for a composition product of homotopy classes to be essential. Some applications follow. Finally, we obtain some information on the properties which non-nilpotent elements in the ring G would necessarily have. The existence of such elements is unknown at the present time.

Prop. 8.1. Let $\alpha \in \pi_q(S^p)$, $\beta \in \pi_r(S^q)$, $p < q < r$. Suppose that in $E^*_{r,0}$ in the spectral sequence for S^q, $\{\beta\} = d_m u$. Then, if $d_m(\bar{\alpha}(u)) \neq 0$, we have $\alpha \circ \beta \neq 0$.

Proof. Since $\bar{\alpha}$ is a map of spectral sequences, we have

$$\{\alpha \circ \beta\} = \bar{\alpha}(d_m(u)) = d_m(\bar{\alpha}(u)).$$

Remark. This condition is not necessary, as is seen by considering degenerate products in the ring G.

Now, the spectral sequence forms a useful tool for investigating the properties of composition of an arbitrary stable element with a fixed stable element. For example, we have

Prop. 8.2. Let $\alpha \in G_k$, $0 \neq \eta^2 \in G_2$. If α has order 2, is not divisible by 2, and $\alpha \circ \eta^2 \neq 0$, then $\alpha \circ \eta^2$ is divisible by 2.

Proof. We shall work in the ring $G \otimes Z_2$. We shall use the same notation for elements as for their images in this ring. Our object is to show that in the ring $G \otimes Z_2$, $\alpha \cdot \eta^2 = 0$. Consider $\bar{\alpha}(\eta^2)$, where now $\bar{\alpha}$ denotes the homomorphism of spectral sequences with Z_2-coefficients. Clearly, η has filtration 1, so that η^2 must be non-degenerate, and η^2 has filtration 1.

Let u be a generator of ${}_SE^1_{0,0}$, and let the generators of ${}_SE^1_{0,1}$ and ${}_SE^1_{0,2}$ be written S^1u and S^2u. Similarly, let η, $S^1\eta$, and $S^2\eta$ be generators of ${}_SE^1_{1,0}$, ${}_SE^1_{1,1}$, and ${}_SE^1_{1,2}$. Clearly, S^i refers to the homology dual of the STEENROD operation S^i_q. We immediately have

$$\eta = d_1(S^2u) \text{ and } \eta^2 = d_1(S^2\eta).$$

Using * to denote the duals in cohomology, we get

$$d_1\eta^* = S^2_q\, u^* \text{ and } d_1\eta^{2*} = S^2_q\, \eta^*.$$

If $\alpha \cdot \eta^2 \neq 0$, there are classes $x_i \in E_1^{k+i,0}$, $i = 0, 1, 2$, such that

$$d_1 x_1 = S^2_q x_0 \text{ and } d_1 x_2 = S^2_q x_1,$$

as well as $\bar{\alpha}^*(x_0) = u^*$, $\bar{\alpha}^*(x_1) = \eta^*$ and $\bar{\alpha}^*(x_2) = \eta^{2*}$.

Now, because $d_1 x_2 = S^2_q x_1$, we must have $d_1(S^2_q x_1) = 0$. On the other hand,

$$d_1(S^2_q\, x_1) = S^2_q\, S^2_q\, x_0 = S^3_q\, S^1_q\, x_0$$

which is not zero, as α has order 2 exactly. Therefore, we must have $x_2 = 0$, which implies that $\alpha \cdot \eta^2 = 0$ in the ring $G \otimes Z_2$.

Remark. This proposition may be observed empirically in the range where G has been computed. Similar methods may be used to study other compositions of fixed elements with arbitrary elements. However, in these cases, if filtrations are greater than 1, there is always the possibility of a degenerate composition. In order to state theorems, it is thus necessary to make some assumptions about the vanishing of certain homotopy groups which guarantee non-degeneracy. Details here are left to the reader.

One knows (see [22]) that the elements $\alpha(2) = \eta \in G_1$ and $\alpha(p) \in G_{2p-3}$ are the lowest dimensional, non-trivial elements in the p-component of G. These elements have minimal possible filtration (Prop. 6.1). By Prop. 7.1, any composition with such an element has minimal filtration. Curiously enough, the converse of this fact is also true.

Prop. 8.3. Let $\beta \in G_k$ be an element, whose order is a power of p, and which is not divisible by p. Assume β has minimal filtration for the p-component. Then, β is divisible, in the sense of composition product, by $\alpha(p)$.

Proof. In a similar fashion to the above, we use the spectral sequence with Z_p-coefficients. Here, we identify $\alpha \in G_e$ with its image in ${}_S E^1_{e,0}$ with coefficients in Z_p, and let $P\alpha$ be dual to the cohomology class $P^1 \alpha^* \in E^1_{e,2(p-1)}$, where P^1 is the usual Steenrod operation relative to the prime p.

We assume, with no loss of generality, that $\beta \in G_k$ is a generator of a cyclic summand. Because β is of minimal filtration, there is an element $\alpha \in E^1_{k-2(p-1)+1,0}$ such that $d_{2p-3}(P\alpha) = \beta$. As above, let $u \in G_0$ generate G_0 with $\bar{\alpha}(u) = \alpha$. The dual to P^1 is a natural homology operation P^{-1} such that

$$P^{-1}(\bar{\alpha}(Pu)) = \alpha,$$

and hence,

$$\bar{\alpha}(Pu) = P\alpha.$$

Now, $d_{2p-3}(Pu) = \alpha(p)$, so that

$$\beta = d_{2p-3}(P\alpha) = \{\alpha \circ \alpha(p)\} \text{ in } E^{2p-3}_{k,0}.$$

But, because d_{2p-3} is the first non-zero differential to map to $E^i_{k,0}$, we conclude that $\beta = \alpha \circ \alpha(p)$, as desired.

Cor. 8.1. Any element β of G_k, which is of order a power of p and which is not divisible by p, and which has minimal filtration for the p-component, is nilpotent. In fact, if p is odd, $\beta^2 = 0$, while if $p = 2$, $\beta^3 = 0$.

Proof. By the previous proposition, these statements are reduced to the corresponding facts about $\alpha(p)$. But $\alpha(2)^3 = \eta^3 = 0$, while for p odd, $\alpha(p)^2 = 0$. (See [22].)

As a final application to compositions, we make a remark concerning non-nilpotent elements. A non-nilpotent element of G means some $\alpha \in G_k$, $k > 0$, such that $\alpha^i \neq 0$, for all i. It is presently unknown whether such elements exist, but it is still of interest to examine what properties they would enjoy. In fact, sufficient information in this direction may lead to results on the non-existence of such elements.

Our result, roughly speaking, states that such elements would generate an ideal of a certain size. More specifically, we have

Prop. 8.4. Let $\alpha \in G_k$, $k > 0$, have order a power of p, where p is an odd prime. Suppose $\alpha^i \neq 0$, for all i. Then there is an element $\beta \in G_l$, $l \not\equiv 0 \pmod{k}$, so that

$$\alpha^i \circ \beta \neq 0, \text{ for all positive integers } i.$$

Proof. $f(\alpha^i)$, the filtration of α^i, is a non-increasing function of i; because $\alpha^i \neq 0$, all i, $f(\alpha^i)$ is bounded below away from zero. Hence, there is some $N > 0$, such that if $i \geq N$, then $f(\alpha^i)$ is constantly at its minimum value. Select some $i \geq N$. Then $\{\alpha^i\} = d_m u$, and by Prop. 6.2, $m \leq f(\alpha) < k$.

Since $\bar{\alpha}(\alpha^i) = \alpha^{i+1}$, and

$$\{\alpha^{i+1}\} = \bar{\alpha}(d_m(u)) = d_m(\bar{\alpha}(u)),$$

we see that $\bar{\alpha}(u) \neq 0$.

In general, we have $\{\alpha^{i+j}\} = d_m(\bar{\alpha}^j(u))$, which implies that $\bar{\alpha}^j(u) \neq 0$, for any $j > 0$. Now, since the map $\bar{\alpha}$ applied to $E^1_{ki-m,*}$ is an induced homomorphism on the homology of an EILENBERG–MACLANE space, there is clearly an element $\beta \in E^1_{ki-m,0}$, with $\alpha^i \cdot \beta \neq 0$ for all i.

Remark. 1. In case of elements whose order is a power of 2, we immediately get the same result except when α has dimension 1, 3, or 7, or when α is divisible by such an element. In these dimensions, there are elements which are of equal dimension and filtration. For such an element, the above mentioned inequality, $m \leq f(\alpha) < k$, does not hold. Thus, for such an element, we cannot rule out the possibility that β is itself a power of α. Fortunately, the elements in G_1 and G_3 are all known to be nilpotent. However, the generator of G_7 has a non-zero cube. Several people have conjectured, but to my knowledge not yet proved, that the fourth power will be zero. If this is the case, Prop. 8.4 is valid for $p = 2$.

2. With special assumptions on the filtration of α, one may extend these results and get other elements which are not annihilated by all powers of α.

9. The WHITEHEAD Product

In the present section, I shall interpret the WHITEHEAD product in the spectral sequence, and then obtain estimates on the filtration of WHITEHEAD product elements. After that, we give some applications.

We shall take, as a definition, the formulation of the WHITEHEAD product given by J.-P. MEYER. To fix ideas, I will first sketch his approach (see [15]).

Let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, $p \geq q$. In a POSTNIKOV decomposition for X, consider the following composite fibration.

$$\begin{array}{ccl} F_{p,\,p+q-2} & \xrightarrow{i} & X_{p+q-2} \\ & & \big\downarrow \pi_p \circ \dots \circ \pi_{p+q-2} \\ & & X_{p-1} \end{array}$$

We use $F_{n,m}$ to denote the fibre of $X_m \to X_n$. MEYER defines a generalized multiplication

$$\mu : F_{p,\,p+q-2} \times F_{q,\,p+q-2} \to F_{p,\,p+q-2}.$$

If $\bar{\alpha}$ and $\bar{\beta}$ denote the (unique) elements in $\pi_p(F_{p,\,p+q-2})$ and $\pi_q(F_{q,\,p+q-2})$ (resp.), which correspond to α and β, we put

$$\gamma = (-1)^{pq} \mu_*(H(\bar{\alpha}), H(\bar{\beta}))$$

where H is the HUREWICZ homomorphism.

Now, let τ denote the transgression in integral homology for the fibration

$$\begin{array}{ccc} K = K(\pi_{p+q-1}(X), p+q-1) & \longrightarrow & F_{p,\,p+q-1} \\ & & \downarrow \\ & & F_{p,\,p+q-2} \end{array}$$

$\pi_* : H_{p+q}(F_{p,\,p+q-1}, K) \xrightarrow{\approx} H_{p+q}(F_{p,\,p+q-2})$, since $p, q > 1$.

We consider

$$\lambda = \pi_*^{-1}(\gamma).$$

Then, according to MEYER [15],

$$H^{-1}\delta_*(\lambda) = [\alpha, \beta].$$

Now, let $(F_{p,\,p+q-1})_{(i-1)}$ denote the $(i-1)$-term in the POSTNIKOV system for $F_{p,\,p+q-1}$. Set J_i equal to the fibre in the fibration

$$\varrho : F_{p,\,p+q-1} \to (F_{p,\,p+q-1})_{(i-1)}.$$

We consider the following commutative diagram

$$\begin{array}{ccc} H_{p+q}(F_{p,\,p+q-1}, K) & & \\ \| & & \\ H_{p+q}(J_p, J_{p+q-1}) & \xrightarrow{m_*} & H_{p+q}(J_p, J_{p+1}) \\ \downarrow \delta^* & & \downarrow \delta_* \\ H_{p+q-1}(J_{p+q-1}) & \xrightarrow{j_*} & H_{p+q-1}(J_{p+1}) \\ \| & & \\ H_{p+q-1}(J_{p+q-1}, J_{p+q}) & & \end{array}$$

where j is the (composite) inclusion $J_{p+q-1} \to J_{p+1}$ and $m = (1, j)$.

Clearly, the element $m_*(\lambda)$ has the property

$$j_*^{-1}\delta_*(m_*(\lambda)) = \{\delta_*(\lambda)\} = \{H([\alpha, \beta])\}$$

when viewed as a class in the spectral sequence for $F_{p,\,p+q-1}$. In other words, in the spectral sequence for $F_{p,\,p+q-1}$, we have

$$d_{q-1}\{m_*(\lambda)\} = \{H([\alpha, \beta])\}.$$

Now, it is clear that F_p and $F_{p,\,p+q-1}$ have the same spectral sequence in this range, so that we have the same relation in the spectral sequence for F_p.

If we let i; $F_p \to X$ denote the inclusion of the fibre, we have a corresponding statement about $i_* (m_* (\lambda))$.

Theorem 9.1. If $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, $p \leq q$, then we have, in the spectral sequence for a POSTNIKOV system for X,

1. $$f(H([\alpha, \beta])) \leqq q - 1$$

2. $$d_{q-1}\{i_* (m_* (\lambda))\} = \{H([\alpha, \beta])\}.$$

Now, we shall apply this result to the determination of WHITEHEAD products and the study of the filtration of known WHITEHEAD product elements. We shall say that elements in a finitely generated abelian group are *independant*, if they lie in distinct summands in the usual sort of decomposition into a direct sum of cyclic groups. An element which generates a cyclic subgroup which is an entire summand will be called a *generating element.*

Theorem 9.2. Let X be an $(n-1)$-connected space, α, $\beta \in \pi_q(X)$, with $q < 2n - 1$. Suppose that α and β have orders that are powers of a fixed prime p or infinite. Assume that either

a) α and β are independant generators,

or

b) The fixed prime p is odd, q is even, and $\alpha = \beta$ is a generator.

In addition, suppose that $H(\alpha) \neq 0$, $H(\beta) \neq 0$, and $H_{2q}(X; Z_p) = 0$.

Then, we have

$$[\alpha, \beta] \neq 0.$$

Proof. By previous identifications, we may take $i_* m_* (\lambda)$ to be an element of $H_{2q}(F_q, F_{q+1})$. There is an isomorphism

$$l: H_{2q}(F_q, F_{q+1}) \to H_{2q}(K(\pi_q(X), q)),$$

under which $i_* m_* (\lambda)$ corresponds to the PONTRJAGIN product $\mu_* (a, b)$, where $a = H(\alpha)$, $b = H(\beta)$.

It is sufficient to take, as coefficients, the prime field with respect to our fixed prime. It follows from our assumptions that $i_* m_* (\lambda)$ or $\mu_* (a, b)$ is not zero (see [6]). By the previous theorem, $d_{q-1}\{i_* m_* (\lambda)\} = \{[\alpha, \beta]\}$. As $H_{2q}(X; Z_p) = 0$, the theorem will follow if we can show that $i_* m_* (\lambda)$ is not an image under any differential.

Consider the diagram

$$\begin{tikzcd} & H_{2q}(F_q) \arrow[r, "k"] \arrow[d, "j_1"] & H_{2q}(F_q, F_{q+1}) \\ H_{2q+1}(F_{q-m}, F_{q-m+1}) \arrow[r, "\delta_*"] & H_{2q}(F_{q-m+1}) \arrow[d, "j"] & \\ & H_{2q}(F_{q-m}) & \end{tikzcd}$$

in which j_1 and j are induced by the obvious inclusions. If $\{l^{-1}\mu_* (a, b)\}$ is in the image of d_m, there exists $x \in H_{2q}(F_q)$, $k(x) = l^{-1}\mu_* (a, b)$, $j_1(x) \neq 0$, and $j j_1(x) = 0$.

Considering the exact sequence of the pair (F_{q-m}, F_q), i.e.

$$H_{2q+1}(F_{q-m}, F_q) \xrightarrow{\delta_*} H_{2q}(F_q) \xrightarrow{j j_1} H_{2q}(F_{q-m}),$$

there is $w \in H_{2q+1}(F_{q-m}, F_q)$, such that $\delta_* w = x$.

On the other hand, $k^{-1} l^{-1}(a)$ and $k^{-1} l^{-1}(b)$ are not in the image of

$$\delta_* : H_{q+1}(F_{q-m}, F_q) \to H_q(F_q)$$

because $H(\alpha)$ and $H(\beta)$ are different from zero.

We now look at the following commutative diagram:

$$\begin{tikzcd} H_{r+1}(F_{q-m}, F_q) \arrow[r, "\delta_*"] & H_r(F_q) \arrow[r, "j \cdot j_1"] \arrow[d, "p_2"] & H_r(F_{q-m}) \arrow[d, "p_1"] \\ & H_r(K(\pi_q(X), q)) \arrow[r, "i_1"] & H_r(F_{q-m, q}) \end{tikzcd}$$

where p_1 and p_2 are induced by projections onto the q^{th}-terms in the respective POSTNIKOV systems, and i_1 is induced by the inclusion of the fibre. In dimension q, p_1 and p_2 are isomorphisms. As was just noted, $j j_1(k^{-1}l^{-1}(a))$ and $j j_1(k^{-1}l^{-1}(b))$ must be different from zero. We thus conclude that $i_1(a)$ and $i_1(b)$ are different from zero.

Now, because $q < 2n - 1$, it follows (see [10]) that $F_{q-m, q}$ is an H-space and i_1 is induced from a multiplicative map. $i_1(a)$ and $i_1(b)$ are either independant elements, or else the same even-dimensional element. For reasons of dimension, neither $i_1(a)$ or $i_1(b)$ is decomposable, with respect to the PONTRJAGIN product

$\bar{\mu}_*$ in $F_{q-m,q}$. Thus, by the structure theorem for HOPF algebras[3]) we must have $\bar{\mu}_*(i_1(a), i_1(b)) \neq 0$. Since i_1 is induced from a multiplicative map, we can conclude that $i_1(\mu_*(a, b)) \neq 0$.

We have shown that $i_1 p_2(x) = i_1(lk(x)) \neq 0$, so that $p_1(j \cdot j_1(x)) \neq 0$. Therefore, $j \cdot j_1(x) \neq 0$, which is a contradiction. We conclude that the class of $i_* m_*(\lambda)$ is not an image under any d_m, and the theorem follows.

Remark. It is easy to give examples of this result. For example, one may consider iterated suspensions of products of spheres.

As a final application, we would like to remark that the unstable homotopy groups of spheres contain elements which have arbitrarily high filtration.

Prop. 9.1. If n is even and $i_n \in \pi_n(S^n)$ is a generator, then $[i_n, i_n]$ has filtration $n - 1$.

Proof. Here, we know that $i_* m_*(\lambda)$ has infinite order. The domain of any other differential whose image might possibly be $\{H([i_n, i_n])\}$, is a finite group [6]. Since $H_{2n}(S^n) = 0$, $d_{n-1}\{i_* m_*(\lambda)\} \neq 0$, proving the proposition.

Remarks. One may use Theorem 9.1 to prove several other results, too long to be included here. For example, if X is an H-space, $\alpha, \beta \in \pi_q(X)$, then $[\alpha, \beta] = 0$. However, $i_* m_*(\lambda)$ is often not zero. Nevertheless, we can show that $i_* m_*(\lambda)$ never persists in the spectral sequence to represent a class in the homology of X, if $H(\alpha) = H(\beta) = 0$. Simple examples are obtained by considering the spaces $(S^n)_{(j)}$, for $j < 2n - 1$.

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