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A 3-Fold Vector Product in R^8

By PETER ZVENGROWSKI

An r -fold vector product in R^n is, according to ECKMANN [2], a continuous map

$$X : R^{rn} = R^n \times R^n \times \dots \times R^n \rightarrow R^n$$

satisfying (i) $X(a_1, a_2, \dots, a_r) \cdot a_i = 0, 1 \leq i \leq n,$
(ii) $\|X(a_1, a_2, \dots, a_r)\|^2 = |a_i \cdot a_j|.$

It can be deduced from [2] and [4] that R^n admits an r -fold vector product in precisely the following cases:

- a) $r = 1$ and n is even,
- b) $r = n - 1,$
- c) $r = 2$ and $n = 3, 7,$
- d) $r = 3$ and $n = 8.$

Explicit formulas for these vector products are known in all but the last case, and the purpose of this note is to give a simple formula for the last case, taking advantage of the CAYLEY numbers. Such a formula was useful to the author in examining homotopy classes of vector fields on S^7 (cf. [5]).

1. The Cayley Numbers

We take the CAYLEY numbers C to be the 8-dimensional algebra over the reals having basis vectors e_1, e_2, \dots, e_8 and suitably defined products. In particular, $e_1 = 1$ is the unit and for $i, j > 1$ one has $e_i e_j = \begin{cases} -1, & i = j \\ -e_j e_i, & i \neq j. \end{cases}$ The subspace of C orthogonal to e_1 is called the pure CAYLEY numbers. Conjugation is defined by writing any CAYLEY number x as $x = r e_1 + x'$, where x' is pure, and setting $\bar{x} = r e_1 - x'$. The CAYLEY numbers inherit the norm and scalar product of R^8 , and

$$x\bar{x} = \bar{x}x = x \cdot x = |x|^2.$$

Lemma 1.1. For any CAYLEY numbers a, b , one has $\overline{\overline{a}} = a$, $\overline{ab} = \overline{b}a$, and $a\overline{b} + b\overline{a} = 2(a \cdot b)$.

Proof: By linearity it suffices to prove these when a, b are the units e_1, \dots, e_8 , and these cases follow from the above remarks on multiplication.

Corollary: $\overline{ab} + \overline{ba} = 2(\overline{a} \cdot \overline{b}) = 2(a \cdot b)$.

The multiplication in C can be explicitly defined by regarding C as a 2-dimensional algebra over the quaternions H , with basis $1, k$, and setting

$$(q_1 + q_2k)(q'_1 + q'_2k) = (q_1q'_1 - \overline{q'_2}q_2) + (q'_2q_1 + q_2\overline{q'_1})k.$$

We shall need this formula to prove the next lemma.

Lemma 1.2. For any CAYLEY numbers a, b, c ,

$$a(\overline{bc}) + b(\overline{ac}) = 2(a \cdot b)c.$$

Proof: Let $a = a_1 + a_2k, b = b_1 + b_2k, c = c_1 + c_2k$, where $a_1, a_2, b_1, b_2, c_1, c_2$ are quaternions. By direct computation,

$$\begin{aligned} a(\overline{bc}) + b(\overline{ac}) &= (b_1\overline{a_1} + a_1\overline{b_1})c_1 + c_1(\overline{a_2}b_2 + \overline{b_2}a_2) + \\ &\quad + [c_2(\overline{a_1}b_1 + \overline{b_1}a_1) + (b_2\overline{a_2} + a_2\overline{b_2})c_2]k. \end{aligned}$$

Applying Lemma 1.1 and its corollary, this equals

$$\begin{aligned} &2[(b_1 \cdot a_1)c_1 + c_1(a_2 \cdot b_2) + (c_2(a_1 \cdot b_1) + (b_2 \cdot a_2)c_2)k] \\ &= 2(a_1 \cdot b_1 + a_2 \cdot b_2)(c_1 + c_2k) \\ &= 2(a \cdot b)c. \end{aligned}$$

According to [1], the subalgebra of C generated by any two elements (and their conjugates) is associative. In particular, for any CAYLEY numbers a, b, c , the product $a(bc)a$ is well defined, and according to [3], we have $a(bc)a = (ab)(ac)$.

Lemma 1.3. For any CAYLEY numbers a, b, c , one has

- (i) $a \cdot (a(\overline{bc})) = |a|^2(b \cdot c)$,
- (ii) $c \cdot (a(\overline{bc})) = |c|^2(a \cdot b)$, and
- (iii) $b \cdot (a(\overline{bc})) = -|b|^2(c \cdot a) + 2(a \cdot b)(b \cdot c)$.

Proof of (i): $2a \cdot (a(\bar{b}c)) = a(\overline{a(\bar{b}c)}) + (a(\bar{b}c))\bar{a} = a((\bar{c}b)\bar{a}) + (a(\bar{b}c))\bar{a}$.
 Since a , \bar{a} , $\bar{c}b$, and $\overline{\bar{c}b} = \bar{b}c$ generate an associative subalgebra, this equals

$$a(\bar{c}b)\bar{a} + a(\bar{b}c)\bar{a} = a(\bar{c}b + \bar{b}c)\bar{a} = 2a(b \cdot c)\bar{a} = 2|a|^2(b \cdot c).$$

Proof of (ii): $2c \cdot (a(\bar{b}c)) = c((\bar{c}b)\bar{a}) + (a(\bar{b}c))\bar{c} = 2r$, say, where r must be a real number. Then making use of the preceding results without specific mention, we have

$$\begin{aligned} rc &= c((\bar{c}b)\bar{a})c + (a(\bar{b}c))|c|^2 \\ &= (c(\bar{c}b))(\bar{a}c) + (a(\bar{b}c))|c|^2 \\ &= |c|^2(b(\bar{a}c) + a(\bar{b}c)) \\ &= 2|c|^2(a \cdot b)c. \end{aligned}$$

Proof of (iii): $2b \cdot (a(\bar{b}c)) = b((\bar{c}b)\bar{a}) + (a(\bar{b}c))\bar{b}$

$$\begin{aligned} &= b[(-\bar{b}c + 2b \cdot c)\bar{a}] + [a(-\bar{c}b + 2b \cdot c)]\bar{b} \\ &= 2(b \cdot c)(b\bar{a} + a\bar{b}) - b((\bar{b}c)\bar{a}) - (a(\bar{c}b))\bar{b} \\ &= 4(b \cdot c)(a \cdot b) - 2r, \text{ for some real number } r. \end{aligned}$$

Computing rb as in (ii), one finds $r = |b|^2(c \cdot a)$.

2. Applications to Vector Products

A 2-fold vector product in R^7 is obtained (cf. [2]) by identifying R^7 with the pure CAYLEY numbers, and for any two vectors b, c in R^7 putting $X_2(b, c) = bc + b \cdot c$. We shall now define a 3-fold vector product in R^8 , which in a certain sense generalizes X_2 .

Theorem 2.1. Let a, b, c be vectors in R^8 , regarded as the CAYLEY numbers. Then

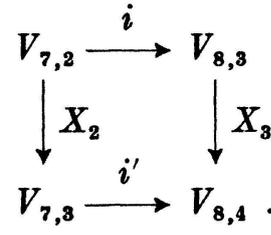
$$X_3(a, b, c) = -a(\bar{b}c) + a(b \cdot c) - b(c \cdot a) + c(a \cdot b)$$

defines a 3-fold vector product in R^8 .

Proof: Lemma 1.3 will be frequently used without specific mention, and we write $X = X_3(a, b, c)$. The continuity of X_3 is obvious. Thus $a \cdot X = -|a|^2(b \cdot c) + |a|^2(b \cdot c) - (a \cdot b)(c \cdot a) + (a \cdot c)(a \cdot b) = 0$, and the proofs that $b \cdot X = 0$, $c \cdot X = 0$, are entirely similar. Finally, writing $d = (a(\bar{b}c))$, we have

$$\begin{aligned}
 |X|^2 &= X \cdot X = |d|^2 + |a|^2(b \cdot c)^2 + |b|^2(a \cdot c)^2 + |c|^2(a \cdot b)^2 - 2(a \cdot d)(b \cdot c) \\
 &\quad + 2(b \cdot d)(c \cdot a) - 2(c \cdot d)(a \cdot b) + (-2 + 2 - 2)(a \cdot b)(b \cdot c)(c \cdot a) \\
 &= |a|^2|b|^2|c|^2 + |a|^2(b \cdot c)^2 + |b|^2(a \cdot c)^2 + |c|^2(a \cdot b)^2 \\
 &\quad - 2|a|^2(b \cdot c)^2 - 2|b|^2(a \cdot c)^2 + 4(a \cdot b)(b \cdot c)(c \cdot a) - 2|c|^2(a \cdot b)^2 \\
 &\quad - 2(a \cdot b)(b \cdot c)(c \cdot a) \\
 &= |a|^2|b|^2|c|^2 - |a|^2(b \cdot c)^2 - |b|^2(a \cdot c)^2 - |c|^2(a \cdot b)^2 + 2(a \cdot b)(b \cdot c)(c \cdot a) \\
 &= \begin{vmatrix} |a|^2 & a \cdot b & a \cdot c \\ b \cdot a & |b|^2 & b \cdot c \\ c \cdot a & c \cdot b & |c|^2 \end{vmatrix}.
 \end{aligned}$$

When restricted to the STIEFEL manifold $V_{7,2}$, the 2-fold product X_2 gives a cross section $X_2: V_{7,2} \rightarrow V_{7,3}$. Similarly, X_3 gives a cross section $X_3: V_{8,3} \rightarrow V_{8,4}$. Then the fact that $X_3(e_1, b, c) = bc + b \cdot c = X_2(b, c)$, where $(e_1, b, c) \in V_{8,3}$ (and hence b, c are pure), implies that the following diagram commutes:



Further details on these vector products can be found in [5].

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