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# Geometric and algebraic intersection numbers

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Let  $M$  be a connected  $2n$ -dimensional differential manifold, not necessarily compact. Let  $x_0 \in M$  be a base point, and  $\alpha \in \pi_n(M, x_0)$  a given homotopy class. It is well known that, unless  $M$  is simply connected, there need not exist any differentiable imbedding  $\varphi: S^n \rightarrow M$  representing  $\alpha$ .

Let  $\bar{M}$  be the universal cover of  $M$  provided with an arbitrary but fixed orientation, and let  $a \in H_n(\bar{M})$  be the (integral) homology class of a lifting of  $\alpha$ .

**Theorem 1.** *Assuming  $n > 2$ , the class  $\alpha \in \pi_n(M, x_0)$  is representable by a differentiable imbedding  $\varphi: S^n \rightarrow M^{2n}$  if and only if for every covering transformation  $\tau \neq 1$  of  $\bar{M}$  the homology intersection number  $a \cdot \tau(a)$  vanishes.*

If  $M$  is oriented, one can define a scalar product

$$H_q(\bar{M}) \otimes H_{m-q}(\bar{M}) \rightarrow \mathbb{Z}[\pi]$$

with values in the integral group ring of  $\pi = \pi_1(M, x_0)$ . (Cf. K. REIDEMEISTER [2] and J. MILNOR [1].) Here  $m = \dim M = \dim \bar{M}$  need not be even, and we assume that the projection map  $p: \bar{M} \rightarrow M$  is orientation preserving. The image of  $x \otimes y$  under the above pairing will be denoted as in MILNOR [1] by  $[x, y]$ .

In terms of this scalar product Theorem 1 can be formulated as follows:

**Theorem 1'.** *Let  $M^{2n}$  be connected and oriented. Assuming that  $n > 2$ , the class  $\alpha \in \pi_n(M, x_0)$  is representable by a differentiable imbedding  $\varphi: S^n \rightarrow M^{2n}$  if and only if*

$$[a, a] - a \cdot a = 0.$$

The proof is given in §1 and §2. In §3 we give conditions under which two imbeddings  $\varphi: X^q \rightarrow M^m$  and  $\psi: Y^{m-q} \rightarrow M^m$  representing the homology classes  $\alpha, \beta$  respectively are diffeotopic to imbeddings  $\varphi_0, \psi_0$  such that the cardinality of the set  $\varphi_0(X^q) \cap \psi_0(Y^{m-q})$  equals the absolute value  $|\alpha \cdot \beta|$  of the homology intersection number  $\alpha \cdot \beta$ . (Cf. Theorem 2 below.)

The proofs of Theorem 1 and Theorem 2 depend on the following well known lemma, essentially due to H. WHITNEY.

Let  $B^r$  denote the open unit ball in  $R^r$ .

**Lemma.** *Let  $V^m$  be a differential manifold, not necessarily compact, and let  $\varphi: B^q \rightarrow V^m$  and  $\psi: B^{m-q} \rightarrow V^m$  be two differentiable imbeddings such that  $\varphi(B^q)$  and  $\psi(B^{m-q})$  intersect transversally at exactly two points  $R = \varphi(P) = \psi(Q)$  and  $R' = \varphi(P') = \psi(Q')$ .*

*Suppose that*

- (i) *both  $q$  and  $m - q$  are larger than 2,*
- (ii) *if  $u: I \rightarrow B^q$  and  $v: I \rightarrow B^{m-q}$  are paths from  $P$  to  $P'$  and  $Q$  to  $Q'$  respectively, then the loop  $\varphi(u) \cdot \psi(v^{-1})$  is freely homotopic in  $V$  to a constant loop,*
- (iii) *with respect to some orientation of a neighborhood of  $\varphi(B^q) \cup \psi(B^{m-q})$  in  $V$  the intersection coefficients of  $\varphi(B^q)$  and  $\psi(B^{m-q})$  at  $R$  and  $R'$  are opposite, i. e.  $\varphi(B^q) \cdot \psi(B^{m-q}) = 0$ .*

*Then there exists a diffeotopy  $\varphi_t: B^q \rightarrow V^m$  such that  $\varphi_1 = \varphi$ ,  $\varphi_t(x)$  is independent of  $t$  for  $|x| > 1 - \varepsilon$  for some positive  $\varepsilon$  and  $\varphi_0(B^q) \cap \psi(B^{m-q}) = \emptyset$ .*

For a proof, see [3] and [4].

## § 1. Proof of Theorem 1

It is easy to see that  $a \cdot \tau(a) = 0$  for all  $\tau \neq 1$  is a necessary condition for the existence of an imbedding  $\varphi: S^n \rightarrow M^{2n}$  representing  $\alpha \in \pi_n(M, x_0)$ . (Recall that  $a$  denotes the homology class of a lifting of  $\alpha$  in the universal cover  $\bar{M}$  of  $M$ .) For let  $\varphi: S^n \rightarrow M^{2n}$  be a mapping representing  $\alpha$ , and  $f: S^n \rightarrow \bar{M}$  a lifting of  $\varphi$ . Let  $\tau: \bar{M} \rightarrow \bar{M}$  be a covering transformation,  $\tau \neq 1$ . We show that if  $P$  is a point in  $f(S^n) \cap \tau f(S^n)$  then  $\varphi$  is not an imbedding. Let  $Q = \tau^{-1}P \in f(S^n)$ . Since  $\tau \neq 1$ , we have  $Q \neq P$ . Choose  $Q', P' \in S^n$  such that  $f(Q') = Q$  and  $f(P') = P$ . Then  $Q' \neq P'$  but  $\varphi(Q') = \varphi(P')$  since  $f$  is a lifting of  $\varphi$ . Hence  $\varphi$  is not bijective. Now, if  $\varphi$  is an imbedding, it follows that  $f(S^n) \cap \tau f(S^n) = \emptyset$  for every  $\tau \neq 1$ , and a fortiori  $a \cdot \tau(a) = 0$ .

Conversely, suppose that  $a \cdot \tau(a) = 0$  for every  $\tau \neq 1$ . Since  $\bar{M}$  is simply connected and  $n > 2$ , WHITNEY's lemma (cf. introduction) implies that  $a$  can be represented by a differentiable imbedding  $f: S^n \rightarrow \bar{M}$ . The projection map  $p: \bar{M} \rightarrow M$  is an immersion. Hence  $f$  projects to an immersion  $p \circ f = \varphi: S^n \rightarrow M$ . We may assume without loss of generality that  $\varphi(S^n)$  intersects itself transver-

sally in a finite number of points where only two sheets of  $\varphi(S^n)$  cross each other. In other words, we may assume that  $\varphi$  is a completely regular immersion in the sense of [4]. (This can be obtained by an arbitrarily close approximation to  $\varphi$  in the  $C^2$ -topology so that the new  $\varphi$  still lifts to an imbedding.)

Let  $S_\varphi$  be the set of pairs of (distinct) points  $(P, Q)$  on  $S^n$  such that  $\varphi(P) = \varphi(Q)$ . If  $S_\varphi \neq \emptyset$ , pick a pair  $(P, Q) \in S_\varphi$ . *Claim:* There exists another pair  $(P', Q') \in S_\varphi$  and  $\varphi$  is regularly homotopic to an immersion  $\Phi: S^n \rightarrow M$  such that

- (i) any lifting  $F: S^n \rightarrow \bar{M}$  of  $\Phi$  is an imbedding,
- (ii)  $S_\Phi = S_\varphi - \{(P, Q), (P', Q')\}$ ,
- (iii)  $\Phi$  and  $\varphi$  coincide outside some neighborhood of a path joining  $P$  to  $P'$  on  $S^n$ .

This will prove the theorem by induction on the number of self-intersection points.

We now prove the claim. Since  $\varphi(P) = \varphi(Q)$  and  $f(P) \neq f(Q)$ , there exists a covering transformation  $\tau \neq 1$  such that  $f(P) = \tau f(Q) = A$ , say. The point  $A$  is a transversal intersection point of  $f(S^n)$  and  $\tau f(S^n)$ . Let  $\varepsilon (\varepsilon = \pm 1)$  be the intersection coefficient. Since  $a \cdot \tau(a) = f(S^n) \cdot \tau f(S^n) = 0$  by assumption, there exists another intersection point  $A'$  of  $f(S^n)$  and  $\tau f(S^n)$  with intersection coefficient  $-\varepsilon$ . Let  $P', Q' \in S^n$  be such that  $f(P') = \tau f(Q') = A'$ , and let  $u: I \rightarrow S^n$  be a path on  $S^n$  from  $P$  to  $P'$ , and similarly  $v: I \rightarrow S^n$  a path on  $S^n$  from  $Q$  to  $Q'$  such that  $u(I) \cap v(I) = \emptyset$ . We may assume moreover that  $u(I)$  and  $v(I)$  are disjoint from the points of the pairs in  $S_\varphi$  except for  $u(0) = P, u(1) = P', v(0) = Q$  and  $v(1) = Q'$ . Then  $\varphi u(I)$  and  $\varphi v(I)$  intersect only at  $\varphi u(0) = \varphi v(0)$  and  $\varphi u(1) = \varphi v(1)$ . Since  $\bar{M}$  is simply connected,  $f u$  and  $\tau f v$  are two homotopic paths from  $A$  to  $A'$ . Hence  $\varphi u$  and  $\varphi v$  are homotopic paths on  $M$  from  $\varphi(P) = \varphi(Q) = R$  to  $\varphi(P') = \varphi(Q') = R'$ . Take disjoint open neighborhoods  $N_u$  and  $N_v$  of  $u(I)$  and  $v(I)$  respectively with diffeomorphisms  $h_u: B^n \rightarrow N_u$  and  $h_v: B^n \rightarrow N_v$ , and such that  $P, Q, P', Q'$  are the only points from  $S_\varphi$  in  $N_u \cup N_v$ . Let  $V = M - \varphi(S^n - N_u \cup N_v)$  and set  $\varphi_1 = \varphi h_u|_{B^n}$  and  $\psi = \varphi h_v|_{B^n}$ . We are now in a position to apply WHITNEY's lemma. The loop  $\varphi_1 h_u^{-1}(u) \cdot \psi h_v^{-1}(v^{-1})$  is homotopic to a constant loop in  $V$  because it is homotopic to a constant loop in  $M$  and the inclusion  $V \rightarrow M$  induces an isomorphism  $\pi_1 V \cong \pi_1 M$ . Thus  $\varphi_1$  is diffeotopic in  $V$ , relative to a neighborhood of the boundary of  $D^n$ , to an imbedding  $\varphi_0: B^n \rightarrow V$  such that  $\varphi_0(B^n) \cap \psi(B^n) = \emptyset$ . Define the immersion  $\Phi: S^n \rightarrow M$  by

$$\Phi(x) = \begin{cases} \varphi_0 h_u^{-1}(x) & \text{if } x \in N_u, \text{ and} \\ \varphi(x) & \text{if } x \in S^n - N_u. \end{cases}$$



It is easily checked that  $\Phi$  satisfies the conditions (ii) and (iii) of the above claim. To see that condition (i) stating that  $\Phi$  lifts to an imbedding  $F: S^n \rightarrow \bar{M}$  is also satisfied, let  $A, B$  be points on  $S^n$  with  $F(A) = F(B)$ . A path  $w$  from  $A$  to  $B$  on  $S^n$  maps to a loop  $Fw$ , and the loop  $\Phi w$  is homotopic to a constant loop in  $M$ . Since  $\Phi(A) = \Phi(B)$  we have  $\varphi(A) = \varphi(B)$  because the self-intersection points of  $\Phi$  are self-intersection points of  $\varphi$ . In fact  $\varphi w$  is also homotopic to a constant loop in  $M$ . (We may assume  $\varphi w = \Phi w$  because, unless  $A = B$ , we have  $A, B \in S^n - N_u$  and we can take  $w(I) \subset S^n - N_u$ .) But then, this means that  $f(A) = f(B)$ , and since  $f$  is an imbedding by construction, it follows that  $A = B$ . So  $F$  is bijective, and hence an imbedding. The proof of Theorem 1 is thus complete.

## § 2. The scalar product

Let  $M$  be a connected, oriented, differential manifold of dimension  $m$ , not necessarily even. Suppose  $M$  is triangulated as a regular cell complex. The triangulation of  $M$  lifts to a triangulation of  $\bar{M}$  invariant under the covering transformations. We denote by  $\xi_i^q, i = 1, \dots, \alpha_q$  the  $q$ -cells of  $M$  and choose for each  $i$  a lifting  $x_i^q$  of  $\xi_i^q$ . Let  $\eta_i^{m-q}$  be the dual cell to  $\xi_i^q$ . ( $\xi_i^q \cdot \eta_i^{m-q} = 1$ .) A lifting  $y_j^{m-q}$  of  $\eta_j^{m-q}$  is then determined by the condition  $x_i^q \cdot y_j^{m-q} = \delta_{ij}$ . When we change the lifting  $x_i^q$  of  $\xi_i^q$  we demand that  $y_i^{m-q}$  be changed accordingly so that  $x_i^q \cdot y_j^{m-q} = \delta_{ij}$  remains valid.

Let  $\pi = \pi_1(M, x_0)$  be the fundamental group of  $M$  at  $x_0$  which we identify with the group of covering transformations of  $\bar{M}$ . A  $q$ -dimensional chain  $x$  of (the triangulation of)  $\bar{M}$  has a unique expression as  $x = \sum_i \lambda_i x_i^q$ , where  $\lambda_i \in Z[\pi]$  and almost all  $\lambda_i$ 's are zero. If  $y = \sum_j \mu_j y_j^{m-q}$ , define

$$[x, y] = [\sum_i \lambda_i x_i^q, \sum_j \mu_j y_j^{m-q}] = \sum_i \lambda_i \bar{\mu}_i,$$

where  $\mu \rightarrow \bar{\mu}$  is the anti-ringhomomorphism of  $Z[\pi]$  onto itself determined by  $\tau \rightarrow \tau^{-1}$  for  $\tau \in \pi$ .

**Theorem.** *The scalar product  $[x, y]$  is independent of the choice of the liftings  $x_i^q$  and induces a pairing*

$$H_q \bar{M} \otimes H_{m-q} \bar{M} \rightarrow Z[\pi].$$

The first statement follows by an easy calculation, using  $\overline{\lambda \cdot \mu} = \bar{\mu} \cdot \bar{\lambda}$ . The bilinearity of the product is obvious, and the second statement follows from

the formula  $[x, dy] = \pm [dx, y]$ , where  $\dim x + \dim y = m + 1$ , and  $d$  is the boundary  $Z[\pi]$ -homomorphism. (Compare [1].)

We now derive a formula which will yield a translation of the condition " $a \cdot \tau(a) = 0$  for all  $\tau \neq 1$ " in terms of the scalar product.

Let  $a_0 = \sum_i \alpha_i x_i^q$ ,  $\alpha_i \in Z[\pi]$ , be a representative  $q$ -chain for  $a \in H_q \bar{M}$  and  $b_0 = \sum_j \beta_j y_j^{m-q}$  a representative of  $b \in H_{m-q} \bar{M}$  in terms of the dual subdivision of  $\bar{M}$ . Then,

$$[a_0, b_0] = \sum_i \alpha_i \bar{\beta}_i.$$

We calculate  $a_0 \cdot \tau(b_0)$ , writing  $x_i$  and  $y_j$  for  $x_i^q$  and  $y_j^{m-q}$  respectively to simplify the notation. We have  $\alpha_i = \sum_{\varrho \in \pi} a_{i,\varrho} \varrho$  and  $\beta_j = \sum_{\sigma \in \pi} b_{j,\sigma} \sigma$ , where  $a_{i,\varrho}, b_{j,\sigma} \in Z[\pi]$  are almost all 0. Then,

$$a_0 \cdot \tau(b_0) = (\sum_{i,\varrho} a_{i,\varrho} \varrho x_i) \cdot \tau(\sum_{j,\sigma} b_{j,\sigma} \sigma y_j) = \sum_{i,\varrho} a_{i,\varrho} b_{i,\tau^{-1}\varrho},$$

and

$$\sum_i \alpha_i \bar{\beta}_i = \sum_{i,\varrho,\sigma} a_{i,\varrho} b_{i,\sigma} \varrho \sigma^{-1} = \sum_{\tau} (\sum_{i,\varrho} a_{i,\varrho} b_{i,\tau^{-1}\varrho}) \tau.$$

In other words, the integer  $a_0 \cdot \tau(b_0)$  is just the coefficient of  $\tau$  in the scalar product  $[a_0, b_0]$ . In formula,

$$(*) \quad [a, b] = \sum_{\tau \in \pi} (a \cdot \tau(b)) \tau.$$

(Compare MILNOR [1] where  $(*)$  is taken as definition of  $[a, b]$ .)

Now, if  $m = 2n$ ,  $q = n$  and  $a = b$ , it follows that the condition " $a \cdot \tau(a) = 0$  for all  $\tau \neq 1$ " is equivalent to

$$[a, a] - a \cdot a = 0.$$

This proves Theorem 1'.

**Remarks.** (1) If we replace  $a$  by  $\tau a$  in Theorem 1',  $[a, a] - a \cdot a$  becomes  $\tau([a, a] - a \cdot a) \tau^{-1}$ . Hence, only the conjugacy class of  $[a, a] - a \cdot a$  is well determined by  $\alpha$ . It seems therefore adequate to choose once and for all a base point  $z_0 \in \bar{M}$  above  $x_0 \in M$  and require all liftings to be liftings at  $z_0$ . We then have a function

$$Q : \pi_n(M, x_0) \rightarrow Z[\pi],$$

$Q(\alpha) = [a, a] - a \cdot a$ , and  $\alpha$  is representable by an imbedded  $n$ -sphere if and only if  $Q(\alpha) = 0$ . The function  $Q$  is somewhat like a quadratic form. Let

$\langle \alpha, \beta \rangle = [a, b] + (-1)^n \overline{[a, b]}$  and let  $\lambda_1$  denote the coefficient of  $1 \in \pi$  in  $\lambda \in Z[\pi]$ . Then

$$Q(\tau\alpha) = \tau Q(\alpha) \bar{\tau} \quad (\tau \in \pi),$$

$$Q(\alpha + \beta) = Q(\alpha) + Q(\beta) + B(\alpha, \beta),$$

where  $B(\alpha, \beta) = \langle \alpha, \beta \rangle - \langle \alpha, \beta \rangle_1$  is bilinear and symmetric.

(2) We have proved a slightly stronger statement than Theorem 1'. A class  $\alpha \in H_n M^{2n}$  is representable by an imbedded manifold  $A^n \subset M^{2n}$  such that  $\pi_1 A \rightarrow \pi_1 M$  is trivial if and only if  $\alpha$  is the projection of a class  $a \in H_n(\bar{M})$  which is representable by a submanifold  $B^n \subset \bar{M}$  and  $[a, a] - a \cdot a = 0$ . This gives a hint for the study of the intersection of submanifolds of  $M$  in §3.

**Example.** Take  $M^{2n}$  to be the connected sum of  $S^1 \times S^{2n-1}$  with  $k$  copies of  $S^n \times S^n$ . Let  $x_i, y_i \in \pi_n(M, x_0)$  be represented by  $S^n \times (\text{point})$  and  $(\text{point}) \times S^n$  in the  $i$ -th copy of  $S^n \times S^n$  (suitably joined to the base point  $x_0$ ). Then,  $\pi_n(M, x_0)$  is the free  $Z[J]$ -module generated by  $x_1, \dots, x_k, y_1, \dots, y_k$ , where  $J$  denotes the (multiplicative) infinite cyclic group. If  $\alpha = \sum_i \alpha_i x_i + \sum_j \beta_j y_j \in \pi_n(M, x_0)$  is a given homotopy class, where  $\alpha_i, \beta_j \in Z[J]$ , it follows from the formula (\*) above that

$$Q(\alpha) = \sum_i (\alpha_i \bar{\beta}_i + (-1)^n \beta_i \bar{\alpha}_i) - \sum_i (\alpha_i \bar{\beta}_i + (-1)^n \beta_i \bar{\alpha}_i)^0,$$

where  $\lambda^0$  is the integer obtained by substituting the value 1 for every element of  $J$  in  $\lambda \in Z[J]$ .

For instance, if we let  $t$  denote a generator of  $J$ , the class  $x + (t + t^{-1})y$  in  $\pi_n M$ , where  $M = S^1 \times S^{2n-1} \# S^n \times S^n$ , is representable by a differentiable imbedding of  $S^n$  into  $M$  if  $n$  is odd, but is not representable if  $n$  is even. The same statement holds for  $x + (t - t^{-1})y$  interverting even and odd.

### § 3. Reducing the geometric intersection of submanifolds

Let  $\varphi: X^q \rightarrow M^m$  and  $\psi: Y^{m-q} \rightarrow M^m$  be differentiable immersions, resp. imbeddings, where  $X, Y, M$  are connected differential manifolds. We assume  $X, Y$  to be compact, without boundary.

Roughly speaking, the problem is to use a deformation of  $\varphi$  so as to reduce the intersection  $\varphi(X) \cap \psi(Y)$  to consist of a number of points equal to the algebraic intersection number of  $\varphi(X)$  and  $\psi(Y)$ .

We assume that  $M$  is oriented. There are then two cases:

*Case 1.*  $X$  and  $Y$  are oriented. Denoting by  $\alpha \in H_q M$  and  $\beta \in H_{m-q} M$  the integral homology classes represented by  $\varphi: X^q \rightarrow M$  and  $\psi: Y^{m-q} \rightarrow M$  respectively, the algebraic (homology) intersection number  $\alpha \cdot \beta$  is then an integer.

*Case 2.* At least one of the manifolds  $X, Y$  is non-orientable. Then  $\varphi: X^q \rightarrow M$  and  $\psi: Y^{m-q} \rightarrow M$  still represent mod 2 homology classes  $\alpha \in H_q(M; \mathbb{Z}_2)$  and  $\beta \in H_{m-q}(M; \mathbb{Z}_2)$ . In this case the intersection number  $\alpha \cdot \beta$  is an integer modulo 2.

We use  $|\alpha \cdot \beta|$  to mean the absolute value in Case 1. In Case 2,  $|\alpha \cdot \beta|$  is the integer 0 or 1 depending on whether  $\alpha \cdot \beta = 0 \in \mathbb{Z}_2$  or  $\alpha \cdot \beta = 1 \in \mathbb{Z}_2$ .

In either case we shall assume that  $\varphi$  and  $\psi$  satisfy the following hypothesis:

(H) *The induced homomorphisms  $\varphi_*: \pi_1 X \rightarrow \pi_1 M$  and  $\psi_*: \pi_1 Y \rightarrow \pi_1 M$  are trivial.*

It follows that  $\varphi$  and  $\psi$  can be lifted to differentiable immersions, resp. imbeddings  $f: X^q \rightarrow \bar{M}$  and  $g: Y^{m-q} \rightarrow \bar{M}$ , where as before  $\bar{M}$  is the universal cover of  $M$ . We let  $a$  and  $b$  denote the homology classes represented by  $f$  and  $g$ . In Case 1,  $a \in H_q \bar{M}$  and  $b \in H_{m-q} \bar{M}$ . In Case 2,  $a \in H_q(\bar{M}; \mathbb{Z}_2)$  and  $b \in H_{m-q}(\bar{M}; \mathbb{Z}_2)$ .

We also assume

$$(H') \quad q > 2 \quad \text{and} \quad m - q > 2.$$

Finally, we use the following notation. If  $\lambda \in Z[\pi]$ ,  $\lambda = \sum_{\tau} n_{\tau} \tau$ , set  $w\lambda = \sum_{\tau} |n_{\tau}|$ .

**Theorem 2.** *With the above notations and hypotheses, including (H) and (H'), the immersion, resp. imbedding  $\varphi: X^q \rightarrow M^m$  is regularly homotopic, resp. diffeotopic to an immersion, resp. imbedding  $\varphi_0: X^q \rightarrow M^m$  such that  $\varphi_0(X) \cap \psi(Y)$  consists of  $|\alpha \cdot \beta|$  points if and only if*

*in Case 1,  $w[a, b] - |\alpha \cdot \beta| = 0$ ,*

*in Case 2,  $[a, b] - a \cdot b = 0$  in  $\mathbb{Z}_2[\pi]$  for some liftings  $a, b$  of  $\alpha, \beta$ .*

**Remark.** Observe that

$$\alpha \cdot \beta = \sum_{\tau \in \pi} a \cdot \tau(b).$$

Thus, in view of (\*), the equation  $w[a, b] - |\alpha \cdot \beta| = 0$  in Case 1 is equivalent to the statement that  $a \cdot \tau(b)$  does not change sign as  $\tau$  runs over  $\pi$ . (More precisely,  $(a \cdot \sigma b)(a \cdot \tau b) \geq 0$  for all  $\sigma, \tau \in \pi$ .) Obviously the condition is independent of the choice of liftings.

An important special case in practice seems to be Case 1 when  $|\alpha \cdot \beta| = 0$  or 1. Then,  $w[a, b] - |\alpha \cdot \beta| = 0$  is equivalent to the nicer looking condition

$$[a, b] - a \cdot b = 0 \text{ for some liftings } a, b \text{ of } \alpha, \beta.$$

(However, in general this last condition is definitely stronger than the former.)

As an illustration to the theorem, let  $M = S^1 \times S^{2n-1} \# S^n \times S^n$ , and  $x, y \in \pi_n(M, x_0)$  be the elements represented by  $S^n \times Q$  and  $P \times S^n$  respectively, with  $x_0 = P \times Q$ . Let  $u, v \in \pi_n(M, x_0)$  be given elements and  $A$  be the 2 by 2 matrix over  $Z[J]$  such that  $\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ , where  $J$  is the multiplicative infinite cyclic group.

Suppose that  $u \cdot v = 1$ , and let  $I_n$  denote the matrix

$$I_n = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

The classes  $u$  and  $v$  can be represented by imbedded spheres with just one intersection point if and only if

$$I_n A I_n^* A^* = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$$

for some integer  $r$ . ( $t$  denotes a generator of  $J$  and  $A^*$  is the conjugate transposed of  $A = (a_{ij})$ , i. e.  $A^* = (a_{ij}^*)$ , where  $a_{ij}^* = \overline{a_{ji}}$ .)

**Proof of Theorem 2.** We may assume that  $\varphi(X)$  and  $\psi(Y)$  intersect transversally in a finite number of points  $S_1, \dots, S_k$  and  $\varphi^{-1}(S_i), \psi^{-1}(S_i)$  each consists of a single point for every  $i = 1, \dots, k$ . Let  $f: X \rightarrow \overline{M}$  and  $g: Y \rightarrow \overline{M}$  be arbitrary liftings of  $\varphi$  and  $\psi$  respectively. (Hypothesis (H).)

*Case 1.* The manifolds  $X$  and  $Y$  are oriented. If  $\tau \in \pi$ , we have  $f(X) \cdot \tau g(Y) = \varepsilon_\tau |a \cdot \tau(b)|$  for some  $\varepsilon_\tau = \pm 1$ . For each  $\tau \in \pi$  we can select intersection points  $R_{\tau,j}, j = 1, \dots, |a \cdot \tau(b)|$ , of  $f(X)$  and  $\tau g(Y)$  so that the intersection coefficient of  $f(X)$  and  $\tau g(Y)$  at  $R_{\tau,j}$  is equal to  $\varepsilon_\tau$ , and thus independent of  $j$ . (If  $a \cdot \tau(b) = 0$ , the set  $\{R_{\tau,j}\}$  is empty.) Let  $(P_{\tau,j}, Q_{\tau,j}) \in X \times Y$  be the uniquely determined pair such that  $f(P_{\tau,j}) = \tau g(Q_{\tau,j}) = R_{\tau,j}$ .

Now, let  $(P, Q) \in X \times Y$  be such that  $\varphi(P) = \psi(Q)$  and  $P \neq P_{\tau,j}$  for all  $\tau, j$ , if any such pair exists. Then, there exists a covering transformation  $\sigma \in \pi$  such that  $f(P) = \sigma g(Q) = R$ , say, and  $R \neq R_{\sigma,i}$  for all  $i$ . ( $1 \leq i \leq |a \cdot \sigma(b)|$ .) Actually  $R \neq R_{\tau,j}$  for all  $\tau, j$ . Since  $f(X) \cdot \sigma g(Y) = \varepsilon_\sigma |a \cdot \sigma(b)|$ , there must exist another pair  $(P', Q') \in X \times Y$  with  $f(P') = \sigma g(Q') = R'$ , where  $R' \neq R_{\sigma,i}$  for all  $i$  and the intersection coefficients of  $f(X)$  and  $\sigma g(Y)$  at

$R$  and  $R'$  are opposite. Again, since  $\psi^{-1}(pR')$  consists of a single point, we actually have  $R' \neq R_{\tau,j}$  for all  $\tau, j$ .

We choose a path  $u: I \rightarrow X$  from  $P$  to  $P'$  such that  $u(I)$  and some neighborhood  $N_u$  of  $u(I)$  in  $X$  are disjoint from any other  $\varphi$ -preimage of an intersection point of  $\varphi(X)$  and  $\psi(Y)$ . Using WHITNEY's lemma as in §1, we can eliminate the intersection points  $\varphi(P) = \psi(Q)$  and  $\varphi(P') = \psi(Q')$  by a diffeotopy of  $\varphi|N_u$  keeping  $\varphi$  fixed near the boundary of  $N_u$ .

It follows that  $\varphi$  is always (i. e. without condition on  $[a, b]$ ) regularly homotopic, resp. diffeotopic, to an immersion, resp. an imbedding  $\varphi_0$  such that

$$|\varphi_0(X) \cap \psi(Y)| = \Sigma_{\tau} |a \cdot \tau(b)|,$$

where  $|\varphi_0(X) \cap \psi(Y)|$  denotes the cardinality of the finite set  $\varphi_0(X) \cap \psi(Y)$ .

If  $a \cdot \tau(b)$  does not change sign, we then have

$$|\varphi_0(X) \cap \psi(Y)| = |\Sigma_{\tau} a \cdot \tau(b)| = |\alpha \cdot \beta|.$$

Conversely, if  $|\varphi_0(X) \cap \psi(Y)| = |\alpha \cdot \beta|$ , then  $|\Sigma_{\tau} a \cdot \tau(b)| = \Sigma_{\tau} |a \cdot \tau(b)|$ , and it follows that  $a \cdot \tau(b) = \varepsilon |a \cdot \tau(b)|$  for all  $\tau$ , where  $\varepsilon = \pm 1$  is independent of  $\tau$ .

*Case 2.* The manifold  $X$ , say, is non-orientable. Then,  $a \in H_q(\bar{M}; \mathbb{Z}_2)$  and we also regard  $b$  as a class in  $H_{m-q}(\bar{M}; \mathbb{Z}_2)$ . For those  $\tau \in \pi$  such that  $a \cdot \tau(b) \neq 0 \pmod{2}$ , we choose an intersection point  $R_{\tau}$  of  $f(X)$  and  $\tau g(Y)$ , and let  $(P_{\tau}, Q_{\tau}) \in X \times Y$  be the uniquely determined pair such that  $f(P_{\tau}) = \tau g(Q_{\tau}) = R_{\tau}$ . Let  $(P, Q) \in X \times Y$  be a pair such that  $\varphi(P) = \psi(Q)$  and  $P \neq P_{\tau}$  for all  $\tau \in \pi$ . Then  $f(P) = \sigma g(Q) = R$ , say, for some  $\sigma \in \pi$ , and since either  $P_{\sigma}$  does not exist or  $P \neq P_{\sigma}$ , there exist another pair  $(P', Q')$  with  $f(P') = \sigma g(Q') = R'$ , say, where  $P' \neq P_{\tau}$  for all  $\tau \in \pi$ . Let  $v$  be a path on  $Y$  from  $Q$  to  $Q'$  such that  $v(I) \cap \{Q_{\tau}\} = \emptyset$ . Choose an orientation of a neighborhood  $N_v$  of  $v(I)$  in  $Y$ . Since  $X$  is non-orientable, there exists a path  $u$  in  $X$  from  $P$  to  $P'$  with  $u(I) \cap \{P_{\tau}\} = \emptyset$ , and an orientation of a neighborhood  $N_u$  of  $u(I)$  in  $X$  such that the intersection number of  $\varphi(N_u)$  and  $\psi(N_v)$  is the integer 0. We can then use WHITNEY's lemma again, as in §1, and eliminate the pairs  $(P, Q)$  and  $(P', Q')$  from the set of intersection pairs by a diffeotopy of  $\varphi|N_u$  keeping  $\varphi$  fixed near the boundary of  $N_u$ . Thus  $\varphi$  can be replaced by  $\varphi_0$  such that

$$|\varphi_0(X) \cap \psi(Y)| = \Sigma_{\tau} |a \cdot \tau(b)|,$$

where  $|a \cdot \tau(b)|$  is the integer 0 if  $a \cdot \tau(b) = 0 \pmod{2}$  and the integer 1 if  $a \cdot \tau(b) = 1 \pmod{2}$ .

If  $[a, b] = a \cdot b$  in  $Z_2[\pi]$  for some liftings  $a, b$ , then  $a \cdot \tau(b) = 0$  for  $\tau \neq 1$ , and therefore, using these liftings in the above argument,  $|\varphi_0(X) \cap \psi(Y)|$  is equal to 0 or 1 depending on whether  $a \cdot b = 0$  or 1 mod 2 respectively. In other words,  $|\varphi_0(X) \cap \psi(Y)| = |a \cdot b|$ . Since  $a \cdot \tau(b) = 0$  for  $\tau \neq 1$  implies  $\alpha \cdot \beta = \sum_{\tau} a \cdot \tau(b) = a \cdot b$ , we have  $|\varphi_0(X) \cap \psi(Y)| = |\alpha \cdot \beta|$  as desired.

Conversely, if  $\varphi(X) \cap \psi(Y) = \emptyset$ , we obviously have  $[a, b] - a \cdot b = 0$  for any liftings of  $\varphi$  and  $\psi$  since both terms are then 0. If  $|\varphi(X) \cap \psi(Y)| = 1$ , we take  $a, b$  to be the classes of liftings of  $\varphi$  and  $\psi$  whose images intersect each other. Then  $a \cdot \tau(b) = \delta_{\tau, 1}$ . Hence  $[a, b] - a \cdot b = 0$  in this case too. This completes the proof of Theorem 2.

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