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# On the Automorphism Group of a G-structure<sup>1</sup>)

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#### 1. Introduction

Throughout this paper M denotes a paracompact differentiable manifold of dimension n. Let G be a Lie subgroup of GL(n,R). The group of diffeomorphisms of M which leave a G-structure invariant is often a Lie group. We shall give a condition on the Lie algebra  $\mathfrak{g}$  of G under which the group of automorphisms of a G-structure is a Lie group. (Cf. Definitions 5 and 6 in Section 3.) The main result is stated as Theorems A and B in Section 5, and examples are given in Sections 8 and 9. To simplify the presentation, 'differentiable' always means 'differentiable of class  $C^{\infty}$ '. It is to be remarked that for every specific G-structure however, differentiability of a suitable degree will be sufficient.

H. Cartan [3] proved in 1935 that the group of all complex analytic transformations of a bounded domain in  $C^n$  is a Lie group. S. Bochner and D. Montgomery [1] proved in 1946 that the group of all complex analytic transformations of a compact complex manifold is a Lie group. This result was extended in 1963 by W. M. BOOTHBY, S. KOBAYASHI and H. C. WANG [2] to the effect that the automorphism group of an almost complex structure on a compact manifold is a Lie group. By introducing a Bergman metric on a bounded domain in  $C^n$ , H. Cartan's result is shown to be a special case of a theorem proved by S. B. Myers and N. Steenrod [9] in 1939. Their theorem states that the group of isometries of a RIEMANNian manifold, i. e. the automorphism group of an O(n)-structure, is a Lie group. In view of the fact that a RIEMANNian manifold has a unique torsion free connection, this result is included in a theorem proved by K. Nomizu [10], J. Hano and A. Morimoto [6]. Their theorem states that the automorphism group of an affinely connected manifold is a Lie group. It will be shown in Section 8 that our main theorem includes all the examples mentioned. Some additional examples will be given in Section 9.

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We now give an outline of the present paper. In Section 3 we construct a sequence of  $G^{(i)}$ -structures induced by a G-structure on a differentiable manifold M. An automorphism  $\varphi$  of a G-structure will be lifted to automorphisms  $\varphi_i$  of the induced  $G^{(i)}$ -structures. This construction gives rise to a system of linear partial differential equations for infinitesimal automorphisms of a G-structure (Section 7). The type of the system will depend on the Lie algebra g of G. We impose conditions on the Lie algebra g so that the vector space of solutions will be finite-dimensional (Section 5). A theorem of g. S. Palais [11] shows that if the space of infinitesimal automorphisms is of finite dimension, then the group of automorphisms can be given a Lie group structure.

# 2. Prolongations of a LIE algebra

Let g be a Lie algebra of endomorphisms of a real n-dimensional vector space V. g may be regarded as a subspace of the tensor product  $V \otimes V^*$ , where  $V^*$  denotes the dual space of V. The first prolongation  $g^{(1)}$  of g is defined to be  $g^{(1)} = g \otimes V^* \cap V \otimes S^2(V^*) \subset V \otimes V^* \otimes V^*$ , where  $S^2(V^*)$  denotes the space of symmetric tensors of degree two over  $V^*$ . With respect to a basis in  $V \otimes V^* \otimes V^*$  an element  $a \in g^{(1)}$  will be given by a matrix  $(a_{j,k}^i)$ . Since  $g \otimes V^* = \text{Hom }(V,g)$ , an element  $a \in \text{Hom }(V,g)$  is in  $g^{(1)}$  if and only if  $a_n(v) = a_n(u)$  for all v,  $u \in V$ .

For each  $a \in \mathfrak{g}^{(1)}$  we define an automorphism  $\overline{a}$  of  $\mathfrak{g} \oplus V$  ( $\oplus$  denotes direct sum) as follows.  $\overline{a}(x) = x$  for  $x \in \mathfrak{g}$ ,  $\overline{a}(u) = a_u + u$  for  $u \in V$ .

**Definition 1.**  $G^{(1)} = \{ \overline{a} \mid a \in \mathfrak{g}^{(1)} \}$ .  $G^{(1)}$  is a commutative Lie group of automorphisms of the vector space  $\mathfrak{g} \oplus V$ .

**Definition 2.** The k-th prolongation  $\mathfrak{g}^{(k)}$  of  $\mathfrak{g}$  is defined to be  $\mathfrak{g}^{(k)} = \mathfrak{g}^{(k-1)} \otimes V^* \cap \mathfrak{g}^{(k-2)} \otimes S^2(V^*) = \mathfrak{g} \otimes V^* \otimes \ldots \otimes V^* \cap V \otimes S^{k+1}(V^*),$  where  $V^* \otimes \ldots \otimes V^*$  denotes the k-fold tensor product. (Note that  $\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(-1)} = V$ .)

**Definition 3.** A Lie algebra g is of finite type if  $g^{(k)} = 0$  for some k.

To  $\mathfrak{g}^{(k)}$  will correspond a commutative Lie group  $G^{(k)}$  of automorphisms of the vector space  $V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \ldots \oplus \mathfrak{g}^{(k-1)}$ , defined as follows. To  $a \in \mathfrak{g}^{(k)}$  define  $\overline{a} \in G^{(k)}$  by setting

$$\overline{a}(x) = x \text{ for } x \in \mathfrak{g} \oplus \ldots \oplus \mathfrak{g}^{(k-1)},$$
  
 $\overline{a}(u) = u + a_u \text{ for } u \in V.$ 

**Definition 4.** The annihilator  $\mathfrak{h}^{(k)}$  of  $\mathfrak{g}^{(k)}$  is defined by:

$$\mathfrak{h}^{(k)} = \{h \mid h \, \epsilon \, V^* \, \otimes S^{k+1}(V) \, , \, \langle h \, , \, g 
angle = O \, \, ext{ for all } \, g \, \epsilon \, \mathfrak{g}^{(k)} \} \, .$$

The annihilator  $\mathfrak{h}^{(k)}$  will be needed in order to state Theorem B.

### 3. G-structure, torsion tensor

Let M be a differentiable manifold of dimension n. A linear frame u at a point  $x \in M$  is an ordered basis  $X_1, \ldots, X_n$  of the tangent space  $T_x(M)$ . Let L(M) be the set of all linear frames at all points of M. Let  $\pi$  be the mapping of L(M) onto M which maps a linear frame u at x into x. L(M) is a principal fiber bundle with structure group GL(n,R). A linear frame u at x can also be defined to be a vector space isomorphism  $u:V \to T_x(M)$ . The two definitions are related in the following way: Let  $e_1,\ldots,e_n$  be a basis in V.  $u:V \to T_x(M)$  is defined by  $u(e_i)=X_i$ . The action of GL(n,R) on L(M) is given by  $u \to u \cdot a$ , where  $u \cdot a:V \xrightarrow{a} V \xrightarrow{u} T_x(M)$ . In the sequel we will think of u as the isomorphism  $u:V \to T_x(M)$ . The notation  $u^{-1}$  therefore makes sense.

**Definition 5.** A G-structure on a differentiable manifold M is a reduction of the structure group GL(n, R) of the bundle of linear frames L(M) to the subgroup G. The reduced bundle, a subbundle of L(M), will be denoted by P(M, G):

$$P(M, G) \xrightarrow{\subset} L(M)$$
 $\pi \downarrow \pi$ 
 $M \xrightarrow{\text{identity}} M$ 

A diffeomorphism  $\varphi$  of M can be lifted to an automorphism  $\Phi$  of the bundle L(M).

**Definition 6.** A diffeomorphism  $\varphi$  of M is called an automorphism of the Gstructure P(M,G) if  $\Phi$  maps P(M,G) onto itself. The restriction of  $\Phi$  to P(M,G) will be denoted by  $\varphi_1$ . (For reference see S. Kobayashi and K. Nomizu [7].)

We now turn to the construction of the torsion tensors associated with a G-structure (cf. S. Sternberg [12]). On L(M) define the canonical form  $\vartheta$  to be the V-valued 1-form

$$\vartheta(X) = u^{-1}\pi_*(X)$$
, where  $X \in T_u(L(M))$ .

The restriction of  $\vartheta$  to P(M,G) will still be denoted by  $\vartheta$ . An n-dimensional subspace  $H \subset T_u(P(M,G))$  is called a *horizontal subspace* if  $\vartheta: H \to V$  is an isomorphism. The exterior derivative  $d\vartheta$  of  $\vartheta$  evaluated at  $u \in P(M,G)$  is a bilinear mapping  $(d\vartheta)_u: \wedge {}^2T_u(P(M,G)) \to V$ . In view of the isomorphism  $\vartheta: H \to V$ ,  $d\vartheta$  restricted to  $H \wedge H$  defines a map  $V \wedge V \to V$ , i.e. an element  $c(u,H) \subset V \otimes V^* \wedge V^*$ .

**Definition 7.** c(u, H) is called the *torsion tensor* corresponding to a pair (u, H).

In the sequel, the dependence of c(u, H) on H will be discussed. The action of G on P, where P stands for P(M, G), induces a homomorphism  $\sigma$  of the Lie algebra  $\mathfrak{F}(P)$  of vector fields on P. For  $A \in \mathfrak{F}$ ,  $\sigma A$  is called the fundamental vector field corresponding to A. Since G acts freely on P, the mapping  $\sigma(u)$  defined by  $A \to (\sigma A)_u$  (( )<sub>u</sub> = evaluation at u) is an isomorphism of the space  $\mathfrak{F}(u)$  onto the tangent space at u of the fiber  $G_u$  through u. Given a horizontal subspace  $H \subset T_u(P)$ , we define n vectors  $Z_i$  such that  $Z_i \in H$  and  $\vartheta(Z_i) = e_i$ ,  $i = 1, 2, \ldots, n$ , where  $e_i$  is the i-th element of a basis in V. For another H' we define  $Z'_i$  in the same fashion. For each i there is a unique  $A_i \in \mathfrak{F}$  such that  $\sigma(u)A_i = Z'_i - Z_i = Y_i$ .

**Definition 8.** S(H, H') is defined to be the map of V into g which sends  $e_i$  into  $A_i$ .

Let  $\hat{Y}_i$  be a vector field in a neighborhood of u in P, such that the evaluation of  $\hat{Y}_i$  at u is equal to  $Y_i$  i.e.  $(\hat{Y}_i)_u = Y_i$ . Likewise find  $\hat{Z}_i$  such that  $(\hat{Z}_i)_u = Z_i$ . The torsion tensor c(u, H) is a map  $V \wedge V \to V$ , given by  $c(u, H)(e_i, e_j) = d\vartheta(Z_i, Z_j) = \frac{1}{2} \{Z_i\vartheta(\hat{Z}_j) - Z_j\vartheta(\hat{Z}_i) - \vartheta([\hat{Z}_i, \hat{Z}_j])\}$ , where  $[\ ,\ ]$  denotes the Lie bracket (cf. S. Kobayashi and K. Nomizu [7], p. 36). Likewise we define c(u, H'). Hence

$$\begin{split} \left(c\left(u\,,\,H'\right)-c\left(u\,,\,H\right)\right)\left(e_{i}\,,\,e_{j}\right) &=d\vartheta\left(Z'_{i},\,Z'_{j}\right)-d\vartheta\left(Z_{i}\,,\,Z_{j}\right)\\ &=d\vartheta\left(Z_{i}\,,\,Y_{j}\right)-d\vartheta\left(Z_{j}\,,\,Y_{i}\right)+d\vartheta\left(Y_{i}\,,\,Y_{j}\right)\,. \end{split}$$

Since  $d\vartheta(Y_i, Y_i)$  is equal to zero (cf. [7], p. 120), we have

$$\begin{split} (c(u,H')-c(u,H))(e_i,e_j) &= \frac{1}{2}\{Z_i\vartheta(\hat{Y}_j)-Y_j\vartheta(\hat{Z}_i)-(Z_j\vartheta(\hat{Y}_i)-Y_i\vartheta(\hat{Z}_j))\\ &-(\vartheta([\hat{Z}_i,\hat{Y}_j])-\vartheta([\hat{Z}_j,\hat{Y}_i]))\} \;. \end{split}$$

Here we note that  $Z_i\vartheta(\hat{Y}_i)=0$  and  $Z_j\vartheta(\hat{Y}_i)=0$  because  $\vartheta$  maps the vectors tangent to the fiber into zero. Now we choose the vector fields  $\hat{Z}_i$  and  $\hat{Y}_i$  such that the brackets  $[\hat{Z}_i, \hat{Y}_j]$  and  $[\hat{Z}_j, \hat{Y}_i]$  vanish, for example, as follows. Let

U be a neighborhood of  $\pi(u) \in M$ ; write  $\pi^{-1}(U) = U \times G$  by choosing a cross section  $U \to P$  through u tangent to  $Z_1, \ldots, Z_n$ . Let  $x^1, \ldots, x^n$  be a coordinate system in U and let  $y^1, \ldots, y^m$  be a coordinate system in a neighborhood of the identity in G such that

$$\left(\frac{\partial}{\partial x^i}, 0\right)_u = Z_i \text{ and } \left(0, \frac{\partial}{\partial y^j}\right)_u = Y_i.$$

Set  $\hat{Z}_i = \left(\frac{\partial}{\partial x^i}, 0\right)$  and  $\hat{Y}_j = \left(0, \frac{\partial}{\partial y^j}\right)$ . With this choice we have:

$$(c(u, H') - c(u, H))(e_i, e_j) = \frac{1}{2} |Y_i \vartheta(\hat{Z}_j) - Y_j \vartheta(\hat{Z}_i)|.$$

We shall prove now that:  $Y_j \vartheta(\hat{Z}_i) = -A_j e_i$ , where  $A_j \epsilon \mathfrak{g}$  is defined by  $Y_j = \sigma(u) A_j$ . We have

$$\vartheta(\widehat{Z}_i)_{ua} = (ua)^{-1}\pi(\widehat{Z}_i) = a^{-1}\vartheta(Z_i) = a^{-1}e_i, \ a \in G.$$

Using the definition of the fundamental vector field  $\sigma A$  we get

$$Y_j(a^{-1}e_i) = \left(\frac{d}{dt}\exp\left(-tA_j\right)\right)_{t=0}e_i = -A_je_i$$
 .

Hence we have

**Proposition 1.**  $(c(u, H') - c(u, H))(e_i, e_j) = \frac{1}{2} |A_j e_i - A_i e_j|$  where  $A_j = S(H, H')e_j$  (cf. Definition 7).

**Proposition 2.**  $c(u, H') - c(u, H) \epsilon \alpha(\mathfrak{g} \otimes V^*) \subset V \otimes V^* \wedge V^*$ , where  $\alpha$  denotes the alternation in the two covariant factors.

Note that the kernel of  $\alpha$  is equal to the first prolongation  $\mathfrak{g}^{(1)}$  of  $\mathfrak{g}$ .

# 4. Induced $G^{(i)}$ -structures

The purpose of this section is to define a sequence of  $G^{(i)}$ -structures and to show that an automorphism of a G-structure can be lifted to automorphisms of the successive  $G^{(i)}$ -structures.

Let P(M, G) be a G-structure on M (see Definition 5). Let  $\mathfrak{g}$  be the Lie algebra of G. We shall choose once and for all a linear subspace  $C \subset V \otimes V^* \wedge V^*$  such that  $V \otimes V^* \wedge V^* = C \oplus \alpha(\mathfrak{g} \otimes V^*)$ . In general there will be no natural way of choosing C. The torsion tensor provides a map of P(M, G) into  $V \otimes V^* \wedge V^* = C \oplus \alpha(\mathfrak{g} \otimes V^*)$  (see Definition 7).

**Definition 9.** The image of c(u, H) in  $\alpha(\mathfrak{g} \otimes V^*)$  will be denoted by k(u, H), where c(u, H) is given in Definition 7.

To a horizontal subspace H at  $u \in P$  we assign a frame

$$z = (Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+m})$$

at u in the following manner. For  $i=1,2,\ldots,n,Z_i$  is defined by  $Z_i \in H$  and  $\vartheta(Z_i)=e_i$ , where  $e_i$  is the i-th element of a basis in V. For  $j=n+1,\ldots,n+m,\,Z_j$  is defined by  $Z_j=\sigma(u)A_{j-n}$ , where  $A_1,\ldots,A_m$  is a basis in  $\mathfrak{g}$ .

**Definition 10.**  $P_1(P, G^{(1)})$  is the set of frames  $z = (Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+m})$  corresponding to horizontal subspaces  $H \subset T_u(P)$  such that  $u \in P$  and k(u, H) = 0.

**Proposition 3.** A G-structure P(M, G) on M gives rise to a uniquely defined  $G^{(1)}$ -structure  $P_1(P, G^{(1)})$  on P = P(M, G).

**Proof.** For  $a \in GL(n+m,R)$  and  $z \in P_1 = P_1(P,G^{(1)}) \subset L(P), z \cdot a$  is defined by  $z \cdot a : V \oplus \mathfrak{g} \xrightarrow{a} V \oplus \mathfrak{g} \xrightarrow{z} T_u(P)$ . Proposition 2, Section 3, shows that  $z \cdot a$  is in  $P_1$  if and only if  $a \in G^{(1)}$ . (See Definition 2 and note that  $\mathfrak{g}^{(1)}$  is the kernel of the map  $\alpha : \mathfrak{g} \otimes V^* \to V \otimes V^* \wedge V^*$ .) The following lemma will conclude the proof of Proposition 3.

**Lemma.**  $P_1(P, G^{(1)})$  is locally, in fact globally, trivial.

**Proof.** We shall construct a distribution of horizontal subspaces  $\mathfrak{H}$ , in fact a connection, on P such that k(u, H) for  $u \in P$  and  $H \in \mathfrak{H}$ , is zero. Since M is paracompact, the bundle P(M, G) has a connection  $\mathfrak{H}'$  giving rise to a differentiable map

$$k(\ ,H'):P(M,G)\rightarrow \alpha(\mathfrak{g}\,\otimes\,V^*)$$
 .

Since  $\mathfrak{g} \otimes V^*$  is isomorphic to kernel  $\alpha \oplus \alpha(\mathfrak{g} \otimes V^*)$ , we may choose a monomorphism  $q: \alpha(\mathfrak{g} \otimes V^*) \to \mathfrak{g} \otimes V^*$ . The composition  $q \circ k(\cdot, H)$  maps P(M, G) into  $\mathfrak{g} \otimes V^*$ . Let  $\mathfrak{H}$  be the distribution defined by the vector fields

$$Z_i = Z'_i - \sigma(u) \left( (q \circ k(u, H))e_i \right), i = 1, 2, \ldots, n.$$

(For definitions of  $Z_i'$  and  $\sigma$  see Section 3.) Since  $k(u, H)(e_i, e_j) = (k(u, H') - \alpha(q \circ k(u, H')))(e_i, e_j) = 0$ , by virtue of Proposition 1, we obtain a global cross section  $(Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+m})$  of  $P_1(P, G^{(1)})$  over P(M, G).

**Proposition 4.** An automorphism  $\varphi$  of the G-structure P(M, G) can be lifted to an automorphism  $\varphi_1$  of the  $G^{(1)}$ -structure  $P_1(P, G^{(1)})$ .

**Proof.** Let  $\varphi_1$  be the map introduced in Definition 6, Section 3. The fundamental 1-form  $\vartheta$  is invariant by  $\varphi_1$  and hence  $d\vartheta(\varphi_{1*}Z_i, \varphi_{1*}Z_j) = d\vartheta(Z_i, Z_j)i$ ,  $j = 1, 2, \ldots, n$ , where  $z = (Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+m}) \epsilon P_1(P, G^{(1)})$ . This together with the fact that  $\varphi_1$  leaves the fundamental vector fields invariant, proves Proposition 4.

**Proposition 5.** An automorphism  $\varphi = \varphi_0$  of the G-structure P(M, G) can be lifted to automorphisms  $\varphi_1, \varphi_2, \ldots$  of the bundles  $P_1, P_2, \ldots$  respectively.

**Proof.** The bundles and automorphisms are defined inductively by applying Propositions 3 and 4 to  $P_i$  and  $\varphi_i$ ,  $i=1,2,\ldots$ , instead of applying them to  $M=P_0$  and  $\varphi=\varphi_0$ .

#### 5. Main Theorems

Let G be a (not necessarily closed) Lie subgroup of GL(n, R) and let M be a differentiable manifold of dimension n.

Our main results are

**Theorem A<sup>1</sup>).** If the Lie algebra g of G is of finite type then the automorphism group of a G-structure P(M, G) on M is a Lie group.

Remark 1. Theorem A applies also to the case where the sequence of bundles starts at the *i*-th stage. In the case i = 1 the theorem reads as follows.

Let  $G^{(1)}$  be a Lie subgroup of  $GL(n, R)^{(1)}$  (first prolongation). If the Lie algebra  $\mathfrak{g}^{(1)}$  of  $G^{(1)}$  is of *finite type*, then the group of diffeomorphisms of M whose lifts to L(M) are automorphisms of a  $G^{(1)}$ -structure on L(M) is a Lie group.

**Theorem B.** Let M be a compact differentiable manifold of dimension n. If there is an integer N and n elements  $_{l}h, l = 1, 2, \ldots, n$ , in the annihilator  $\mathfrak{h}^{(N)}$  of  $\mathfrak{g}^{(N)}$  such that the determinant of the matrix

$$_{l}h_{i}=\sum\limits_{j_{1}\ldots j_{N+1}}{_{l}h_{i}^{j_{1}\ldots j_{N+1}}\xi_{j_{1}}\ldots\xi_{j_{N+1}}}$$

is nonvanishing for every  $\xi = (\xi_1, \ldots, \xi_n) \neq 0$ ,  $\xi \in \mathbb{R}^n$ , then the automorphism group of a G-structure P(M, G) is a Lie group.

Corollary to Theorem B. Let M be a compact differentiable manifold of dimension n. If there is a  $q = (q^{jk}) \in S^2(V)$  such that  $\sum_{jk} q^{jk} \xi_j \xi_k$  is positive definite and  $V^* \otimes q \subset \mathfrak{h}^{(1)} \subset V^* \otimes S^2(V)$ , then the automorphism group of a G-structure P(M, G) is a  $L_{IE}$  group.

<sup>1)</sup> Added in proof: This theorem is already known. (Cf. S. Sternberg, Lectures on Differential Geometry. Prentice Hall, 1964.)

**Remark 2.** If a Lie subgroup G of GL(n, R) satisfies the requirements of Theorem A or B, then so does every Lie subgroup of G.

# 6. Two Lemmas on Partial Differential Equations

In this section, we prepare two lemmas which will be needed in the proof of Theorems A and B.

Consider a system of linear partial differential equations

$$\frac{\partial u^{\sigma}}{\partial x^{j}} = \sum_{\lambda} a^{\sigma}_{j\lambda}(x^{1}, \ldots, x^{r}) u^{\lambda} \quad \sigma, \lambda = 1, 2, \ldots, s, j = 1, 2, \ldots, r,$$
 (1)

for s functions  $u^{\sigma} = u^{\sigma}(x^1, \ldots, x^r)$  with initial conditions

$$u^{\sigma}(0) = u_0^{\sigma}. \tag{2}$$

**Lemma 1.** The system (1) with initial conditions (2) has at most one solution.

**Proof.** Assume it has two solutions  $(u^{\sigma})$  and  $(v^{\sigma})$  such that for  $x^{i} = a^{i}$  we have the following inequality  $u^{\sigma}(a^{1}, \ldots, a^{r}) \neq v^{\sigma}(a^{1}, \ldots, a^{r})$ . By setting  $x^{i} = a^{i}t$  and  $u^{\sigma} = u^{\sigma}(t)$ , a system of ordinary differential equations

$$rac{du^{\sigma}}{dt} = \sum\limits_{i} rac{\partial u^{\sigma}}{\partial x^{j}} rac{dx^{j}}{dt} = \sum\limits_{\lambda i} a^{\sigma}_{\lambda j} a^{j} u^{\lambda} \, ,$$

with initial conditions  $u^{\sigma}(0) = u_0^{\sigma}$  is obtained. The *uniqueness theorem* on ordinary differential equations implies  $u^{\sigma}(1) = v^{\sigma}(1)$ , i.e.  $u^{\sigma}(a^1, \ldots, a^r) = v^{\sigma}(a^1, \ldots, a^r)$ . This contradiction proves Lemma 1.

In the proof of *Theorem A* the following system of differential equations will occur.

$$\frac{\partial^{d+1}}{\partial x^{i_1} \dots \partial x^{i_{d+1}}} X^p = \sum_k L^p_{i_1 \dots i_{d+1} k}(x, D) X^k, \tag{1*}$$

where  $LP_{i_1...i_{d+1}k}$  is a polynomial in the differential operators  $D: \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$  of degree smaller than or equal to d with variable coefficients.

Introducing new variables  $Y_{i_1...i_k}^p = \frac{\partial^k}{\partial x^{i_1}...\partial x^{i_k}} X^p$ , k = 0, 1, ..., d, we

obtain a system of differential equations of first order. By adding the initial conditions  $YP_{i_1...i_k}(0) = YP_{i_1...i_k0}$ , we obtain a system (1), (2). According to Lemma 1, this system has at most one solution.

Let M be a differentiable manifold and let  $\mathfrak{X}$  be a vector space of infinitesimal transformations on M such that every point of M has a coordinate neighborhood with a system (1\*) of differential equations which is satisfied by all

infinitesimal transformations X in  $\mathfrak{X}$ . Let r be the number of linearly independent initial conditions  $(Y_{i_1...i_k}(0))$  at an arbitrary point  $0 \in M$ .

**Lemma A.** The dimension of the vector space  $\mathfrak{X}$  is smaller than or equal to r.

**Proof.** Locally use Lemma 1 and note that the continuation of a solution along a curve in M is unique if it exists at all.

In order to prove *Theorem B*, a lemma on *elliptic partial differential equations* will be needed (cf. A. Douglis and L. Nirenberg [4]). Let

$$\sum_{j=1}^{n} l_{ij} X^{j} = 0 \tag{3}$$

be a system of linear partial differential equations in n independent variables  $x^1, \ldots, x^n$  and n functions  $X^1, \ldots, X^n$ . The  $l_{ij}$ 's are linear differential operators which may be expressed as polynomials,  $l_{ij}(x, D)$ , in the differential operators  $D: (\partial/\partial x^1, \ldots, \partial/\partial x^n)$  with variable coefficients,  $a_{ij}, \varrho$ . Let  $l'_{ij}(x, D)$  represent the sum of the terms in  $l_{ij}(x, D)$  which are of order S, where S denotes the order of the system (3). For arbitrary scalars  $\xi = (\xi_1, \ldots, \xi_n)$  the characteristic matrix of (3) is defined to be the  $n \times n$  matrix  $l'_{ij}(x, \xi)$ . The determinant,  $L(x, \xi)$ , is a homogeneous polynomial in  $\xi$  of degree  $n \cdot S$ . The system is called elliptic if  $L(x, \xi)$  is nonvanishing for every  $\xi \neq 0$ .

#### Lemma 2. Assume that

- (i)  $L(x,\xi) \geq K \cdot |\xi|^{n \cdot S}$  for some K > 0;
- (ii) There exists a constant  $L_1$  such that

$$\left| rac{\partial^k a_{ij,\,\varrho}}{\partial \, x^{i_1} \ldots \partial \, x^{i_{m k}}} 
ight| < L_1 \; ext{ for } \; k=0\,,\,1\,,\,2\,;$$

(iii)  $X = (X^1, ..., X^n)$  is a solution of (3) in a domain D and there exists a constant  $L_2$  such that

$$\left| rac{\partial^k X^p}{\partial x^{i_1} \ldots \partial x^{i_k}} 
ight| < L_2 \; ext{ for } \; k=0,1,\ldots,S+1$$
 .

Then for any compact subset  $F \subset D$  there exists a constant C depending only on  $L_1, L_2$ , and K such that

$$\left| \frac{\partial^{S+1} X^p(P)}{\partial x^{i_1} \dots \partial x^{i_{S+1}}} - \frac{\partial^{S+1} X^p(Q)}{\partial x^{i_1} \dots \partial x^{i_{S+1}}} \right| < C \cdot |P - Q|,$$

where P and Q are arbitrary points in F.

Let  $\mathfrak{X}' = \{(XP)\}$  be a family of functions subject to the conditions in Lemma 2. The family of functions  $\mathfrak{X}'$  and their partial derivatives through the S+1-st order is bounded and equicontinuous in F. By Arzela's theorem

every sequence in X', if restricted to F, has a subsequence which is convergent with respect to the topology of uniform convergence of functions together with their partial derivatives through the S+1-st order.

Let M be a compact differentiable manifold and let  $\mathfrak{X}$  be a vector space of infinitesimal transformations on M such that every point of M has a coordinate neighborhood with a system (3) of partial differential equations which is satisfied by all infinitesimal transformations X in  $\mathfrak{X}$ . In addition, X is subject to the assumptions in Lemma 2. In local coordinates  $X \in \mathfrak{X}$  is given by

$$X = \sum_{p=1}^{n} X^{p} \frac{\partial}{\partial x^{p}}$$
. By choosing an arbitrary Riemannian metric we define

$$||X|| = \max_{P \in M} |X| + \ldots \max_{P \in M} |\nabla^{S+1}X|$$
,,

where  $\nabla$  denotes the covariant derivative defined by this metric and | denotes the norm obtained by extending the Riemannian metric.

The norm || || makes  $\mathfrak{X}$  into a Banach space. Since convergence in this norm is equivalent to uniform convergence of functions together with their partial derivatives through the S+1-st order, and since M is compact, the following lemma is obtained.

**Lemma B.** The Banach space  $\mathfrak{X}$  is locally compact and hence finite-dimensional.

#### 7. Proof of the Main Theorems

The proof consists of the following steps. First a system of linear partial differential equations for infinitesimal automorphisms of a G-structure is established. Under the assumptions of Theorems A and B we shall prove that the space of solutions is finite-dimensional. Then a theorem of R. S. Palais [11] shows that in this case the group of automorphisms is a Lie group.

Let the vector fields  $\{Z_j, j=1, 2, \ldots, n+m\}$  be a cross section of the bundle  $P_1(P, G^{(1)})$ . By Proposition 4, Section 4, an automorphism  $\varphi$  of P(M, G) can be lifted to an automorphism  $\varphi_1$  of  $P_1(P, G^{(1)})$ . Hence, for  $u \in P(M, G)$ ,  $\{\varphi_1 * ((Z_j)_u), j=1, 2, \ldots, n+m\}$  will be an element of  $P_1(P, G^{(1)})$  at  $\varphi_1(u)$ . Recalling the proof of Proposition 3, Section 4, we have

**Proposition 6.** For each  $u \in P$  there is an  $a \in \mathfrak{g}^{(1)}$  such that

$$\varphi_{1}^{*}((Z_{j})_{u}) = (Z_{j})_{\varphi_{1}(u)} + \sigma(\varphi_{1}(u))(ae_{j}), \text{ for } j = 1, 2, ..., n.$$

Note that  $\sigma(\varphi_1(u))(ae_i)$  is the fundamental vector field, evaluated at  $\varphi_1(u)$ , which corresponds to  $ae_i \in \mathfrak{g}$ . Let  $(x^1, \ldots, x^n)$  be a coordinate system in  $U \subset M$ . With respect to the coordinates  $(x^i, x_i^i)$  in  $U \times GL(n, R)$ ,

 $Z_{j}$ , for  $j=1,2,\ldots,n$ , is given by  $Z_{j}=\left(\sum_{i}c_{j}^{i}(u)\frac{\partial}{\partial x^{i}},\sum_{kl}c_{lj}^{k}(u)\frac{\partial}{\partial x_{l}^{k}}\right)$ . For  $u=(x^{i},x_{j}^{i}),\ \varphi_{1}(u)$  is expressed by  $\left(\varphi^{i}(x),\sum_{q}\frac{\partial\varphi^{i}(x)}{\partial x^{q}}\cdot x_{j}^{q}\right)$ . Proposition 6 yields

$$\left(\sum_{ik} c_{j}^{i}(u) \frac{\partial \varphi^{k}}{\partial x^{i}} \frac{\partial}{\partial x^{k}}, \sum_{pl} \left(\sum_{iq} c_{j}^{i}(u) \frac{\partial^{2} \varphi^{p}(x)}{\partial x^{q} \partial x^{i}} x_{l}^{q} + \sum_{q} c_{lj}^{q}(u) \frac{\partial \varphi^{p}(x)}{\partial x^{q}}\right) \frac{\partial}{\partial x_{l}^{p}}\right) = \\
= \left(\sum_{i} c_{j}^{i}(\varphi_{1}(u)) \frac{\partial}{\partial x^{i}}, \sum_{kl} c_{lj}^{k}(\varphi_{1}(u)) \frac{\partial}{\partial x_{l}^{k}}\right) + \sigma(\varphi_{1}(u)) (a e_{j}). \tag{1}$$

As in Section 2, an element  $a \in \mathfrak{g}^{(1)}$  is given by a matrix  $a = (a_{l,j}^p)$ . For  $u = (x^i, \delta_j^i)(\delta_j^i = \text{Kronecker delta})$  we get  $\sigma(u)(ae_j) = \sum_{pl} a_{l,j}^p \frac{\partial}{\partial x_l^p}$ . Since this is true for  $u = (x^i, \delta_j^i)$  only, we evaluate the above equation at  $w = (y^i, y_j^i) = \left(\left(\varphi^{-1}(x)\right)^i, \sum_k \left(\frac{\partial \varphi^k(x)}{\partial x^j}\right)^{-1} \delta_k^i\right)$ . Thus we have  $\varphi_1(w) = (x^i, \delta_j^i)$ . The components in the fiber direction of equation (1) yield

$$\sum_{i,q} c_{i}^{i}(w) \frac{\partial^{2} \varphi^{p}(y)}{\partial x^{q} \partial x^{i}} y_{l}^{q} + \sum_{q} c_{lj}^{q}(w) \frac{\partial \varphi^{p}(y)}{\partial x^{q}} = c_{lj}^{p}(x^{i}, \delta_{j}^{i}) + a_{l,j}^{p}, \qquad (2)$$

where the  $a_{l,i}^p$ 's depend on  $\varphi(x)$ .

In order to obtain a system of differential equations, satisfied by all automorphisms  $\varphi$ , we let an element  $h = (h_p^{lj}) \epsilon \mathfrak{h}^{(1)}$  operate on equation (2). (For  $\mathfrak{h}^{(1)}$  see Definition 4.) Thus we get

$$\sum_{p \mid j} h_p^{lj} \left( \sum_{i,q} c_j^i(w) \frac{\partial^2 \varphi^p(y)}{\partial x^q \partial x^i} y_l^q + \sum_{q} c_{lj}^q(w) \frac{\partial \varphi^p(y)}{\partial x^q} - c_{lj}^p(x) \right) = \sum_{p \mid j} h_p^{lj} a_{l,j}^p = 0. \quad (3)$$

Let X be a vector field on M. In a coordinate neighborhood U write  $X = \sum_{p} X^{p} \frac{\partial}{\partial x^{p}}$ . X generates a local 1-parameter group of local transformations  $\varphi: U' \times I_{\varepsilon} \to U$ , where U' is an open subset of U and  $I_{\varepsilon}$  is an open neighborhood of zero in R. X is called an *infinitesimal automorphism* of the G-structure P(M, G) if for  $t \in I_{\varepsilon}$   $\varphi(x, t)$  is an isomorphism of P(U', G) onto  $P(\varphi_{t}(U'), G)$ .

In this case,  $\varphi(x,t)$  for  $t \in I_{\varepsilon}$  is a solution of equation (3). By taking the derivative of equation (3) with respect to t for t = 0, and noting that

$$\varphi^p \ (x, 0) = x^p \ ext{ and } \frac{\partial^{k+1} \varphi(x, 0)}{\partial t \, \partial x^{i_1} \dots \partial x^{i_k}} = \frac{\partial^k X^p}{\partial x^{i_1} \dots \partial x^{i_k}} ext{ for } k = 0, 1, \dots,$$

we get

$$\sum_{ijlp} h_p^{lj} c_j^i(x) \frac{\partial^2 X^p}{\partial x^l \partial x^i} + \sum_{jp} h_p^j(x) \frac{\partial X^p}{\partial x^j} + \sum_{p} h_p(x) X^p = 0, \qquad (4)$$

where  $c_j^i(x)$ ,  $h_p^j(x)$ , and  $h_p(x)$  depend on  $x=(x^1,\ldots,x^n)$  only, while  $h_p^{lj}$  is a constant. Thus we see that every element  $h=(h_p^{lj}) \epsilon \mathfrak{h}^{(1)}$  gives rise to an equation (4). By lifting an automorphism  $\varphi$  to a  $G^{(d)}$ -structure, we see that an element  $h=(h_p^{j_1,\ldots,j_{d+1}}) \epsilon \mathfrak{h}^{(d)}$  gives rise to a linear partial differential equation of order d+1. The highest order term will be

$$\sum_{ij_1\ldots j_{d+1}p} h_p^{j_1\ldots j_{d+1}} c_{j_{d+1}}^i(x) \frac{\partial^{d+1} X^p}{\partial x^i \partial x^{j_1}\ldots \partial x^{j_d}}.$$
 (5)

If the Lie algebra g of G is of finite type, then there is an integer d such that  $g^{(d)} = 0$ , i. e.  $\mathfrak{h}^{(d)}$  equals  $V^* \otimes S^{d+1}(V)$ . This yields the system of linear partial differential equations:

$$\frac{\partial^{d+1}}{\partial x^{j_1} \dots \partial x^{j_{d+1}}} X^p = \sum_k L^p_{j_1 \dots j_{d+1} k}(x, D) X^k.$$

(For a definition of D see equation [1\*] in Section 6.) By Lemma A, Section 6, the vector space of all infinitesimal automorphisms of a G-structure P(M, G) is of finite dimension. An upper bound of this dimension is given by the following:  $n + \dim \mathfrak{g} + \dim \mathfrak{g}^{(1)} + \ldots + \dim \mathfrak{g}^{(d)}$ .

The condition on the Lie algebra  $\mathfrak{g}$  of G in Theorem B is to insure that condition (i) of Lemma 2, Section 6 is fulfilled. Note that it is possible to choose a coordinate system such that  $c_j^i(0) = \delta_j^i$ . Condition (ii) can be satisfied by restricting the vector field X to a smaller neighborhood if necessary. Lemma B therefore shows that under the assumptions in Theorem B the vector space of infinitesimal automorphisms of a G-structure P(M, G) is of finite dimension.

An application of the following theorem of R. S. Palais [11] concludes the proof of Theorems A and B. Let H be a group of differentiable transformations acting on a differentiable manifold M. Let  $\mathfrak{h}'$  be the set of all vector fields on M which generate a global 1-parameter group of transformations belonging to H. Let  $\mathfrak{h}$  be the subalgebra generated by  $\mathfrak{h}'$  in the Lie algebra  $\mathfrak{X}(M)$  of all differentiable vector fields on M.

**Theorem.** If  $\mathfrak{h}$  is finite-dimensional, then H admits a Lie group structure (such that the map  $H \times M \to M$  is differentiable), and  $\mathfrak{h} = \mathfrak{h}'$ . The Lie algebra of H is naturally isomorphic to  $\mathfrak{h}$ .

Now let H be the group of all automorphisms of a G-structure P(M, G). Then  $\mathfrak{h}'$  and  $\mathfrak{h}$  are contained in the Lie algebra of *infinitesimal automorphisms* of P(M, G). This Lie algebra has been proved to be of finite dimension under the assumptions of Theorems A and B respectively.

# 8. Examples Mentioned in the Introduction

#### 1. RIEMANNian manifold.

Let O(n) denote the group which leaves a given nondegenerate symmetric bilinear form (,) on V (of arbitrary signature) invariant. A Riemannian structure is an O(n)-structure on M. We will show that the Lie algebra  $\mathfrak{o}(n)$  of O(n) is of finite type, in fact,  $\mathfrak{o}(n)^{(1)}=0$ . Thus Theorem A will apply. The maximal dimension of the automorphism group is  $n+\dim\mathfrak{o}(n)=\frac{n(n+1)}{2}$ . The following computation is taken from V. W. Guillemin and S. Sternberg [5]. A linear transformation a of V is in  $\mathfrak{o}(n)$  if and only if (au, v) + (u, av) = 0; for all  $u, v \in V$ . For any  $a \in \mathfrak{o}(n)^{(1)}$  and any  $u, v, w \in V$  we have

$$(awv, u) = (avw, u) = -(avu, w) = -(auv, w) = (auw, v) =$$
  
=  $(awu, v) = -(awv, u)$ .

Thus (awu, v) = 0, which implies a = 0 because (,) is nonsingular.

# 2. Conformal structure on a manifold of dimension $n \geq 3$ .

Let (,) be as in Example 1.  $\mathfrak{co}(n)$  denotes its conformal algebra.

$$a \in \mathfrak{co}(n)$$
 if and only if  $(au, v) + (u, av) = \lambda \cdot (u, v)$  for all  $u, v \in V$ ,

where  $\lambda$  is a scalar depending on a. V. W. Gullemin and S. Sternberg [5] show that  $\mathfrak{co}(n)^{(1)}$  is of dimension n by establishing a vector space isomorphism  $\mathfrak{co}(n)^{(1)} \to V^*$ . For  $a \in \mathfrak{co}(n)^{(2)}$  and u, v, x, y in V we get

$$(auvx, y) + (x, auvy) = (\Lambda uv)(x, y),$$

where  $\Lambda$  is an element of  $V^* \otimes V^*$  which depends on a. By a computation similar to that used in Example 1,  $\Lambda$  is shown to be zero (cf. [5]). This implies that a=0 because in this case  $a \in \mathfrak{o}(n)^{(2)}=0$ . Hence  $\mathfrak{co}(n)^{(2)}=0$ , the conformal structure is of finite type and Theorem A applies.

#### 3. Manifold with an affine connection.

An affine connection is a  $G^{(1)}$ -structure on the bundle of frames L(M), where  $G^{(1)}$  consists of the identity in  $GL(n,R)^{(1)}$  alone, i.e.,  $\mathfrak{g}^{(1)}=0$  (cf. Remark 1, Section 5). Theorem A applies; hence, the maximal dimension of the automorphism group of an affine connection will be

$$N = n + \dim \mathfrak{gl}(n, R) = n + n^2$$
.

# 4. Almost complex structure

Let M be a compact differentiable manifold of dimension n = 2m, and let  $G = GL(m, C) \subset GL(2m, R)$ .

$$a = (a_q^p) \epsilon g$$
 if and only if  $a_j^i = a_{j+m}^{i+m}, a_j^{m+i} = -a_{m+j}^i;$   
 $i, j = 1, 2, ..., m; p, q = 1, 2, ..., 2m.$ 

Computation of  $g^{(1)}$  (cf. Section 2):

$$\begin{array}{lll} a^i_{j,k} & = & a^{i+m}_{j+m,k} = & a^{i+m}_{k,j+m} = -a^i_{k+m,j+m} = -a^i_{j+m,k+m} \,, \\ \\ a^{i+m}_{j,k} & = -a^i_{j+m,k} = -a^i_{k,j+m} = -a^{i+m}_{k+m,j+m} = -a^{i+m}_{j+m,k+m} \,. \end{array}$$

Since  $a_{i,j}^p + a_{i+m,j+m}^p = 0$ , we obtain

$$h = (h_p^{ql}) = (a_p \cdot \delta^{ql}) \subset \mathfrak{h}^{(1)}$$
 (cf. Definition 4).

The Kronecker Delta,  $\delta^{ql}$ , is a unit matrix and hence, positive definite; therefore, the corollary to *Theorem B* applies. The *automorphism group* of an almost complex structure on a compact differentiable manifold is a Lie group.

# 9. Further Examples

# 1. Tensor product structure on a manifold M.

Let M be a manifold of dimension  $p \cdot q$  where  $p, q \geq 2$ . Let G be the Lie subgroup of  $GL(p \cdot q, R)$  whose Lie algebra is given by the tensor product representation of  $\mathfrak{g}_1 \oplus \mathfrak{g}_2$  on  $V_1 \otimes V_2$ . The action of  $\mathfrak{g}$  on  $V = V_1 \otimes V_2$  is given by  $(a, b)(v_1 \otimes v_2) = av_1 \otimes v_2 + v_1 \otimes bv_2,$ 

where  $a = (a_j^i) \epsilon \mathfrak{g}_1$  and  $b = (b_l^k) \epsilon \mathfrak{g}_2$ .

V has a basis  $e_{(i,k)} = e_i \otimes f_k$ , where  $(e_i)$  and  $(f_k)$  are bases in  $V_1$  and  $V_2$  respectively.

An element in g is denoted by  $A = (A_{(j,l)}^{(i,k)}) = (a_j^i \delta_l^k + \delta_j^i b_l^k)$ . This gives rise to the following four equations:

For 
$$i \neq j$$
,  $k \neq l$ ,  $A_{(j,l)}^{(i,k)} = 0$ ; (1)

For 
$$i \neq j$$
,  $A_{(i,n)}^{(i,n)} = A_{(i,m)}^{(i,m)}$ ;  $A_{(m,i)}^{(m,i)} = A_{(n,i)}^{(n,j)}$ ; (2)

$$A_{(i,m)}^{(i,m)} - A_{(j,m)}^{(j,m)} = A_{(i,n)}^{(i,n)} - A_{(j,n)}^{(j,n)}; \tag{3}$$

$$A_{(i,m)}^{(i,m)} - A_{(i,n)}^{(i,n)} = A_{(i,m)}^{(i,m)} - A_{(i,n)}^{(i,n)}. \tag{4}$$

We show that  $g^{(2)} = 0$ . Let  $A = (A_{(i,m),(j,k),(k,o)}^{(h,l)})$  be an element in  $g^{(2)}$ . It is easy to show that unless all index pairs coincide, the corresponding component of A vanishes as the following computation illustrates. Assume  $l \neq m, i \neq j, k \neq l$ .

$$A_{(i,m),(i,l),(i,l)}^{(i,l)} = A_{(i,m),(i,l),(i,l)}^{(i,l)} = A_{(i,l),(i,m),(i,l)}^{(i,l)} = A_{(i,k),(i,l),(i,l)}^{(i,l)} = 0.$$

This is true because of equations (1) and (2). If all index pairs coincide, equation (3) or (4) is used to reduce this case to the previously solved case. The *group of automorphisms* of a tensor product structure is therefore, by *Theorem A*, a Lie group.

# 2. G-structures for which the Lie algebra g of G acts irreducibly on V.

Let g be an irreducible Lie algebra of endomorphisms of a real vector space V of dimension n. There are six classes of Lie algebras which are of infinite type (see Y. Matsushima [8]):

$$g = gl(n, R)$$
 Lie algebra of all endomorphisms of  $V$ ; (1)

$$g = \mathfrak{sl}(n, R)$$
 Lie algebra of all endomorphisms of  $V$  of trace zero; (2)

 $g = \mathfrak{sp}(2m, R) n = 2m$ , g is the Lie algebra of endomorphisms of V which leave the following skew symmetric bilinear form of maximal rank, Q(x, y), invariant.

$$Q(x, y) = x_1 y_2 - x_2 y_1 + \ldots + x_{n-1} y_n - x_n y_{n-1};$$
 (3)

$$g = \mathfrak{sp}(2m, R) + Z$$
, where  $Z = \text{center of } \mathfrak{gl}(2m, R)$ ; (4)

 $g = \mathfrak{sl}(m, C) + U \subset \mathfrak{gl}(2m, R)$ , where U is a certain real subspace of the center of  $\mathfrak{gl}(m, C)$ ; (5)

$$g = \mathfrak{sp}(2m, C) + U \subset \mathfrak{gl}(4m, R)$$
, where U is a certain real subspace of the center of  $\mathfrak{gl}(2m, C)$ .

Let M be a compact manifold of dimension  $n \geq 2$ , and let P(M, G) be a G-structure on M. If the Lie algebra  $\mathfrak{g}$  of G is one of the Lie algebras in (5) or (6), it follows immediately from Example 4, Section 8, that the automorphism group of P(M, G) is a Lie group. If the Lie algebra  $\mathfrak{g}$  of G is one of the Lie algebras in (1), (2), (3), and (4), the group of automorphisms of a G-structure P(M, G) is not in general a  $L_{IE}$  group.

Counterexample. All groups corresponding to (1), (2), (3), and (4) contain  $SL(2,R)\times I_{n-2}$  as a Lie subgroup. SL(2,R) denotes the special linear group acting on the space spanned by the first two elements of a basis in V;  $I_{n-2}$  is the identity on the space spanned by the last n-2 elements of the same basis. Let T be the torus obtained from  $R^2$  by identifying the points (x,y) and (x+p,y+q), where p and q are integers. The frame field  $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$  defines a SL(2,R) structure on T. The vector field  $f(y)\frac{\partial}{\partial x}$ , where f(p)=f(q),p,q integers, is an infinitesimal automorphism of this SL(2,R) structure. The vector space of infinitesimal automorphisms is of infinite

dimension. The automorphism group is therefore not a Lie group. (Since T is compact, every infinitesimal automorphism generates a 1-parameter group of automorphisms.)

Additional applications of Theorem B can be obtained by considering non-irreducible Lie algebras.

Let M be a compact differentiable manifold of dimension n = 2m and let  $\mathfrak{g}$  be a commutative Lie algebra of endomorphisms on a 2m-dimensional vector space V such that for any element  $a = (a_a^p) \in \mathfrak{g}$  the following equations hold:

$$a_j^i = -a_{j+m}^{i+m}, \quad a_j^{i+m} = a_{j+m}^i; \quad i,j = 1, 2, \ldots, m.$$

Let G denote the Lie subgroup of GL(n, R) whose Lie algebra is equal to  $\mathfrak{g}$ . Computation of  $\mathfrak{g}^{(1)}$ :

$$a^{i}_{j,k} = -a^{i+m}_{j+m,k} = -a^{i+m}_{k, j+m} = -a^{i}_{k+m, j+m} = -a^{i}_{j+m, k+m},$$
  $a^{i+m}_{j, k} = a^{i}_{j+m, k} = a^{i}_{k, j+m} = -a^{i+m}_{k+m, j+m} = -a^{i+m}_{j+m, k+m}.$ 

For every  $a = (a_p) \in V^*$ ,  $h = (a_p \cdot \delta^{ql}) \in \mathfrak{h}^{(1)}$ . The corollary to Theorem B applies. The automorphism group of a G-structure is a Lie group.

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