

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 38 (1963-1964)

Artikel: A Note on the Intrinsic Join of STIEFEL Manifolds.
Autor: Hussein, S.Y.
DOI: <https://doi.org/10.5169/seals-29435>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A Note on the Intrinsic Join of STIEFEL Manifolds

by S. Y. HUSSEINI

Introduction

In [1] BOTT defines a map,

$$\lambda^E : S(O_{n,k} \wedge O_{m,k}) \rightarrow O_{n+m,k},$$

of the suspension of the reduced join $O_{n,k} \wedge O_{m,k}$ of the real (complex, quaternionic) STIEFEL manifolds of the orthonormal k -frames in real (complex, quaternionic) n - and m -space to $O_{n+m,k}$, the STIEFEL manifold of orthonormal k -frames in real (complex, quaternionic) $(n + m)$ -space. JAMES defines ([2], or [1], p. 256) a map

$$\lambda^J : O_{n,k} * O_{m,k} \rightarrow O_{n+m,k},$$

of the join of $O_{n,k}$ and $O_{m,k}$ to $O_{n+m,k}$. λ^E and λ^J have the same range; and since $S(O_{n,k} \wedge O_{m,k})$ and $O_{n,k} * O_{m,k}$ are of the same homotopy type, λ^E and λ^J have equivalent domains of definition as well. BOTT showed in [1] that λ^E and λ^J are equivalent when $k = 1$ and asked about their relation in general. The aim of this note is to answer his question, by first defining an auxiliary map λ^B (in (1.2)) and showing that it is equivalent to λ^E , and then proving that λ^B is equivalent to λ^J . The precise statement is given in Proposition (4.1) below.

1. Suppose that K is the real, complex or quaternionic field. Let K^{n+m} be the vector space of $(n + m)$ -tuples over K with the usual inner product. Let a_1, \dots, a_{n+m} be the usual basis. We shall find it convenient to write $W(i_1, \dots, i_k)$ for the subspace of K^{n+m} generated by the basic vectors a_{i_1}, \dots, a_{i_k} . Denote by O_{n+m} the group of orthogonal transformations of K^{n+m} . Thus O_{n+m} is the real, unitary or symplectic group when K is the real, complex or quaternionic field. Let $i : O_n \rightarrow O_{n+m}$ be the imbedding induced by the map which takes K^n onto $W(1, \dots, n)$ and $i' : O_m \rightarrow O_{n+m}$, the imbedding induced by the map which takes K^m onto $W(n - k + 1, \dots, n + m - k)$. Define χ_t to be the orthogonal transformation which leaves $W(1, \dots, n - k) + W(n + 1, \dots, n + m - k)$ pointwise fixed and is a rotation through an angle of $\frac{\pi}{2}t$ in each of the planes $W(n - k + j, n + m - k + j)$ for $j = 1, \dots, k$. Thus

$$\chi_t \cdot a_{n-k+j} = \cos \frac{\pi}{2}t a_{n-k+j} + \sin \frac{\pi}{2}t a_{n+m-k+j}$$

$$\chi_t \cdot a_{n+m-k+j} = -\sin \frac{\pi}{2} t a_{n-k+j} + \cos \frac{\pi}{2} t a_{n+m-k+j}$$

for $j = 1, \dots, k$.

Suppose g^o is the frame $(a_{n+m-k+1}, \dots, a_{n+m})$. Then $O_{n+m,k}$, the STIEFEL manifold of k -frames in K^{n+m} , can be identified with O_{n+m}/O_{n+m-k} by making the coset xO_{n+m-k} correspond to the frame xg^o . We wish to define a map of $O_m/O_{m-k} * O_n/O_{n-k}$, the join of O_m/O_{m-k} and O_n/O_{n-k} into $O_{n+m,k}$. So let us recall the necessary notions.

Definition (1.1). Let A and B be two countable CW -complexes, and consider the disjoint union $A \cup A \times B \times I \cup B$, where I is the unit interval $[0, 1]$. Identify $(a, B, 0)$ with a in A and $(A, b, 1)$ with b in B . The resulting space is the *join* $A * B$. Following JAMES we denote the image of (a, b, t) in $A * B$ by the same symbol. If a_0 and b_0 are the basepoints of A and B , then we take $(a_0, b_0, 1)$ to be the basepoint in $A * B$. There is an alternative way of looking at the join. By $A \wedge B$ one understands the space obtained from $A \times B$ by collapsing the sum $A \vee B = A \times b_0 \cup a_0 \times B$ to a point. $A \wedge B$ is called the *reduced join*. The suspension of $A \wedge B$ is obtained from $(A \wedge B) \times I$ by collapsing the subset $(A \wedge B) \times 0 \cup (A \wedge B) \times 1 \cup (a_0 \wedge b_0) \times I$ to a point. The natural map $q: A * B \rightarrow S(A \wedge B)$ is a homotopy equivalence.

Consider the map

$$\lambda: O_m \times O_n \times I \rightarrow O_{n+m,k}$$

such that $\lambda(y, x, t) = \chi_t y' \chi_t^{-1} x \chi_t y'^{-1} \chi_t^{-1} x^{-1} \cdot g^o$, where x stands for ix and y' stands for $i'y$. Notice that $\lambda(y, x, t)$ does not vary when y is changed by an element in O_{m-k} , the subgroup of O_m which leaves $W(n-k+1, \dots, n)$ pointwise fixed. Also $\lambda(y, x, t)$ does not vary when x is changed by an element in O_{n-k} , the subgroup of O_n which leaves $W(n-k+1, \dots, n)$ pointwise fixed. Moreover, $\lambda(y, x, 0)$ is independent of x and $\lambda(y, x, 1)$ is independent of y , since x and $\chi_1 y' \chi_1^{-1}$ commute. Hence λ induces a map

$$\lambda^B: O_m/O_{m-k} * O_n/O_{n-k} \rightarrow O_{n+m,k}. \quad (1.2)$$

Take e_m , the identity in O_m , to be the basepoint, and its image, \tilde{e}_m , to be the basepoint in O_m/O_{m-k} ; hence $(\tilde{e}_m, \tilde{e}_n, 1)$ is the basepoint in $O_m/O_{m-k} * O_n/O_{n-k}$, and λ^B takes $(\tilde{e}_m, \tilde{e}_n, 1)$ to \tilde{e}_{n+m} . We claim that λ^B is equivalent to the BOTT map λ^E of [1]. Our argument is as follows: it is easy to see that λ also induces a map $\tilde{\lambda}^B: S(O_m/O_{m-k} \wedge O_n/O_{n-k}) \rightarrow O_{n+m,k}$, such that $\tilde{\lambda}^B \circ q = \lambda^B$, where $q: O_m/O_{m-k} * O_n/O_{n-k} \rightarrow S(O_m/O_{m-k} \wedge O_n/O_{n-k})$ is the natural equivalence. Consider now the map

$$r: O_n \times O_m \times I \rightarrow O_m \times O_n \times I$$

such that $r(x, y, t) = (y, x, t)$. Then $\lambda \circ r$ induces the map

$$\lambda^E : S(O_n/O_{n-k} \wedge O_m/O_{m-k}) \rightarrow O_{n+m,k},$$

which BOTT introduced in [1] (p. 252). Observe that $\lambda^E = \tilde{\lambda}^B \circ r$ where $\tilde{r} : S(O_n/O_{n-k} \wedge O_m/O_{m-k}) \rightarrow S(O_m/O_{m-k} \wedge O_n/O_{n-k})$ is the homeomorphism induced by r . Hence

$$\lambda^B = \lambda^E \circ \tilde{r}^{-1} \circ q, \text{ and } q \text{ and } \tilde{r}^{-1} \text{ are} \quad (1.3)$$

homotopy equivalences.

2. Consider next the homeomorphism

$$T : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_m/O_{m-k} * O_n/O_{n-k} \quad (2.1)$$

which takes (x, y, t) to $(y, x, 1-t)$. (Observe that T does not preserve basepoints; but, since $(e_m, e_n, 1)$ and $(e_m, e_n, 0)$ are connected by a contractible segment, T can (in most cases) be regarded as preserving basepoints. Hence

$$\lambda^B \circ T : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k}$$

is induced by the map of $O_n \times O_m \times I \rightarrow O_{n+m,k}$, which takes (x, y, t) to

$$\chi_{1-t} y' \chi_{1-t}^{-1} x \chi_{1-t} y'^{-1} \chi_{1-t}^{-1} x^{-1} \cdot g^o = \chi_t^{-1} y'' \chi_t x \chi_t^{-1} y''^{-1} \chi_t x^{-1} \cdot g^o$$

where $y'' = \chi_1 y' \chi_1^{-1}$. Notice that $\lambda^B \circ T$ preserves basepoints although T does not.

Let us consider now the map

$$F : (O_n \times O_m \times I) \times I \rightarrow O_{n+m,k}$$

such that $F((x, y, t), s) = \chi_{(1-s)t}^{-1} y'' \chi_{(1-s)t} x \chi_t^{-1} y''^{-1} \chi_{s+(1-s)t} x^{-1} g^o$. Observe that $F((x, y, t), s)$ does not change when x and y are varied by an element in O_{n-k} and O_{m-k} respectively. Moreover, $F((x, y, 0), s)$ is independent of y since x and y'' commute, and $F((x, y, 1), s)$ is independent of x since $x \chi_1^{-1} y''^{-1} \chi_1 x^{-1} \cdot g^o = g^o$. Thus F induces a homotopy

$$\tilde{F}_s : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k} \quad (2.2)$$

such that $\tilde{F}_0 = \lambda^B \circ T$, and the basepoint is stationary during the homotopy, i.e., $F(\tilde{e}_n, \tilde{e}_m, 1) = g^o$, for $0 \leq s \leq 1$.

Define $\mu = \tilde{F}_1$. Then

$$\mu : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k} \quad (2.3)$$

is induced by the map of $O_n \times O_m \times I \rightarrow O_{n+m,k}$ which takes (x, y, t) to $y'' x \chi_t^{-1} (y''^{-1} \chi_1 g^o) = y'' x \chi_t^{-1} (\chi_1 g^o)$, since

$$\begin{aligned} y''^{-1} \chi_1 &= \chi_1 y'^{-1} \chi_1^{-1} \cdot \chi_1 \\ &= \chi_1 y'^{-1}, \text{ and } y'^{-1} g^o = g^o. \end{aligned} \text{ Summing up we have}$$

Proposition (2.2). The maps $\mu, \lambda^B \circ T : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k}$ are homotopic. Here λ^B is the map defined in (1.2) and T and μ are the maps defined above in (2.1) and (2.3).

3. Let $O_{n,k}$ and $O_{m,k}$ be the STIEFEL manifolds of orthonormal k -frames in $W(1, \dots, n) = K^n$ and $W(n+1, \dots, n+m) = K^m$, respectively. Suppose that f° is the k -frame which consists of the last k basic vectors of the image of K^n , i.e., $f^\circ = (a_{n-k+1}, \dots, a_n)$. Then $O_{n,k}$ can be identified with O_n/O_{n-k} by making the coset xO_{n-k} correspond to the k -frame xf° . Similarly $O_{m,k}$ can be identified with O_m/O_{m-k} by making the coset yO_{m-k} correspond to the k -frame $y''g^\circ$, where $y'' = \chi_1 y' \chi_1^{-1} = \chi_1 i' y \chi_1^{-1}$. We claim

Proposition (3.1.) If $f = (f_1, \dots, f_k) \in O_{n,k}$ and $g = (g_1, \dots, g_k) \in O_{m,k}$, and if $x \in O_n$ and $y \in O_m$ be such that $xf^\circ = f$ and $y''g^\circ = g$, then

$$\mu(\tilde{x}, \tilde{y}, t) = \left(-\cos \frac{\pi}{2} t \right) f + \left(\sin \frac{\pi}{2} t \right) g,$$

where \tilde{x} and \tilde{y} are the images of x and y in O_n/O_{n-k} and O_m/O_{m-k} .

Proof. By the definition of μ in (2.3)

$$\mu(\tilde{x}, \tilde{y}, t) = y'' x \chi_t^{-1} (\chi_1 g^\circ) = -y'' x \chi_t^{-1} f^\circ,$$

since $\chi_1 g^\circ = -f^\circ = (-a_{n-k+1}, \dots, -a_n)$. Notice that $x \cdot g^\circ = g^\circ$ and $y'' \cdot f^\circ = f^\circ$. Hence

$$\mu(\tilde{x}, \tilde{y}, t) = \left(-\cos \frac{\pi}{2} t \right) f + \left(\sin \frac{\pi}{2} t \right) g.$$

4. Suppose that ϱ_0 is the element of O_n which has -1 along the diagonal and zeroes everywhere else. Then ϱ_0 induces a homeomorphism of $O_n/O_{n-k} * O_m/O_{m-k}$ by sending $(\tilde{x}, \tilde{y}, t)$ to $(\varrho_0 \tilde{x}, \tilde{y}, t)$. Denote this homeomorphism (which is related to what JAMES calls in [2] a ‘‘row operation’’) by the same letter ϱ_0 . Proposition (3.1) implies that the JAMES map λ^J (the map h' of [2], p. 513) is $\mu\varrho_0$. Now we can conclude

Proposition (4.1). If $\lambda^B : O_m/O_{m-k} * O_n/O_{n-k} \rightarrow O_{n+m,k}$ is the map of (1.2) and

$$\lambda^J : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k}$$

is the JAMES map ([2], p. 513), then $\lambda^B \circ T$ is homotopic to $\lambda^J \varrho_0$.

Since λ^B and λ^E are equivalent (see (1.3)), Proposition (4.1) implies that λ^E and λ^J are equivalent.

Remark. If $k = 1$, and if d is the dimension of K over the real field, then $O_{n,1}$, $O_{m,1}$ and $O_{n+m,1}$ become the unit spheres S^{dn-1} , S^{dm-1} , and $S^{d(n+m)-1}$

in K^n , K^m , and K^{n+m} , respectively. Moreover, λ^J reduces to the usual join of sphere

$$h' : S^p * S^q \rightarrow S^{p+q+1},$$

where h' takes (x, y, t) to $\left(x \cos \frac{\pi}{2} t, y \sin \frac{\pi}{2} t\right)$ ([2], p. 512). If h' is used to orient $S^{dm-1} * S^{dn-1}$ and $S^{dn-1} * S^{dm-1}$, then T becomes a map of degree $(-1)^{(dn)(dm)}$ ([2], p. 512) and ϱ_0 is a map of degree $(-1)^{(dn)}$. Hence when K is the complex or quaternionic field, λ^B and λ^J have the same degree. In the real case they also agree if $n(m+1)$ is even; otherwise they agree up to a sign.

The University of Wisconsin
Madison, Wis.

BIBLIOGRAPHY

- [1] R. BOTT, *A note on the SAMUELSON Product in the classical groups*, Comment. Math. Helv. 34 (1960), 249–256.
 [2] I. JAMES, *The intrinsic join: A study of the homotopy groups of STIEFEL manifolds*, Proc. London Math. Soc. 8 (1958), 507–535.

(Received January 8, 1963)