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**Autor:** Hussein, S.Y.  
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# A Note on the Intrinsic Join of STIEFEL Manifolds

by S. Y. HUSSEINI

## Introduction

In [1] BOTT defines a map,

$$\lambda^E : S(O_{n,k} \wedge O_{m,k}) \rightarrow O_{n+m,k},$$

of the suspension of the reduced join  $O_{n,k} \wedge O_{m,k}$  of the real (complex, quaternionic) STIEFEL manifolds of the orthonormal  $k$ -frames in real (complex, quaternionic)  $n$ - and  $m$ -space to  $O_{n+m,k}$ , the STIEFEL manifold of orthonormal  $k$ -frames in real (complex, quaternionic)  $(n + m)$ -space. JAMES defines ([2], or [1], p. 256) a map

$$\lambda^J : O_{n,k} * O_{m,k} \rightarrow O_{n+m,k},$$

of the join of  $O_{n,k}$  and  $O_{m,k}$  to  $O_{n+m,k}$ .  $\lambda^E$  and  $\lambda^J$  have the same range; and since  $S(O_{n,k} \wedge O_{m,k})$  and  $O_{n,k} * O_{m,k}$  are of the same homotopy type,  $\lambda^E$  and  $\lambda^J$  have equivalent domains of definition as well. BOTT showed in [1] that  $\lambda^E$  and  $\lambda^J$  are equivalent when  $k = 1$  and asked about their relation in general. The aim of this note is to answer his question, by first defining an auxiliary map  $\lambda^B$  (in (1.2)) and showing that it is equivalent to  $\lambda^E$ , and then proving that  $\lambda^B$  is equivalent to  $\lambda^J$ . The precise statement is given in Proposition (4.1) below.

1. Suppose that  $K$  is the real, complex or quaternionic field. Let  $K^{n+m}$  be the vector space of  $(n + m)$ -tuples over  $K$  with the usual inner product. Let  $a_1, \dots, a_{n+m}$  be the usual basis. We shall find it convenient to write  $W(i_1, \dots, i_k)$  for the subspace of  $K^{n+m}$  generated by the basic vectors  $a_{i_1}, \dots, a_{i_k}$ . Denote by  $O_{n+m}$  the group of orthogonal transformations of  $K^{n+m}$ . Thus  $O_{n+m}$  is the real, unitary or symplectic group when  $K$  is the real, complex or quaternionic field. Let  $i : O_n \rightarrow O_{n+m}$  be the imbedding induced by the map which takes  $K^n$  onto  $W(1, \dots, n)$  and  $i' : O_m \rightarrow O_{n+m}$ , the imbedding induced by the map which takes  $K^m$  onto  $W(n - k + 1, \dots, n + m - k)$ . Define  $\chi_t$  to be the orthogonal transformation which leaves  $W(1, \dots, n - k) + W(n + 1, \dots, n + m - k)$  pointwise fixed and is a rotation through an angle of  $\frac{\pi}{2}t$  in each of the planes  $W(n - k + j, n + m - k + j)$  for  $j = 1, \dots, k$ . Thus

$$\chi_t \cdot a_{n-k+j} = \cos \frac{\pi}{2}t \, a_{n-k+j} + \sin \frac{\pi}{2}t \, a_{n+m-k+j}$$

$$\chi_t \cdot a_{n+m-k+j} = -\sin \frac{\pi}{2} t a_{n-k+j} + \cos \frac{\pi}{2} t a_{n+m-k+j}$$

for  $j = 1, \dots, k$ .

Suppose  $g^o$  is the frame  $(a_{n+m-k+1}, \dots, a_{n+m})$ . Then  $O_{n+m,k}$ , the STIEFEL manifold of  $k$ -frames in  $K^{n+m}$ , can be identified with  $O_{n+m}/O_{n+m-k}$  by making the coset  $xO_{n+m-k}$  correspond to the frame  $xg^o$ . We wish to define a map of  $O_m/O_{m-k} * O_n/O_{n-k}$ , the join of  $O_m/O_{m-k}$  and  $O_n/O_{n-k}$  into  $O_{n+m,k}$ . So let us recall the necessary notions.

**Definition (1.1).** Let  $A$  and  $B$  be two countable  $CW$ -complexes, and consider the disjoint union  $A \cup A \times B \times I \cup B$ , where  $I$  is the unit interval  $[0, 1]$ . Identify  $(a, B, 0)$  with  $a$  in  $A$  and  $(A, b, 1)$  with  $b$  in  $B$ . The resulting space is the *join*  $A * B$ . Following JAMES we denote the image of  $(a, b, t)$  in  $A * B$  by the same symbol. If  $a_0$  and  $b_0$  are the basepoints of  $A$  and  $B$ , then we take  $(a_0, b_0, 1)$  to be the basepoint in  $A * B$ . There is an alternative way of looking at the join. By  $A \wedge B$  one understands the space obtained from  $A \times B$  by collapsing the sum  $A \vee B = A \times b_0 \cup a_0 \times B$  to a point.  $A \wedge B$  is called the *reduced join*. The suspension of  $A \wedge B$  is obtained from  $(A \wedge B) \times I$  by collapsing the subset  $(A \wedge B) \times 0 \cup (A \wedge B) \times 1 \cup (a_0 \wedge b_0) \times I$  to a point. The natural map  $q: A * B \rightarrow S(A \wedge B)$  is a homotopy equivalence.

Consider the map

$$\lambda: O_m \times O_n \times I \rightarrow O_{n+m,k}$$

such that  $\lambda(y, x, t) = \chi_t y' \chi_t^{-1} x \chi_t y'^{-1} \chi_t^{-1} x^{-1} \cdot g^o$ , where  $x$  stands for  $i x$  and  $y'$  stands for  $i' y$ . Notice that  $\lambda(y, x, t)$  does not vary when  $y$  is changed by an element in  $O_{m-k}$ , the subgroup of  $O_m$  which leaves  $W(n-k+1, \dots, n)$  pointwise fixed. Also  $\lambda(y, x, t)$  does not vary when  $x$  is changed by an element in  $O_{n-k}$ , the subgroup of  $O_n$  which leaves  $W(n-k+1, \dots, n)$  pointwise fixed. Moreover,  $\lambda(y, x, 0)$  is independent of  $x$  and  $\lambda(y, x, 1)$  is independent of  $y$ , since  $x$  and  $\chi_1 y' \chi_1^{-1}$  commute. Hence  $\lambda$  induces a map

$$\lambda^B: O_m/O_{m-k} * O_n/O_{n-k} \rightarrow O_{n+m,k}. \quad (1.2)$$

Take  $e_m$ , the identity in  $O_m$ , to be the basepoint, and its image,  $\tilde{e}_m$ , to be the basepoint in  $O_m/O_{m-k}$ ; hence  $(\tilde{e}_m, \tilde{e}_n, 1)$  is the basepoint in  $O_m/O_{m-k} * O_n/O_{n-k}$ , and  $\lambda^B$  takes  $(\tilde{e}_m, \tilde{e}_n, 1)$  to  $\tilde{e}_{n+m}$ . We claim that  $\lambda^B$  is equivalent to the BOTT map  $\lambda^E$  of [1]. Our argument is as follows: it is easy to see that  $\lambda$  also induces a map  $\tilde{\lambda}^B: S(O_m/O_{m-k} \wedge O_n/O_{n-k}) \rightarrow O_{n+m,k}$ , such that  $\tilde{\lambda}^B \circ q = \lambda^B$ , where  $q: O_m/O_{m-k} * O_n/O_{n-k} \rightarrow S(O_m/O_{m-k} \wedge O_n/O_{n-k})$  is the natural equivalence. Consider now the map

$$r: O_n \times O_m \times I \rightarrow O_m \times O_n \times I$$

such that  $r(x, y, t) = (y, x, t)$ . Then  $\lambda \circ r$  induces the map

$$\lambda^E : S(O_n/O_{n-k} \wedge O_m/O_{m-k}) \rightarrow O_{n+m,k},$$

which BOTT introduced in [1] (p. 252). Observe that  $\lambda^E = \tilde{\lambda}^B \circ r$  where  $\tilde{r} : S(O_n/O_{n-k} \wedge O_m/O_{m-k}) \rightarrow S(O_m/O_{m-k} \wedge O_n/O_{n-k})$  is the homeomorphism induced by  $r$ . Hence

$$\lambda^B = \lambda^E \circ \tilde{r}^{-1} \circ q, \text{ and } q \text{ and } \tilde{r}^{-1} \text{ are} \quad (1.3)$$

homotopy equivalences.

2. Consider next the homeomorphism

$$T : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_m/O_{m-k} * O_n/O_{n-k} \quad (2.1)$$

which takes  $(x, y, t)$  to  $(y, x, 1-t)$ . (Observe that  $T$  does not preserve basepoints; but, since  $(e_m, e_n, 1)$  and  $(e_m, e_n, 0)$  are connected by a contractible segment,  $T$  can (in most cases) be regarded as preserving basepoints. Hence

$$\lambda^B \circ T : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k}$$

is induced by the map of  $O_n \times O_m \times I \rightarrow O_{n+m,k}$ , which takes  $(x, y, t)$  to

$$\chi_{1-t} y' \chi_{1-t}^{-1} x \chi_{1-t} y'^{-1} \chi_{1-t}^{-1} x^{-1} \cdot g^o = \chi_t^{-1} y'' \chi_t x \chi_t^{-1} y''^{-1} \chi_t x^{-1} \cdot g^o$$

where  $y'' = \chi_1 y' \chi_1^{-1}$ . Notice that  $\lambda^B \circ T$  preserves basepoints although  $T$  does not.

Let us consider now the map

$$F : (O_n \times O_m \times I) \times I \rightarrow O_{n+m,k}$$

such that  $F((x, y, t), s) = \chi_{(1-s)t}^{-1} y'' \chi_{(1-s)t} x \chi_t^{-1} y''^{-1} \chi_{s+(1-s)t}^{-1} x^{-1} g^o$ . Observe that  $F((x, y, t), s)$  does not change when  $x$  and  $y$  are varied by an element in  $O_{n-k}$  and  $O_{m-k}$  respectively. Moreover,  $F((x, y, 0), s)$  is independent of  $y$  since  $x$  and  $y''$  commute, and  $F((x, y, 1), s)$  is independent of  $x$  since  $x \chi_1^{-1} y''^{-1} \chi_1 x^{-1} \cdot g^o = g^o$ . Thus  $F$  induces a homotopy

$$\tilde{F}_s : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k} \quad (2.2)$$

such that  $\tilde{F}_0 = \lambda^B \circ T$ , and the basepoint is stationary during the homotopy, i.e.,  $F(\tilde{e}_n, \tilde{e}_m, 1) = g^o$ , for  $0 \leq s \leq 1$ .

Define  $\mu = \tilde{F}_1$ . Then

$$\mu : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k} \quad (2.3)$$

is induced by the map of  $O_n \times O_m \times I \rightarrow O_{n+m,k}$  which takes  $(x, y, t)$  to  $y'' x \chi_t^{-1} (y''^{-1} \chi_1 g^o) = y'' x \chi_t^{-1} (\chi_1 g^o)$ , since

$$\begin{aligned} y''^{-1} \chi_1 &= \chi_1 y'^{-1} \chi_1^{-1} \cdot \chi_1 \\ &= \chi_1 y'^{-1}, \text{ and } y'^{-1} g^o = g^o. \end{aligned} \text{ Summing up we have}$$

**Proposition (2.2).** The maps  $\mu, \lambda^B \circ T : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k}$  are homotopic. Here  $\lambda^B$  is the map defined in (1.2) and  $T$  and  $\mu$  are the maps defined above in (2.1) and (2.3).

**3.** Let  $O_{n,k}$  and  $O_{m,k}$  be the STIEFEL manifolds of orthonormal  $k$ -frames in  $W(1, \dots, n) = K^n$  and  $W(n+1, \dots, n+m) = K^m$ , respectively. Suppose that  $f^o$  is the  $k$ -frame which consists of the last  $k$  basic vectors of the image of  $K^n$ , i.e.,  $f^o = (a_{n-k+1}, \dots, a_n)$ . Then  $O_{n,k}$  can be identified with  $O_n/O_{n-k}$  by making the coset  $xO_{n-k}$  correspond to the  $k$ -frame  $xf^o$ . Similarly  $O_{m,k}$  can be identified with  $O_m/O_{m-k}$  by making the coset  $yO_{m-k}$  correspond to the  $k$ -frame  $y''g^o$ , where  $y'' = \chi_1 y' \chi_1^{-1} = \chi_1 i' y \chi_1^{-1}$ . We claim

**Proposition (3.1.)** If  $f = (f_1, \dots, f_k) \in O_{n,k}$  and  $g = (g_1, \dots, g_k) \in O_{m,k}$ , and if  $x \in O_n$  and  $y \in O_m$  be such that  $xf^o = f$  and  $y''g^o = g$ , then

$$\mu(\tilde{x}, \tilde{y}, t) = \left( -\cos \frac{\pi}{2} t \right) f + \left( \sin \frac{\pi}{2} t \right) g,$$

where  $\tilde{x}$  and  $\tilde{y}$  are the images of  $x$  and  $y$  in  $O_n/O_{n-k}$  and  $O_m/O_{m-k}$ .

*Proof.* By the definition of  $\mu$  in (2.3)

$$\mu(\tilde{x}, \tilde{y}, t) = y'' x \chi_t^{-1} (\chi_1 g^o) = -y'' x \chi_t^{-1} f^o,$$

since  $\chi_1 g^o = -f^o = (-a_{n-k+1}, \dots, -a_n)$ . Notice that  $x \cdot g^o = g^o$  and  $y'' \cdot f^o = f^o$ . Hence

$$\mu(\tilde{x}, \tilde{y}, t) = \left( -\cos \frac{\pi}{2} t \right) f + \left( \sin \frac{\pi}{2} t \right) g.$$

**4.** Suppose that  $\varrho_0$  is the element of  $O_n$  which has  $-1$  along the diagonal and zeroes everywhere else. Then  $\varrho_0$  induces a homeomorphism of  $O_n/O_{n-k} * O_m/O_{m-k}$  by sending  $(\tilde{x}, \tilde{y}, t)$  to  $(\varrho_0 \tilde{x}, \tilde{y}, t)$ . Denote this homeomorphism (which is related to what JAMES calls in [2] a ‘‘row operation’’) by the same letter  $\varrho_0$ . Proposition (3.1) implies that the JAMES map  $\lambda^J$  (the map  $h'$  of [2], p. 513) is  $\mu\varrho_0$ . Now we can conclude

**Proposition (4.1).** If  $\lambda^B : O_m/O_{m-k} * O_n/O_{n-k} \rightarrow O_{n+m,k}$  is the map of (1.2) and

$$\lambda^J : O_n/O_{n-k} * O_m/O_{m-k} \rightarrow O_{n+m,k}$$

is the JAMES map ([2], p. 513), then  $\lambda^B \circ T$  is homotopic to  $\lambda^J \varrho_0$ .

Since  $\lambda^B$  and  $\lambda^E$  are equivalent (see (1.3)), Proposition (4.1) implies that  $\lambda^E$  and  $\lambda^J$  are equivalent.

*Remark.* If  $k = 1$ , and if  $d$  is the dimension of  $K$  over the real field, then  $O_{n,1}$ ,  $O_{m,1}$  and  $O_{n+m,1}$  become the unit spheres  $S^{dn-1}$ ,  $S^{dm-1}$ , and  $S^{d(n+m)-1}$

in  $K^n$ ,  $K^m$ , and  $K^{n+m}$ , respectively. Moreover,  $\lambda^J$  reduces to the usual join of sphere

$$h' : S^p * S^q \rightarrow S^{p+q+1},$$

where  $h'$  takes  $(x, y, t)$  to  $\left(x \cos \frac{\pi}{2} t, y \sin \frac{\pi}{2} t\right)$  ([2], p. 512). If  $h'$  is used to orient  $S^{dm-1} * S^{dn-1}$  and  $S^{dn-1} * S^{dm-1}$ , then  $T$  becomes a map of degree  $(-1)^{(dn)(dm)}$  ([2], p. 512) and  $\varrho_0$  is a map of degree  $(-1)^{(dn)}$ . Hence when  $K$  is the complex or quaternionic field,  $\lambda^B$  and  $\lambda^J$  have the same degree. In the real case they also agree if  $n(m+1)$  is even; otherwise they agree up to a sign.

The University of Wisconsin  
Madison, Wis.

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