Fixed Point Sets and Equivalence of Differentiable Transformation Groups

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MONTGOMERY and SAMELSON have shown that given a compact LIE group G there exists an integer n such that there are a countable number of differentiable actions of G on S^n, the n-sphere with mutually nonhomeomorphic fixed point sets. The problem then arises as to what can be said if the fixed point set is specified in advance. In particular in the case of S^n we might wish to determine the actions with fixed point set diffeomorphic to a sphere S^k.

In the first three sections we will be interested in the more general aspects of the relations between the fixed point set and the differentiable action. In § 1 we consider the isotropy representations at points in the fixed point sets and the normal bundle of the fixed point set. In § 2 we restrict our attention to the circle group operating differentiably with two types of orbits. In particular, we show that in order that the action have the simplest possible form near the fixed point set F it is necessary that F be in the zero cobordism class.

In § 3 we apply recent results of SMALE in differential topology to study free actions of G on $M \times D^k$, D^k a disk.

Finally in § 4 we give a recipe for obtaining all possible differentiable actions of the circle group on S^m with two types of orbits and fixed point sets diffeomorphic to S^q, with suitably severe restrictions on q and m (see Theorem 4.2). We mention here that the recipe might possibly give the same action (up to equivalence) several times. One can proceed further then we have here and actually reduce the classification of such actions to a problem in the extension of diffeomorphisms. Since at this point nothing precisely calculable comes out we did not undertake this here. It does, however, seem likely to the author that there are probably at most a finite number of equivalence classes of these actions for suitable dimensions.

§ 1. By a manifold we shall mean a differentiable manifold of class C^∞ with or without boundary. A manifold is said to be closed if it is compact and without boundary. If G is a compact LIE group, M a manifold, then by a group action of G on M we will mean a differentiable function (of class C^∞) $\varphi : G \times M \to M$ satisfying the usual composition rules

$$\varphi(g_1 g_2 ; x) = \varphi(g_1 ; \varphi(g_2 ; x)) \tag{1.1}$$
$$\varphi(e ; x) = x$$

[1]) The author is supported by the National Science Foundation under contract No. G-22126.

e the identity of G. We will also write $\varphi_g(x) = \varphi(g; x)$. We denote by M^* the orbit space M/G. Recall that if the action is free (i.e. $g \neq e$ then $\varphi_g(x) \neq x$) then M^* is a manifold and the natural map $p: M \to M^*$ is the projection of a differentiable, principal G-bundle. In the general case we can, by averaging over G a RIEMANNian metric on M, obtain a RIEMANNien metric Φ such that each φ_g is an isometry. We shall suppose that Φ is given and fixed throughout and shall then use freely the terms and notations of normed vector spaces, e.g. length, orthogonality etc.

The set of x in M for which $\varphi(g; x) = x$ for all g will be denoted $F(\varphi)$, the fixed point set. If M_x is the tangent space to M at $x \in F(\varphi)$ we have a linear, orthogonal representation α_x of G in M_x via the differential i.e.

$$\alpha_x(g) = (d\varphi_g)_x ,$$

called the isotropy representation of φ at x. If $(u_1, \ldots, u_m)$ are normal coordinates at x and U a sufficiently small disk in these coordinates we see that U is invariant under φ and $\varphi | U$ is equivalent to α_x restricted to a disk in M_x, (each φ_g sends geodesics to geodesics). In particular, $F(\varphi)$ is a submanifold of M.

Lemma 1. 1. *If $F(\varphi)$ is connected and $x_1, x_2 \in F(\varphi)$, then the isotropy representations of φ at x_1 and x_2 are equivalent representations of G in the orthogonal group $O(m)$, $m = \dim M$.*

Proof. Let $S(\alpha)$ be the set of points of $F(\varphi)$ at which the isotropy representation is equivalent to a fixed representation $\alpha: G \to O(m)$. Since at each $x \in F(\varphi)$ the action φ is locally linear $S(\alpha)$ is clearly an open and closed set, and hence the lemma.

Suppose now that G is connected and let T be a maximal torus of G. Let ψ be the restriction of φ to $T \times M$.

Proposition 1. 1. *If $F(\psi)$ is connected and $x_1, x_2 \in F(\varphi)$ then the isotropy representation of φ at x_1 and x_2 are equivalent as representations of G in $Gl(m, C)$.*

Indeed, $(d\psi)_{x_1}$ and $(d\psi)_{x_2}$ are equivalent by lemma 1.1. But these are precisely the restrictions of α_{x_1} and α_{x_2} to T. Since every complex representation of G is determined by its weights, [5], which are linear functionals on the universal covering space of T, a complex representation of G is determined by its restriction to T and the proposition follows.

Corollary. If φ is a differentiable action of a compact, connected LIE *group on a contractible manifold, the dimension of every component of $F(\varphi)$ is the same.*

For such manifolds we know that $F(\psi)$ is connected by SMITH theory ([6]) and the dimension of $F(\varphi)$ is then the number of times the trivial representation occurs in an isotropy representation (considered as a complex representation of G).

Remark. One would hope, of course, that on such manifolds $F(\varphi)$ is connected. This would seem to be related to the question of whether fixed points exist at all on contractible manifolds. For example, in the continuous case, if there exists a continuous action φ of G on euclidean space R^n without fixed points then there is a continuous action φ_1 on R^{n+1} with $F(\varphi_1)$ the disjoint union of two half rays. One simply adjoins ∞ to R^n and extends φ to an action of G on S^n. Taking the suspension of this action to S^{n+1} and deleting ∞ we obtain φ_1.

We wish to determine now what the action φ looks like near $F(\varphi)$. We assume that there exists a fixed orthogonal representation $\alpha: G \to O(k)$, $m = n + k$, such that for each $x \in F(\varphi)$ coordinates exist $(u_1, \ldots, u_m)$ so that

(1) $(u_1, \ldots, u_n, 0, \ldots, 0)$ forms a coordinate system at x in $F(\varphi)$

(2) $\dfrac{\delta}{\delta u_{n+1}}, \ldots, \dfrac{\delta}{\delta u_m}$ forms an orthonormal base of the subspace of M_z orthogonal to the tangent space of $F(\varphi)$ at z, for z sufficiently near $x, z \in F(\varphi)$.

(3) α_z restricted to the space spanned by $\dfrac{\delta}{\delta u_{n+1}}, \ldots, \dfrac{\delta}{\delta u_m}$ is the representation α when expressed in this base.

It is not difficult to see that this will be the case if $F(\varphi)$ is connected. We will also assume that M is compact.

Let N be the total space of the normal vector bundle of $F(\varphi)$ in M. N is the set of all pairs (x, y), $x \in F(\varphi)$, $y \in M_x$ orthogonal to the tangent space of $F(\varphi)$ at x. N_ε will denote the total space of the associated disk bundle characterized by $||y|| \leq \varepsilon$. Let ψ be the action of G on N_ε defined by

$$\psi_g(x, y) = (x, \alpha_x(g) \cdot y) .$$

Lemma 1. 2. *There exists a tubular neighborhood V of $F(\varphi)$ diffeomorphic to N_ε, invariant under φ and $\varphi | G \times V$ is equivalent to the action ψ.*

Proof. For each pair $(x, y) \in N$ we have a unique geodesic $\gamma(t)$ in M such that $\gamma(0) = x$, $\gamma'(0) = y$. Recall then that the map $\operatorname{Exp}: N_\varepsilon \to M$ defined by $\operatorname{Exp}(x, y) = \gamma(1)$ is a C^∞ map and that for sufficiently small ε it is a diffeomorphism of N_ε onto a tubular neighborhood V of $F(\varphi)$ [3]. Again since φ_g is an isometry the map Exp is equivariant and hence the lemma.

Theorem 1. 1. *The structural group of the normal bundle of $F(\varphi)$ in M is reducible to the centralizer P of $\alpha(G)$ in $O(k)$.*

Proof. Let $F(\varphi)$ be covered by coordinate neighborhoods U_i such that for $x \in W_i = U_i \cap F(\varphi)$, U_i satisfies the hypotheses (1), (2), (3) above. For each $z \in W_i$, the last k coordinates then determine a basis $A^i(z)$ of the fibre over z of the normal bundle of $F(\varphi)$. From (3) we see that for $z \in W_i \cap W_j, \alpha_z(g)$

restricted to the subspace of M_z normal to $F(\varphi)_z$, is $\alpha(g)$ when expressed in either of the bases $A^i(z)$ or $A^j(z)$. It follows that if $f_{ji}: W_j \cap W_i \to O(k)$ are the transition functions which assign to $z \in W_i \cap W_j$ the transformation sending $A^i(z)$ to $A^j(z)$ we have

$$f_{ji}(z) \cdot \alpha(g) = \alpha(g) \cdot f_{ji}(z), \; g \in G \tag{1.2}$$

These transition function values therefore lie in P and hence the theorem.

We remark that in case α is complex irreducible it follows from SCHUR's lemma that the normal bundle is then the WHITNEY sum of one or two dimensional bundles.

Let D^k denote the unit disk of euclidean k-space. If α is as above then we define an action, also denoted α, on $F \times D^k$ by $\alpha(g; (x, y)) = (x, \alpha(g)(y))$.

Definition. We say an action φ is totally linear at $F(\varphi)$ if there exists a tubular neighborhood V of $F(\varphi)$ invariant under φ and $\varphi | V$ is equivalent to α.

A totally linear action φ at $F(\varphi)$ has the simplest possible form near its fixed point set. Now by reasoning completely analogous to the proof of theorem 1.1 we can obtain

Theorem 1.2. *φ is totally linear at $F(\varphi)$ if and only if the P-bundle defined in Theorem 1.1 is the trivial P-bundle.*

§ **2.** In this section we shall assume that G is the circle group S^1, which we represent as the reals modulo 1, acting on a closed orientable manifold M. It is then easy to see that $F(\varphi)$ is an orientable manifold. On the action φ we make the rather severe restriction

(A) If $x \notin F(\varphi)$ and $\varphi_t(x) = x$ then $t \equiv 0 \pmod 1$.

If $x \in F(\varphi)$ and N_x is the subspace of M_x orthogonal to $F(\varphi)_x$ we see that $k = \dim N_x$ is even, $k = 2S$. Let R^{2S} be given a decomposition as the direct sum of S orthogonal planes $R^{2S} = L_1 + \cdots + L_S$. We define an orthogonal representation α of G in R^{2S} by

$$\alpha(t) | L_i = \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix}.$$

It follows easily from the hypothesis (A) that α_x is equivalent to the representation α for each $x \in F(\varphi)$. The centralizer of $\alpha(G)$ in $O(2S)$ is clearly just the unitary group $U(S)$. Thus the action φ on M assigns a complex vector bundle over $F(\varphi)$.

Example. Let $P^n(C)$ denote the complex projective space of complex dimension n, $(z_0, \ldots, z_n)$ homogeneous coordinates in $P^n(C)$. Define the action $\varphi_n^{(k)}$ of G on $P^n(C)$ by

$$\varphi_n^{(k)}(t; (z_0, \ldots, z_n)) = (z_0, \ldots, z_k, e^{2\pi i t} \cdot z_{k+1}, \ldots, e^{2\pi i t} \cdot z_n) \tag{2.1}$$

$\varphi_n^{(k)}$ satisfies (A) and $F(\varphi_n^{(k)})$ is just $P^k(C)$. Note that for every k the complex vector bundle over $P^k(C)$ induced by $\varphi_n^{(k)}$ is non-trivial (its first CHERN class is non-zero), and consequently the action is not totally linear at $F(\varphi_n^{(k)})$. We shall show that in half the cases this last fact is due to the character of the fixed point set rather than the ambient space.

Theorem 2. 1. *If φ is totally linear at $F(\varphi)$ then $F(\varphi)$ bounds a compact, orientable manifold. Conversely if F is a manifold in the zero cobordism class there exists a closed orientable manifold M and an action φ of G on M satisfying (A) with $F(\varphi)$ diffeomorphic to F and φ is totally linear at $F(\varphi)$.*

Proof. Suppose that φ is totally linear at $F(\varphi)$ and let V be chosen as in the definition of total linearity. The boundary ∂V of V is then clearly $B = F(\varphi) \times S^{2S-1}$. Denoting by W the complement of the interior of V in M we see that W is an orientable, compact manifold with boundary B and φ is a free action af G on W. Thus W^* is a compact manifold with $\delta W^* = = B^* = F(\varphi) \times P^{S-1}(C)$. The remainder of the proof is divided into two cases. (a) If S is odd it follows that $F(\varphi) \times P^{S-1}(C)$ bounds a compact, orientable manifold W^*. Now according to WALL ([8]) it follows that both the STIEFEL-WHITNEY and the PONTRJAGIN numbers of $F(\varphi) \times P^{S-1}(C)$ vanish. Since $P^{S-1}(C)$ for $S-1$ even has both non-zero STIEFEL-WHITNEY numbers and non-zero PONTRJAGIN numbers [2] it follows that these numbers vanish for $F(\varphi)$ which, again according to Wall, shows that $F(\varphi)$ bounds. (b) We consider now the case S even. Let $r > 2 \cdot \dim W^* + 1$. Let $f: B^* \to P^r(C)$ be a characteristic map for the bundle $B \to B^*$. It is clear that f can be chosen transverse regular on

$$P^{r-1}(C) \text{ (see [2]) and } f^{-1}(P^{r-1}(C)) = F(\varphi) \times P^{S-2}(C).$$

Since $B \to B^*$ is a sub-bundle of $W \to W^*$ and $P^r(C)$ is an $(r-1)$ universal base space we see that f can be extended to a map $g: W^* \to P^r(C)$ which is characteristic for $W \to W^*$. Further ([2, page 101]) g can be supposed transverse regular on $P^{r-1}(C)$. Then $g^{-1}(P^{r-1}(C))$ is a compact, orientable manifold with boundary $F(\varphi) \times P^{S-2}(C)$, We proceed then just as in (a) to show that $F(\varphi)$ determines the zero cobordism class.

For the converse statement we suppose Q is a compact, orientable manifold with $\delta Q = F$. Then we have a free action of G on $Q \times S^{2S-1}$ simply by taking the action α on the second factor. Taking the union of $Q \times S^{2S-1}$ with $F \times D^{2S}$ and identifying boundaries we obtain M and the asserted action φ.

§ **3.** In this section we apply the results of SMALE in differential topology to the study of transformation groups. Again M will denote a closed manifold. I will be the closed interval $[0, 1]$. If φ is an action on $M \times I$ we shall denote by φ_0 the action $\varphi | G \times (M \times \{0\})$.

Lemma 3. 1. *If φ is a free action of a compact, connected group G on $W = M \times I$ $\dim W - \dim G \geq 6$ and $\Pi_1(M) = 0$ then φ is differentiably equivalent to the action $\psi(g; (m, t)) = (\varphi_0(g; m), t)$.*

Proof. Set $M_0 = M \times \{0\}$. First we see that M_0^* is a deformation retract of W^*. For $M_0^* \subset W^*$ and for their homotopy sequences we have the commutative diagram:

$$\begin{array}{ccccccc}
\dashrightarrow \Pi_i(W) & \to & \Pi_i(W^*) & \to & \Pi_{i-1}(G) & \dashrightarrow \\
\uparrow & & \uparrow & & \uparrow & \\
\dashrightarrow \Pi_i(M_0) & \to & \Pi_i(M_0^*) & \to & \Pi_{i-1}(G) & \dashrightarrow
\end{array}$$

the vertical maps being induced by inclusions. Since $\Pi_i(M_0) \to \Pi_i(W)$ is bijective we see by the five lemma that $\Pi_i(M_0^*) \to \Pi_i(W^*)$ is bijective. It follows (for example, by obstruction theory) that M_0^* is a deformation retract of W^*. Further, since G is connected M_0^* is simply connected. Then ([7]) there exists a C^∞ real valued function f on W^* without critical points such that

$$f(M_0^*) = 0, \; f((M \times \{1\})^*) = 1. \text{ If } \pi: W \to W^*$$

is the natural map, we set $h = f \cdot \pi$. h therefore has no critical points and further h is invariant under the action φ. Let Φ be an invariant. RIEMANNian metric on W and let $q(t; m)$ $m \in M$ be the integral curve of gradient h such that $q(0; m) = m$. In the usual way (see [7]) q is a diffeomorphism of W onto itself. Let φ' be the action $q \cdot \varphi \cdot q^{-1}$. φ' is then an action of the type described in [4]. It follows that φ' is differentiably equivalent to ψ.

Now suppose φ is a free action of a connected, compact group G on $W = M^n \times D^k$, $k > 2$, $\dim W - \dim G > 6$, $\Pi_1(M^n) = 0$. Further assume that $M_0 = M \times \{0\}$ is invariant under φ M_0^* has trivial normal bundle in W^*.

Lemma 3. 2. *Under the above conditions the action φ is differentiably equivalent to ψ, $\psi_g(x, y) = (\varphi_g(x), y)$, $(x, y) \in M^n \times D^k$.*

Proof. Let N^* be a tubular neighborhood of M^* in W^* diffeomorphic to $M^* \times D^k$ and let V^* be the complement in W^* of the interior of N^*. We have δN^* diffeomorphic to $M_0^* \times S^{k-1}$, which is simply connected since $k > 2$. We establish just as in the previous lemma that δN^* is a deformation retract of V^*. Proceeding then just as before we conclude that V^* is diffeomorphic to $\delta N^* \times I$. Since the integral curves which produce this diffeomorphism could be chosen to be differentiable continuations of the "radial lines" in N^* we see easily that W^* is diffeomorphic to $M_0^* \times D^k$.

Now let $E_G \to B_G$ be a universal G-bundle, $f: W^* \to B_G$ a characteristic map inducing $W \to W^*$, and $h = f \mid M_0^*$. On $W^* = M_0^* \times D^k$ define h' by $h'(x, y) = h(x)$. Then f is homotopic to h' by the homotopy

$$H(x, y, s) = f(x, sy) .$$

But clearly h' induces the bundle $\pi: M_0 \times D^k \to M_0^* \times D^k$ $\pi(x, y) = (\pi(x), y)$ and the lemma follows easily.

§ 4. We consider now the case of the circle group G acting on the sphere S^m with fixed point set diffeomorphic to S^q and satisfying (A). In this case we see that m has the form $q + 2S$. We will assume throughout that $m > 6$. We first consider the case $q = 0$.

Theorem 4. 1. *If φ is an action of G on the cell D^m satisfying (A) and with exactly one fixed point then φ is differentiably equivalent to the linear action with these properties.*

Proof. We may suppose the fixed point is the origin O. Let U be a neighborhood of O diffeomorphic to a cell on which the action φ is equivalent to the linear action of the isotropy representation. Then $D^m - U$ is diffeomorphic to $S^{m-1} \times I$. By lemma 3.1 the action φ restricted to $D^m - U$ is equivalent to the action $(\varphi \mid \delta U) \times I$. By an argument similar to the one used in the proof of lemma 3.2 we see the equivalence can be extended over U and obtain the theorem.

Corollary. If φ acts on S^m with exactly two fixed points then φ is topologically equivalent to a linear action.

If $x_1 \in F(\varphi)$ then we let U_1 be a cell about x_1 on which φ is linear. $\varphi | S^m - U_1$ is an action as in the theorem and hence equivalent to a linear action. The equivalence can be extended to a topological equivalence over U_1 simply by regarding U_1 as the cone over δU_1.

We turn now to the case $q > 1$. We will assume $m = q + n + 1$ with $q \leq n$. In this case we have $F(\varphi)$ isotopic to the standard imbedding of S^q in S^m. In particular, $F(\varphi)$ has trivial normal bundle in S^m. Further, if N is a tubular neighborhood of $F(\varphi)$, we have $S^m - N$ diffeomorphic to $D^{q+1} \times S^n$, N diffeomorphic to $S^q \times D^{n+1}$.

The first problem we encounter is whether φ is totally linear at $F(\varphi)$. It is easily seen to be equivalent to whether or not there is a nontrivial complex vector bundle whose underlying real vector bundle is trivial and such that the associated fibre bundle with fibre $P^r, (2r = 1 + n)$, is diffeomorphic to

$S^q \times P^r$. As far as I know this is quite possible even in the stable range. However, by simply examining BOTT's periodicity theorems [1] we see that:

Lemma 4. 1. *For* $q \neq 4k+2$, φ *is totally linear at* $F(\varphi)$.

Let N_1 be a tubular neighborhood of S^q and K a diffeomorphism of N_1 onto $S^q \times D^{n+1}$ carrying $\varphi|N_1$ onto α (as defined in § 2). Let N be the tubular neighborhood whose image under K has second coordinate of length $\leq \frac{1}{2}$. The closure of $S^m - N$ then is diffeomorphic to $D^{q+1} \times S^n$. Further if $x_0 \in S^q$ and $S_0^n = \{y| \;||y|| = \frac{3}{4}\}$ we have $Q = K^{-1}(\{x_0\} \times S_0^n)$ invariant under φ and φ is equivalent to α in a tubular neighborhood of Q. Applying lemma 3.2 we find a diffeomorphism $H: W \to D^{q+1} \times S^n$ which carries $\varphi|W$ to α. Let h, k denote the restriction of H and K to boundaries. We have then

$$h \cdot k^{-1}: S^q \times S^n \to S^q \times S^n \tag{4.1}$$

$$(h \cdot k^{-1}) \cdot \alpha = \alpha \cdot (h \cdot k^{-1}) \,. \tag{4.2}$$

Conversely if P is a diffeomorphism of $S^q \times S^n$ satisfying (4.2) we can define a differentiable action φ on S^m satisfying (A) and having S^q for fixed point set. Thus:

Theorem 4. 2. *For* $m > 6$, $q \neq 4k+2$, 1, $2q < m$ *we can obtain every differentiable action* φ *of* G *on* S^m *having* S^q *diffeomorphic to* $F(\varphi)$ *and satisfying* (A) *by identifying the boundary of* $D^{q+1} \times S^n$ *with the boundary of* $S^q \times D^{n+1}$ *by a diffeomorphism which commutes with* α.

We mention that it is possible that S^m is given an exotic differentiable structure by such a diffeomorphism. In order that the action be differentiable in the ordinary structure on S^m we have only to restrict the diffeomorphism to lie in the proper class.

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(Received May 20, 1962)