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On a Certain Property of Closed Hypersurfaces in an EINSTEIN Space

by YOSHIE KATSURADA, Sapporo

Introduction.

The following theorem, due to H. LIEBMANN (1900) [1], has been, and still is, the starting point of various interesting investigations within the Differential Geometry in the Large:

The only ovaloids with constant mean curvature H in Euclidean space E^3 are the spheres.

The analogous theorem for convex m -dimensional hypersurfaces in E^{m+1} has been proved by W. SÜSS (1929), [2], (cf. also [3], p. 118, and [4]). Recently (1958), A.D. ALEXANDROV has achieved the striking result that the convexity is not necessary for the validity of the LIEBMANN-SÜSS theorem, [5]: the theorem holds for arbitrary closed m -dimensional surfaces (hypersurfaces) without double points in E^{m+1} (i.e., for 1-1-imbeddings of closed m -manifolds). Already previously (1951), H. HOPF had shown that, for $n = 2$ and for surfaces of genus 0, the theorem holds even without the hypothesis that there are no double points (i.e., it holds for immersions, not necessarily one-one, of 2-spheres into E^3), [6]. It remains an open question whether there exists an immersion, not one-one, of a closed surface of higher genus into E^3 such that $H = \text{constant}$.

There are also interesting investigations about generalizing the condition $H = \text{constant}$ in LIEBMANN's theorem. But we shall not discuss these problems here.

It is the aim of the present author to investigate the question whether the mentioned theorems, especially the LIEBMANN-SÜSS theorem, are special cases of theorems which hold in more general RIEMANN spaces. One step in this direction has already been made in a previous paper dealing with RIEMANN spaces with constant RIEMANN curvature [7]. The present paper deals with EINSTEIN spaces and generalizes the paper [7], without making use of it. Our result is Theorem 3.1 which, as is easily seen, contains the LIEBMANN-SÜSS theorem as special case (so does, by the way, also the main theorem of [7]).

It is well known that the LIEBMANN-SÜSS theorem is closely related to classical integral formulas of MINKOWSKI (cf. the paper of SÜSS). The base of our proof of Theorem 3.1 is a formula of MINKOWSKI type which holds in arbitrary RIEMANN spaces (formula (I) in § 1). This formula had already been established in [7]; a new proof is given in § 1 below. In § 2, some integral formulas for

hypersurfaces with $H = \text{constant}$ in EINSTEIN spaces are derived, and in § 3, the main theorem is proved.

The author wishes to express to Professor HEINZ HOPF her very sincere thanks for his valuable advice and suggestions.

§ 1. Another proof of the generalized MINKOWSKI formula (I).

In this section, we shall give a different proof of the generalized MINKOWSKI formula (I) derived in the previous paper ([7], p. 288).

We consider a RIEMANN space R^{m+1} ($m + 1 \geq 3$) of class C^ν ($\nu \geq 3$) which admits an one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x) \delta\tau \quad (1.1)$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a contravariant vector ξ). We suppose that the paths of these transformations cover R^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. If ξ is a KILLING vector, a homothetic KILLING, a conformal KILLING vector etc. ([8], p. 32), then the group G is called isometric, homothetic, conformal etc., respectively.

We now consider a closed orientable hypersurface V^m of class C^3 imbedded in R^{m+1} , locally given by

$$x^i = x^i(u^\alpha); \quad (1.2)$$

here and henceforth, Latin indices run from 1 to $m + 1$ and Greek indices from 1 to m .

To the vector ξ introduced above, there belongs a covariant vector $\bar{\xi}$ of V^m with the components

$$\bar{\xi}_\alpha = \frac{\partial x^i}{\partial u^\alpha} \xi_i$$

where ξ_i are the covariant components of ξ ; we shall compute its covariant derivatives along V^m : by virtue of the fact that the covariant derivatives of $\frac{\partial x^i}{\partial u^\alpha}$ are

$$\frac{\delta}{\delta u^\beta} \left(\frac{\partial x^i}{\partial u^\alpha} \right) = b_{\alpha\beta} n^i$$

where $b_{\alpha\beta}$ is the second fundamental tensor and n^i is the unit normal vector of V^m , we find

$$\bar{\xi}_{\alpha;\beta} = b_{\alpha\beta} n^i \xi_i + \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \xi_{i;j} \quad (1.3)$$

(the symbol " ; " always means the covariant derivative).

Multiplying (1.3) by the contravariant metric tensor $g^{\alpha\beta}$ of V^m and contracting, we get

$$g^{\alpha\beta} \bar{\xi}_{;\beta} = m H_1 n^i \xi_i + \frac{1}{2} g^{\alpha\beta} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \cdot \mathcal{L}_\xi g_{ij}, \quad (1.4)$$

where H_1 is the first mean curvature $\frac{1}{m} g^{\alpha\beta} b_{\alpha\beta}$ of V^m and $\mathcal{L}_\xi g_{ij}$ is the LIE derivative of the fundamental tensor g_{ij} of R^{m+1} with respect to the infinitesimal transformation (1.1) (cf. [8], p. 5). If we put

$$\frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \mathcal{L}_\xi g_{ij} = \mathcal{L}_\xi g_{\alpha\beta}$$

then (1.4) rewritten is as follows:

$$\frac{1}{m} \bar{\xi}^\alpha_{;\alpha} = H_1 n^i \xi_i + \frac{1}{2m} g^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta}.$$

dA being the area element of V^m , there holds

$$\int_{V^m} \dots \int \bar{\xi}^\alpha_{;\alpha} dA = 0$$

because V^m is closed and orientable ([9], p. 31). Thus we obtain the integral formula

$$\int_{V^m} \dots \int H_1 n^i \xi_i dA + \frac{1}{2m} \int_{V^m} \dots \int g^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} dA = 0 \quad (I)$$

which is nothing but the formula (I) of the previous paper [7], p. 288.

Let the group G be conformal, that is, ξ^i satisfy the equation

$$\mathcal{L}_\xi g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij}$$

(cf. [8], p. 32), then (I) becomes

$$\int_{V^m} \dots \int H_1 n^i \xi_i dA + \int_{V^m} \dots \int \Phi dA = 0; \quad (I)_c$$

let G be homothetic, that is, $\Phi \equiv C = \text{constant}$, then

$$\int_{V^m} \dots \int H_1 n^i \xi_i dA + C \int_{V^m} \dots \int dA = 0; \quad (I)_h$$

and let G be isometric, that is, $C = 0$, then

$$\int_{V^m} \dots \int H_1 n^i \xi_i dA = 0. \quad (I)_i$$

§ 2. Some integral formulas for a closed hypersurface with $H_1 = \text{constant}$ in an EINSTEIN space.

Hereafter we shall assume that the RIEMANN space R^{m+1} is an EINSTEIN space and V^m is a closed orientable hypersurface with $H_1 = \text{constant}$.

If we take the covariant vector of the hypersurface V^m , defined by

$$\eta_\alpha = n^i_{;\alpha} \xi_i$$

and calculate its covariant derivatives along V^m , we have

$$\eta_{\alpha;\beta} = n^i_{;\alpha;\beta} \xi_i + n^i_{;\alpha} \xi_{i;j} \frac{\partial x^j}{\partial u^\beta}.$$

Remembering the following formulas for hypersurfaces

$$n^i_{;\alpha} = - b_\alpha^\gamma \frac{\partial x^i}{\partial u^\gamma}, \quad (2.1)$$

$$\frac{\delta}{\partial u^\beta} \left(\frac{\partial x^i}{\partial u^\gamma} \right) = b_{\beta\gamma} n^i, \quad (2.2)$$

where b_α^γ means $g^{\gamma\delta} b_{\alpha\delta}$ ([10] p. 136, p. 127), we find that

$$\eta_{\alpha;\beta} = - \left(\xi_i b_\alpha^\gamma \frac{\partial x^i}{\partial u^\gamma} + b_\alpha^\gamma b_{\gamma\beta} \xi_i n^i + \xi_{i;j} b_\alpha^\gamma \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^\beta} \right).$$

Multiplying by $g^{\alpha\beta}$ and contracting, we obtain

$$g^{\alpha\beta} \eta_{\alpha;\beta} = - g^{\alpha\beta} \left(\xi_i b_\alpha^\gamma \frac{\partial x^i}{\partial u^\gamma} + b_\alpha^\gamma b_{\gamma\beta} \xi_i n^i + \xi_{i;j} b_\alpha^\gamma \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^\beta} \right). \quad (2.3)$$

We shall first calculate the first term of the right-hand side of (2.3):

$$g^{\alpha\beta} \xi_i b_\alpha^\gamma \frac{\partial x^i}{\partial u^\gamma} = g^{\alpha\beta} g^{\gamma\delta} b_{\alpha\delta} \xi_i \frac{\partial x^i}{\partial u^\gamma}. \quad (2.4)$$

As well-known, an hypersurface in a RIEMANN space has the following property

$$b_{\alpha\delta;\beta} - b_{\alpha\beta;\delta} = - R_{ijkl} \frac{\partial x^i}{\partial u^\alpha} n^j \frac{\partial x^k}{\partial u^\delta} \frac{\partial x^l}{\partial u^\beta}, \quad ([10] p. 138)$$

where R_{ijkl} is the curvature tensor of R^{m+1} . Multiplying both sides of this equation by $g^{\alpha\beta}$ and contracting, we get

$$g^{\alpha\beta} (b_{\alpha\delta;\beta} - b_{\alpha\beta;\delta}) = - R_{ijkl} \frac{\partial x^i}{\partial u^\alpha} n^j \frac{\partial x^k}{\partial u^\delta} \frac{\partial x^l}{\partial u^\beta} g^{\alpha\beta}; \quad (2.5)$$

substituting

$$\frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^l}{\partial u^\beta} g^{\alpha\beta} = g^{il} - n^i n^l$$

into the right-hand side of (2.5), we obtain

$$g^{\alpha\beta}(b_{\alpha\delta;\beta} - b_{\alpha\beta;\delta}) = -R_{jk}n^j \frac{\partial x^k}{\partial u^\delta} \quad (2.6)$$

R_{jk} being the Ricci tensor of R^{m+1} ($R_{jk} = g^{il}R_{ijkl}$). Because R^{m+1} is an EINSTEIN space and V^m has the property $H_1 = \text{constant}$, the right-hand side of (2.6) and the second term of the left-hand side vanish; it follows that

$$g^{\alpha\beta}b_{\alpha\delta;\beta} = 0. \quad (2.7)$$

Therefore, and with respect to (2.4), the first term of the right-hand side of (2.3) is equal to zero.

Next, we discuss the second term of the right-hand side of (2.3):

$$g^{\alpha\beta}b_\alpha^\gamma b_{\gamma\beta} n^i \xi_i = g^{\alpha\beta}g^{\gamma\delta}b_{\alpha\delta}b_{\gamma\beta} n^i \xi_i. \quad (2.8)$$

Let k_1, k_2, \dots, k_m be the principal curvatures at a point P of V^m , and let H_2 be the second mean curvature of V^m at the point P which is defined to be the second elementary symmetric function of k_1, k_2, \dots, k_m divided by the number of terms, that is.

$$\binom{m}{2} H_2 = \sum_{(\alpha, \beta)} k_\alpha k_\beta \quad (\alpha < \beta);$$

since, furthermore the following relation holds

$$\frac{1}{2} (g^{\alpha\delta}g^{\gamma\beta}b_{\alpha\delta}b_{\gamma\beta} - g^{\alpha\beta}g^{\gamma\delta}b_{\alpha\delta}b_{\gamma\beta}) = \binom{m}{2} H_2,$$

(2.8) can be written as follows

$$g^{\alpha\beta}b_\alpha^\gamma b_{\gamma\beta} n^i \xi_i = \{m^2 H_1^2 - 2 \binom{m}{2} H_2\} n^i \xi_i. \quad (2.9)$$

At last, for the third term of the right-hand side of (2.3), we calculate as follows

$$\begin{aligned} g^{\alpha\beta}b_\alpha^\gamma \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^\beta} \xi_{i;j} &= g^{\alpha\beta}g^{\gamma\delta}b_{\alpha\delta} \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^\beta} \xi_{i;j} \\ &= \frac{1}{2} g^{\alpha\beta}g^{\gamma\delta}b_{\alpha\delta} \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^\beta} (\xi_{i;j} + \xi_{j;i}) \\ &= \frac{1}{2} H^{\beta\gamma} \mathcal{L}_\xi g_{\beta\gamma} \end{aligned} \quad (2.10)$$

where $H^{\beta\gamma}$ denotes $b_{\alpha\delta}g^{\alpha\beta}g^{\gamma\delta}$.

Accordingly, from (2.7), (2.9), and (2.10), (2.3) becomes

$$\frac{1}{m} \eta^\alpha_{;\alpha} = - \{(m H_1^2 - (m-1)H_2) n^i \xi_i + \frac{1}{2m} H^{\beta\gamma} \mathcal{L}_\xi g_{\beta\gamma}\}.$$

And also on making use of

$$\int_{V^m} \dots \int \eta^\alpha_{;\alpha} dA = 0$$

by virtue of V^m being closed orientable, we finally reach the integral formula

$$\int_{V^m} \dots \int \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA + \frac{1}{2m} \int_{V^m} \dots \int H^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} dA = 0. \quad (\text{II})$$

If the group G of transformation is conformal, (II) becomes

$$\int_{V^m} \dots \int \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA + \int_{V^m} \dots \int \Phi H_1 dA = 0; \quad (\text{II})_c$$

if G is homothetic (i.e. $\Phi \equiv \text{constant} = C$),

$$\int_{V^m} \dots \int \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA + C \int_{V^m} \dots \int H_1 dA = 0; \quad (\text{II})_h$$

and if G is isometric (i.e. $\Phi \equiv 0$),

$$\int_{V^m} \dots \int \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA = 0. \quad (\text{II})_i$$

§ 3. Closed orientable hypersurfaces with $H_1 = \text{constant}$ in an EINSTEIN space.

In this section, we shall prove the following theorem:

Theorem 3.1. *Let R^{m+1} be an EINSTEIN space, V^m a closed orientable hypersurface with $H_1 = \text{constant}$ in R^{m+1} ; we suppose that there exists a continuous one-parameter group G of conformal transformations of R^{m+1} such that the scalar product $\tilde{p} = n^i \xi_i$ of the normal vector n of V^m and the vector ξ belonging to G does not change the sign (and is not $\equiv 0$) on V^m . Then every point of V^m is umbilic.*

Proof. Multiplying the formula (I)_c in § 1 by H_1 ($= \text{const.}$), we obtain

$$\int_{V^m} \dots \int H_1^2 \tilde{p} dA + \int_{V^m} \dots \int \Phi H_1 dA = 0,$$

and subtracting this formula from the formula (II)_c in § 2, we find

$$\int_{V^m} \dots \int (m-1)(H_1^2 - H_2) \tilde{p} dA = 0. \quad (3.1)$$

From

$$H_1^2 - H_2 = \frac{1}{m^2} \cdot (\sum k_\alpha)^2 - \frac{2}{m(m-1)} \sum_{\alpha, \beta} k_\alpha k_\beta = \frac{1}{m^2(m-1)} \sum (k_\alpha - k_\beta)^2 \quad (3.2)$$

(with $\alpha \neq \beta$) we see that

$$H_1^2 - H_2 \geq 0. \quad (3.3)$$

From (3.1), (3.3) and the fact that \tilde{p} has a fixed sign we conclude that

$$H_1^2 - H_2 = 0$$

and therefore, because of (3.2), that

$$k_1 = k_2 = \dots = k_m$$

at each point of V^m . This means, that each point of V^m is umbilic.

We wish now to show that the LIEBMANN-SÜSS Theorem is a special case of our Theorem 3.1. Because in euclidean E^{m+1} an hypersurface is a sphere if all its points are umbilical we have only to verify that, for a convex V^m in E^{m+1} , there exists a vector field ξ having the properties formulated in Theorem 3.1. We take a point in the interior of V^m as origin of the euclidean coordinates x^i and attach to each point x the vector $\xi(x)$ with the components $\xi^i = x^i$ (i.e. the position vector of x). Then, the transformations (1.1) are homothetic, thus conformal; furthermore, for $x \in V^m$, $\tilde{p}(x)$ is the support function and, because V^m is convex, $\tilde{p}(x) \neq 0$.

REFERENCES

- [1] H. LIEBMANN: *Über die Verbiegung der geschlossenen Flächen positiver Krümmung*, Math. Ann. 53 (1900), 91–112.
- [2] W. SÜSS: *Zur relativen Differentialgeometrie V.*, Tôhoku Math. J. 30 (1929), 202–209.
- [3] T. BONNESEN und W. FENCHEL: *Theorie der konvexen Körper* (Springer, Berlin 1934).
- [4] C. C. HSIUNG: *Some integral formulas for closed hypersurfaces*, Math. Scand. 2 (1954), 286–294.
- [5] A. D. ALEXANDROV: *Uniqueness theorems for surfaces in the large*, V. Vestnik Leningrad University 13 (1958), 5–8 (Russian, with English summary).
- [6] H. HOPF: *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math. Nachr. 4 (1951), 232–249.
- [7] Y. KATSURADA: *Generalized Minkowski formulas for closed hypersurfaces in a RIEMANN space*, Annali di Matematica, Serie IV, 57 (1962), 283–294.
- [8] K. YANO: *The theory of LIE derivatives and its applications* (Amsterdam 1957).
- [9] K. YANO and S. BOCHNER: *Curvature and Betti numbers* (Princeton, Annals of Math. Studies, 1953).
- [10] C. E. WEATHERBURN: *An introduction to RIEMANN geometry and the tensor calculus* (Cambridge University Press 1938).

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