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# On a characterization of quasiconformal mappings[1]

by EDGAR REICH

## 1. Introduction

Let $M(Q)$ denote the module of the quadrilateral $Q$ (a JORDAN domain with four distinguished boundary points and two non-adjacent distinguished sides), A sense-preserving homeomorphism $f$ of a region $\Omega$ onto the image region $f(\Omega)$ is said to be a quasiconformal mapping with maximal dilatation $K(\Omega) = K_f(\Omega)$ if

$$\sup_{\bar{Q} \subset \Omega} \frac{M(f(Q))}{M(Q)} = K(\Omega) < \infty . \tag{1.1}$$

($f(Q)$ denotes the quadrilateral with preimage $Q$.)

Instead of considering the effect of $f$ on quadrilaterals it is natural to study the effect of $f$ on ring domains (doubly connected regions). The modulus $\mu(R)$ of a ring $R$ is defined as $(2\pi)^{-1} \log (r_2/r_1)$, where $\{r_1 < |w| < r_2\}$ is an annulus conformally equivalent to $R$. It is well known [1] that if (1.1) holds then

$$\frac{\mu(f(R))}{\mu(R)} \leq K(\Omega) , \ \bar{R} \subset \Omega . \tag{1.2}$$

In fact, the following has been proved by GEHRING and VÄISÄLÄ ([2], Theorem 3):

$$\sup_{\bar{R} \subset \Omega} \frac{\mu(f(R))}{\mu(R)} = K(\Omega) . \tag{1.3}$$

Thus quasiconformal mappings $f$ with maximal dilatation $K(\Omega)$ may be characterized by (1.3) instead of (1.1).

In the proof of (1.3) in [2] essential use is made of the deep and rather difficult "analytic" characterization of quasiconformal maps (See, for instance, [4], Chapter 4). In view of the significance of (1.3) it appears desirable to obtain a more direct and more elementary proof of (1.3), starting with the definition (1.1). It is the object of the present note to provide such a proof.

## 2. A class of ring domains

Let $Q$ be a quadrilateral with distinguished sides $\alpha_1, \alpha_2$. A ring $R$ will be said to *link* $Q$ if every closed curve $\gamma$, in $R$, separating the boundary com-

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ponents of $R$, contains an arc lying in $Q$ which joins $\alpha_1$ and $\alpha_2$. (See Fig. 1 where the heavy lines are the boundary conponents of $R$, and the dashed line represents $Q$.)

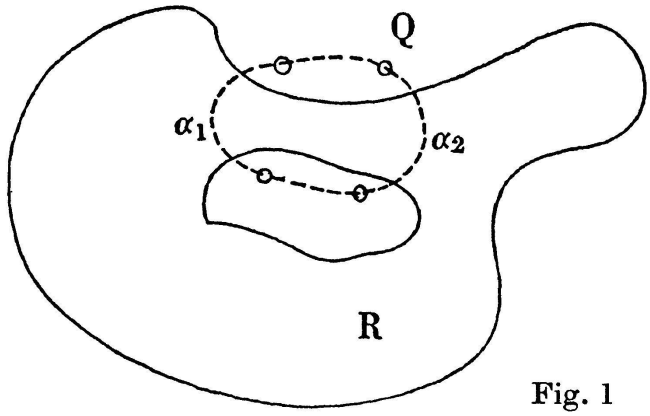


Fig. 1

We will first show that

$$\mu(R) \leq M(Q)\,, \text{ if } R \text{ links } Q\,. \tag{2.1}$$

Let $\varrho^*(z), z \in Q$, be the extremal metric for $Q$; that is,

$$M(Q) = \frac{A_{\varrho^*}(Q)}{L^2_{\varrho^*}(Q)}\,,\ A_{\varrho^*}(Q) = \iint_Q \varrho^{*2}\,|\,dz\,|^2,\ L_{\varrho^*}(Q) = \inf_\gamma \int_\gamma \varrho^*\,|\,dz\,|\,,$$

where $\{\gamma\}$ is the set of locally rectifiable curves in $Q$ joining $\alpha_1, \alpha_2$. On the other hand, (See, for instance, [3], Chapter 2),

$$\mu(R) = \inf_\varrho \frac{A_\varrho(R)}{L^2_\varrho(R)} \tag{2.2}$$

where, this time, $\{\gamma\}$ is the set of closed curves in $R$ separating the boundary components. For $z \in R$, let

$$\varrho(z) = \begin{cases} \varrho^*(z), z \in R \cap Q \\ 0, z \in R - Q \end{cases}.$$

Clearly, $$A_\varrho(R) \leq A_{\varrho^*}(Q).$$

Since $R$ links $Q$, $L_\varrho(R) \geq L_{\varrho^*}(Q)$. Hence, by (2.2),

$$\mu(R) \leq \frac{A_{\varrho^*}(Q)}{L^2_{\varrho^*}(Q)} = M(Q)\,,$$

as was to be shown.

In general, to obtain a bound on the modulus of a ring domain $R$ from below, one attempts to make use of the fact ([3], Chapter 2) that

$$\frac{1}{\mu(R)} = \inf_\varrho \frac{A_\varrho(R)}{L^2_\varrho(R)} \tag{2.3}$$

where, now, $\{\gamma\}$ is the set of locally rectifiable curves in $R$ *joining* the boun-

dary components. Fortunately, we shall require a bound on $\mu(R)$ from below, for rings linking $Q$, only in the very special case considered next.

Suppose the set of positive numbers, $a, p, h, s$, is given, and

$$2p < \min\ (a, 1),\ h < a - 2p,\ p < s < a - p - h,\ h < 1\ .$$

We consider the ring $R'$ (Fig. 2) obtained as follows.

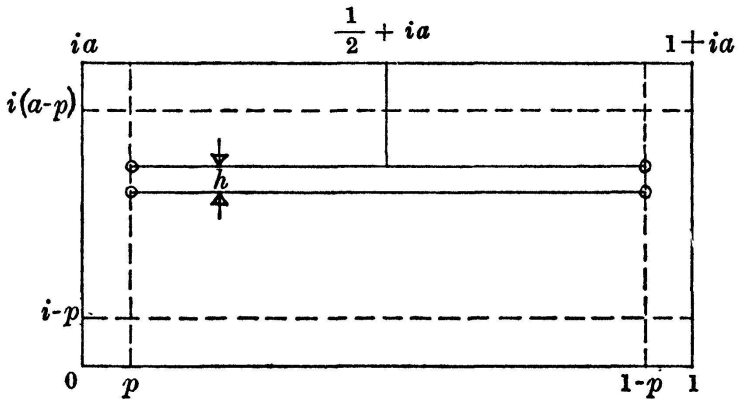


Fig. 2

The outer boundary component of $R'$ consists of the union of (a) the boundary of the rectangle with vertices $0, 1, 1 + ia, ia$ (b) the vertical segment joining the points $\frac{1}{2} + ia, \frac{1}{2} + i(a - s)$ (c) the horizontal segment $I_1 = \{z \mid \Im z = a - s,\ p \leq \Re z \leq 1 - p\}$.

The inner boundary component of $R'$ is the horizontal segment

$$I_2 = \{z \mid \Im z = a - s - h,\ p \leq \Re z \leq 1 - p\}\ .$$

Let $T$ be the rectangle with horizontal sides $I_1, I_2$, with the vertical sides (of length $h$) distinguished. The ring $R'$ links $T$, and, therefore, by (2.1), $\mu(R') \leq h/(1 - 2p)$. We shall now show that

$$\mu(R') \geq \frac{h}{1 + ah^{\frac{1}{2}}},\quad \text{if } h \leq p^4\ . \tag{2.4}$$

Let

$$\varrho(z) = \begin{cases} 1, & 0 < \Re z < 1,\ a - s - h < \Im z < a - s \\ h^{\frac{3}{4}}, & \text{elsewhere in } R'. \end{cases}$$

Let $\{\gamma\}$ be the family of curves in $R'$ joining $I_2$ to the outer boundary component of $R'$. Clearly,

$$L_\varrho(R') = \min\ (h, h^{\frac{3}{4}} p),\ A_\varrho = h + h^{\frac{3}{2}}(a - h) \leq h + ah^{\frac{3}{2}}\ .$$

Hence, by (2.3),

$$\frac{1}{\mu(R')} \leq \frac{A_\varrho(R')}{L_\varrho^2(R')} \leq \frac{h + ah^{\frac{3}{2}}}{h^2} = \frac{1 + ah^{\frac{1}{2}}}{h}\ ,\ \text{if } h^{\frac{3}{4}} p \geq h\ .$$

This is equivalent to (2.4).

## 3. Proof of (1.3)

Let $\epsilon$, $0 < \epsilon < \frac{1}{2}$, be given. In the notation of Section 1, there exists a quadrilateral $Q, \bar{Q} \subset \Omega$, with

$$\frac{M(f(Q))}{M(Q)} \geq (1 - \varepsilon) K(\Omega).$$

Let the transformation $\Phi$ map $f(Q)$ conformally onto a rectangle $V$ with vertices $0, 1, 1 + ia, ia$, with the vertical sides distinguished. Thus

$$a = M(V) = M(f(Q)).$$

Let $V_p$ be the rectangle, interior to $V$ and oriented as $V$, such that the distance between corresponding sides of $V_p$ and $V$ is $p$, $0 < p < \frac{1}{2}\min(a, 1)$ Let $Q_p = f^{-1}\Phi^{-1}(V_p)$. There exists [1] a number $\delta(\epsilon) > 0$ such that

$$\frac{M(V_p)}{M(Q_p)} = \frac{a - 2p}{(1 - 2p) M(Q_p)} \geq (1 - 2\varepsilon) K(\Omega), \quad \text{if } p \leq \delta(\varepsilon);$$

that is, $Q_p$ has nearly the same module as $Q$ if $p$ is small.

We now divide $V_p$ into $n$ equal horizontal strips, $\sigma_k$, $k = 1, 2, \ldots, n$. $\sigma_k$ is an $h_n \times 1 - 2p$ rectangle, $h_n = (a - 2p) n^{-1}$, whose vertical sides are distinguished. The quadrilaterals

$$q_k = f^{-1}\Phi^{-1}(\sigma_k), \; k = 1, 2, \ldots$$

which lie in $\Omega$, subdivide $Q_p$. According to the subadditivity property for modules [1],

$$\sum_{k=l}^{n} M(q_k) \leq M(Q_p) \leq \frac{a - 2p/1 - 2p}{(1 - 2\varepsilon) K(\Omega)}.$$

Hence for some $k$, say $k = k_n$,

$$M(q_{k_n}) \leq \frac{a - 2p/1 - 2p}{(1 - 2\varepsilon) nK(\Omega)} = \frac{h_n/1 - 2p}{(1 - 2\varepsilon) K(\Omega)}.$$

If we define $N(p)$ as $N(p) = (a - 2p) p^{-4}$, then $n \geq N(p)$ implies $h_n \leq p^4$.

Let $R'_{pn}$ be the ring linking $\sigma_{k_n}$ formed as in Fig. 2, with the present $h_n$ serving as the $h$ of Fig. 2. Then, by (2.4),

$$\mu(R'_{pn}) \geq \frac{h_n}{1 + ah_n^{\frac{1}{2}}}, \quad \text{if } n \geq N(p).$$

Consider the ring $R_{pn} = f^{-1}\Phi^{-1}(R'_{pn})$, in $\Omega$. Since $R_{pn}$ links $q_{k_n}$, we have, by (2.1),

$$\mu(R_{pn}) \leq M(q_{k_n}) \leq \frac{h_n/1 - 2p}{(1 - 2\varepsilon)K(\Omega)}, \quad \text{if } p \leq \delta(\varepsilon).$$

Now $\mu(f(R_{pn})) = \mu(R'_{pn})$. Therefore,

$$\frac{\mu(f(R_{pn}))}{\mu(R_{pn})} \geq \frac{(1 - 2\varepsilon)(1 - 2p)K(\Omega)}{1 + ah_n^{\frac{1}{2}}}, \quad \text{if } p \leq \delta(\varepsilon), n \geq N(p). \quad (3.1)$$

Once $p$ is chosen, $h_n$ can be made arbitrarily small by taking $n$ sufficiently large. Thus (3.1) shows that there exist rings $R, \bar{R} \subset \Omega$, such that

$$\frac{\mu(f(R))}{\mu(R)} \geq (1 - 3\varepsilon)K(\Omega).$$

Together with (1.2) the above establishes (1.3).

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December 1961

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