Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 37 (1962-1963)

Artikel: On Locally Symmetric Spaces of Non-negative Curvature and certain other Locally Homogeneous Spaces.

Autor: Wolf, Joseph A.

DOI: https://doi.org/10.5169/seals-28623

On Locally Symmetric Spaces of Non-negative Curvature and certain other Locally Homogeneous Spaces

by Joseph A. Wolf[1], Princeton (N. J.)

To Professor Georges de Rham on his sixtieth birthday

1. Introduction and summary

This paper is a study of the global structure of the complete connected locally symmetric Riemannian manifolds N in which every sectional curvature is non-negative. Our main result is that the fundamental group $\pi_1(N)$ is a finite 2-group if the Euler-Poincaré characteristic (singular theory) $\chi(N) \neq 0$. In fact, that result is proved under slightly weaker conditions on N.

The first principle result (Theorem 3.1) states that there is a real analytic covering $N' \to N$ of finite multiplicity and a real analytic deformation retraction of N onto a compact totally geodesic submanifold, such that $N' = E \times T \times M'$ where E is a Euclidean space, T is a torus, M' is a compact simply connected Riemannian symmetric space, and the deformation retraction of N lifts to a deformation retraction of N' onto $T \times M'$. In particular, the betti numbers (singular theory) of N are finite and the Euler-Poincaré characteristic $\chi(N)$ is defined. Theorem 3.1 then states that $\chi(N) \geqq 0$, and that the fundamental group $\pi_1(N)$ is a finite 2-group if $\chi(N) \neq 0$.

The second principle result (Theorem 3.2) gives a general method of constructing all manifolds N with $\chi(N) \neq 0$. Application of this method is a combinatorial problem which requires a classification (up to global isometry) of the space forms of the irreducible compact simply connected Riemannian symmetric manifolds S with $\chi(S) \neq 0$. That classification problem is solved in § 5. We first prove (Theorem 5.1) that S is equal to any of its space forms unless S is a Grassmann manifold, $\mathbf{SO}(2n)/\mathbf{U}(n)$, $\mathbf{Sp}(n)/\mathbf{U}(n)$, $\mathbf{E}_7/A_7$ or $\mathbf{E}_7/E_6 \cdot T^1$. We have already classified the space forms of Grassmann manifolds of nonzero characteristic [13]; the result is recalled as Theorem 5.3. We then (Theorems 5.4 – 5.7) classify the space forms of the other possibilities of S. From these classification theorems we are able (Theorem 6.2) to give a necessary and sufficient condition on the set of factors of a product M'

[1]) The author thanks the National Science Foundation for fellowship support during the period of preparation of this paper. His present address is: Department of Mathematics, University of California, Berkeley 4, California.

of manifolds S, that every space form of M' have abelian fundamental group. If M' is irreducible, the condition is automatically fulfilled. Finally, in § 6.5, we give a good description of the possibilities for a manifold N with $\chi(N) \neq 0$ when, in the universal RIEMANNian covering manifold $M_0 \times M'$ (M_0 EUCLIDean and M' compact; this is the M' which occurs in N'), M' satisfies the commutativity conditions of Theorem 6.2.

Some parts of Theorem 3.1 do not fully use the hypotheses on N. This leads us to define a *RIEMANNian nilmanifold* to be a RIEMANNian manifold which admits a transitive nilpotent group of isometries. We prove (Theorem 4.2) that a connected RIEMANNian nilmanifold is isometric to a connected nilpotent Lie group in a left invariant RIEMANNian metric, that the nilradical of its connected group of isometries is the only connected transitive nilpotent group of isometries, and that its full group of isometries is the semidirect product of this nilradical with an isotropy group. Now let N be a RIEMANNian manifold with universal RIEMANNian covering manifold of the form $M_0 \times M'$ where M_0 is a RIEMANNian nilmanifold and M' is a compact RIEMANNian homogeneous manifold. Theorem 4.1 provides a real analytic covering

$$N' = E \times N'' \times M' \to N$$

of some finite multiplicity $r > 0$, where E is a EUCLIDean space and N'' is a compact nilmanifold. While I am unfortunately unable to retract N onto a compact submanifold unless M_0 is a EUCLIDean space (and so cannot prove the betti numbers of N to be finite, and so cannot assert that $\chi(N)$ is defined) it is shown (Proposition 4.4) that $\chi^*(N) = \frac{1}{r}\chi(N')$ is a topological invariant of N. Theorem 4.1 then states that $\chi^*(N)$ is an integer, that $\chi^*(N) \geqq 0$, that $\chi^*(N) = \chi(N)$ if M_0 is a EUCLIDean space, that $\pi_1(N)$ is finite if $\chi^*(N) \neq 0$, and that $\pi_1(N)$ is a finite 2-group if $\chi^*(N) \neq 0$ and M' is RIEMANNian symmetric.

The "rational EULER-POINCARÉ characteristic" χ^* was invented by C.T.C. WALL [10] in another context. D.B.A. EPSTEIN suggested that I use it here, and gave valuable suggestions for adapting it to noncompact spaces and then proving it well defined.

By Theorem 3.1, we mean the theorem in § 3.1. Similarly, Theorem 3.9 is the theorem in § 3.9, etc.

Added in proof: By different methods, J. C. SANWAL has obtained the flat case of the fourth corollary of § 4.2 and has shown that the fundamental group of a complete flat RIEMANNian manifold is isomorphic to that of a compact flat RIEMANNian manifold, special case of our Theorem 3.1.

2. Preliminaries and notation

We will assume familiarity with LIE groups and discrete subgroups, RIEMANNian manifolds, and covering spaces.

2. 1. LIE groups and algebras. If G is a LIE group, then G_0 will denote its identity component, $\mathfrak{G}$ will denote its LIE algebra, $\exp : \mathfrak{G} \to G_0$ will be the exponential mapping, and adjoint representation of G on $\mathfrak{G}$ will be denoted by "ad". If H is a LIE subgroup of G, then $\mathfrak{H}$ is viewed as a subalgebra of $\mathfrak{G}$. If $\mathfrak{H}$ is a subalgebra of $\mathfrak{G}$, then the *corresponding analytic subgroup of* G is the analytic (= connected LIE) subgroup generated by the image of $\mathfrak{H}$ under the exponential mapping of G.

If G and H are LIE groups and $\beta : H \to \mathrm{Aut}(G)$ is a continuous homomorphism of H into the group of automorphisms of G, then the *semidirect product* $G \cdot {}_\beta H$ (denoted $G \cdot H$ when there is no possibility of confusion) is the manifold $G \times H$ with group structure $(g_1, h_1)(g_2, h_2) = (g_1 \cdot \beta(h_1) g_2, h_1 h_2)$. $G \cdot H$ is a LIE group, G and H are closed subgroups under identifications $g \to (g, 1)$ and $h \to (1, h)$ (we always use 1 to denote the group identity), and G is a normal subgroup. The two extreme cases are when β is trivial, so $G \cdot H$ is the direct product $G \times H$, and when β is faithful (trivial kernel), so H may be viewed as a group of automorphisms of G if $\beta(H)$ is closed in $\mathrm{Aut}(G)$.

The compact classical groups are the orthogonal groups $\mathbf{O}(n)$ in n real variables, the identity components, $\mathbf{SO}(n)$, the special (= determinant 1) orthogonal groups, the unitary groups $\mathbf{U}(n)$ in n complex variables and the special unitary groups $\mathbf{SU}(n)$, the symplectic groups $\mathbf{Sp}(n)$ which are the unitary groups in n quaternion variables, and the universal covering groups $\mathbf{Spin}(n)$ of $\mathbf{SO}(n)$. T^m will denote an m-torus. $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7$ and E_8 will refer both to the CARTAN classification types and to compact connected groups of those types. In boldface, these letters will denote the compact simply connected groups. For example, $\mathbf{A}_n = \mathbf{SU}(n+1)$, $\mathbf{B}_n = \mathbf{Spin}(2n+1)$, $\mathbf{C}_n = \mathbf{Sp}(n)$, $\mathbf{D}_n = \mathbf{Spin}(2n)$, and $\mathbf{F}_4$ is the group of isometries of the CAYLEY elliptic plane.

2. 2. Discrete groups. A subgroup Γ of a topological group G is called *discrete* if it is a discrete subset, i.e., if there is a neighborhood U of $1 \in G$ such that $\Gamma \cap U = \{1\}$. Γ is called *uniform in* G if ($\bar{\Gamma}$ denotes the topological closure) $G/\bar{\Gamma}$ is compact.

Let Γ be a topological group and let X be a topological space. An *action* of Γ on X is a homomorphism of Γ into the group of homeomorphisms of X such that the associated map $\Gamma \times X \to X$ is continuous. We write $\gamma(x)$ for the image of (γ, x). The action is *effective* if $1 \neq \gamma \in \Gamma$ implies $\gamma(x) \neq x$ for some element $x \in X$; the action is *free* if $1 \neq \gamma \in \Gamma$ implies $\gamma(x) \neq x$ for every element $x \in X$; the action is *properly discontinuous* if every $x \in X$ has a neighborhood which meets its transforms by only a finite number of elements of Γ.

Let Γ and K be subgroups of G, K closed in G. Then there is a natural action $\gamma : gK \rightarrow \gamma gK$ of Γ on the coset space G/K. If G is a LIE group, or even locally compact with only finitely many components, and if K is compact, then the action is properly discontinuous if and only if Γ is discrete in G. In any case, the action is free if and only if 1 is the only element of Γ conjugate (in G) to an element of K, and the identification space of G/K under Γ is the double coset space $\Gamma \backslash G/K$.

2. 3. Isometries and product structure. An *isometry* of a RIEMANNian manifold is an automorphism of the RIEMANNian structure. If M is a RIEMANNian manifold, then its *full group of* (= group of all) *isometries* is a LIE group denoted $\mathbf{I}(M)$; the *connected group of isometries* is the identity component $\mathbf{I}(M)_0$; following tradition, we write $\mathbf{I}_0(M)$ for $\mathbf{I}(M)_0$. M is *homogeneous* if $\mathbf{I}(M)$ is transitive on the points of M. If M is homogeneous and connected, and if $x \in M$, then $g \rightarrow g(x)$ induces differentiable homeomorphisms of M with the coset spaces $\mathbf{I}(M)/K$ and $\mathbf{I}_0(M)/(\mathbf{I}_0(M) \cap K)$ where $K = \{g \in \mathbf{I}(M) : g(x) = x\}$ is the *isotropy subgroup of* $\mathbf{I}(M)$ *at* x; K is compact. If $s \in \mathbf{I}(M)$ has square 1 and has $x \in M$ as an isolated fixed point, then s is a *symmetry to* M *at* x; if M is connected, s is unique because it induces $-I$ (I = identity) on the tangentspace M_x. M is *symmetric* if it has a symmetry at each of its points. If M is connected and symmetric, then any two points x and y can be joined by a broken geodesic, and the product of the symmetries to M at the midpoints of the geodesic segments will send x to y; thus M is homogeneous.

Let M be complete and simply connected. Then [7] M is isometric to a product $M_0 \times M_1 \times \ldots \times M_t$ where M_0 is a EUCLIDean space (the *EUCLIDean factor of* M) and the other M_i, the *irreducible factors of* M, are irreducible, i.e., are non-EUCLIDean and not locally products of lower dimensional manifolds. This *DE RHAM decomposition* is unique up to the order of the factors. M is homogeneous (resp. symmetric) if and only if each of the M_i is homogeneous (resp. symmetric). Identifying M with $M_0 \times \ldots \times M_t$ and letting $\mathbf{I}(M_i)$ act on M by acting on M_i in the usual way and by acting trivially on the other M_j, $\mathbf{I}(M)$ is generated by the $\mathbf{I}(M_i)$ and by all permutations on sets of mutually isometric factors M_i. In particular, $\mathbf{I}_0(M) = \mathbf{I}_0(M_0) \times \times \mathbf{I}_0(M_1) \times \ldots \times \mathbf{I}_0(M_t)$.

2. 4. Curvature, characteristic and submanifolds. If S is a two dimensional subspace of a tangentspace M_x to a RIEMANNian manifold M, then in a neighborhood of x the geodesics of M through x tangent to S form a surface; the *sectional curvature of M at (S, x)* is the GAUSSIAN curvature of that surface at x. In a EUCLIDean space, every sectional curvature is zero. In a compact RIEMANNian symmetric space, every sectional curvature is $\geqq 0$. In a noncompact irreducible RIEMANNian symmetric space, every sectional curvature is $\leqq 0$ and some are < 0. In particular, if M is a complete simply connected RIEMANNian symmetric space, then M has every sectional curvature $\geqq 0$ if and only if every irreducible factor of M is compact.

A submanifold of M is *totally geodesic* if and only if every geodesic of the submanifold is a geodesic of M, i.e., if and only if the submanifold contains every geodesic of M to which it is tangent. If X is a totally geodesic submanifold of M, $x \in X$ and S is a two dimensional subspace of X_x, then it is clear that M and X have the same sectional curvature at (S, x). In particular, the sectional curvatures of X satisfy any bounds satisfied by those of M.

The *rank* of a compact LIE group is the common dimension of its maximal toral subgroups. If K is a closed subgroup of a compact LIE group G, then [8] the EULER-POINCARÉ characteristic (in any homology or cohomology theory) $\chi(G/K) \geqq 0$, and $\chi(G/K) > 0$ if and only if rank. $G =$ rank. K.

2. 5. RIEMANNian coverings and locally symmetric spaces. A *RIEMANNian covering* is a covering $\pi : M \to N$ of *connected* RIEMANNian manifolds where π is a local isometry. It is then easily seen that the group Γ of deck transformations of π (homeomorphisms $\gamma : M \to M$ with $\pi = \pi \cdot \gamma$) is a discrete subgroup of $\mathbf{I}(M)$ acting freely and properly discontinuously on M. If M is simply connected, then Γ is identified with the fundamental group $\pi_1(N)$ and N is identified with the quotient space M/Γ. Conversely, if M is a connected RIEMANNian manifold and Γ is a subgroup of $\mathbf{I}(M)$ acting freely and properly discontinuously, then M/Γ admits a unique RIEMANNian structure such that the projection $M \to M/\Gamma$ is a RIEMANNian covering.

A RIEMANNian manifold M is *locally symmetric* if every $x \in M$ has an open neighborhood which, in the induced RIEMANNian structure, admits a symmetry at x. This is the case if M is symmetric, if M is a RIEMANNian covering manifold of a locally symmetric RIEMANNian manifold, or if M admits a RIEMANNian covering by a locally symmetric RIEMANNian manifold. M is complete, connected and locally symmetric, if and only if its universal RIEMANNian covering manifold is symmetric. In particular, *M is a complete connected locally symmetric RIEMANNian manifold with every sectional curvature $\geqq 0$, if and only if the universal RIEMANNian covering manifold of M is the product of a EUCLIDean space and a compact simply connected RIEMANNian symmetric space.* This is the sort of manifold with which we shall concern ourselves here.

3. The structure theorems for locally symmetric spaces

Our main results on the structure of locally symmetric spaces of non-negative curvature are:

3. 1. Topological Structure Theorem. *Let N be a complete connected locally symmetric* Riemannian *manifold with every sectional curvature $\geqq 0$. Then:*

1. *There is a real analytic covering $N' \to N$ of finite multiplicity where N' is the product of a* Euclidean *space, a torus, and a compact simply connected* Riemannian *symmetric space. This covering need not be* Riemannian. *In particular, the fundamental group $\pi_1(N)$ has a free abelian subgroup of finite index.*

2. *There is a real analytic deformation retraction of N onto a compact totally geodesic submanifold which lifts to a deformation retraction of N' onto the product of its toral and compact simply connected factors. In particular the betti numbers of N are finite for singular homology and cohomology, and the* Euler-Poincaré *characteristic $\chi(N)$, alternating sum of the betti numbers, is a well defined integer. We have $\chi(N) \geqq 0$.*

3. *If $\chi(N) \neq 0$, then $\pi_1(N)$ is a finite 2-group (finite of some order 2^a).*

Given the first and second statements above, it is easily seen that $\pi_1(N)$ must be finite when $\chi(N) \neq 0$, but it is a bit surprising that $\pi_1(N)$ must be a 2-group. This comes from an examination of the universal covering of N and the form of the elements of $\pi_1(N)$, and from É. Cartan's determination [4] of the full groups of isometries of symmetric spaces:

3. 2. Geometric Structure Theorem. *Let $M = M_0 \times M_1 \times \ldots \times M_t$ where M_0 is a* Euclidean *space and each $M_i (i > 0)$ is a compact connected simply connected irreducible* Riemannian *symmetric space with $\chi(M_i) > 0$. Let Σ be a group of isometries acting freely on $M_1 \times \ldots \times M_t$, let f be a homomorphism of Σ into the orthogonal group of M_0, and let Γ be the group of isometries of M consisting of all $\gamma = f(\sigma) \times \sigma$. Then Γ is isomorphic to Σ and is a finite 2-group, and M/Γ is a complete connected locally symmetric* Riemannian *manifold with every sectional curvature $\geqq 0$ and* Euler-Poincaré *characteristic $\chi(M/\Gamma) > 0$. If an element of Σ has order 2^{u+1}, then it induces a transformation*

$$(x_1, \ldots, x_m) \to (\tau x_m, x_1, \ldots, x_{m-1})$$

on a product of m distinct mutually isometric factors M_i of M, where either $m = 2^u$ and τ is a fixed point free involutive isometry, or $m = 2^{u-1}$ and (for some $n \geqq 2$) each of these M_i is isometric to the oriented real Grassmann *manifold $\mathbf{SO}(4n)/\mathbf{SO}(2n) \times \mathbf{SO}(2n)$, and τ^2 is a fixed point free involutive isometry.*

Conversely, every complete connected locally symmetric Riemannian *manifold, with all curvatures $\geqq 0$ and nonzero characteristic, is isometric to a manifold M/Γ described above.*

3. 3. Outline of proof. The remainder of § 3 is devoted to proving Theorems 3.1 and 3.2.

We identify $\pi_1(N)$ with the group Γ of deck transformations of the universal RIEMANNian covering $M \to N$. To obtain the finite covering and the retraction of N, we find a free abelian subgroup Δ of finite index in Γ and submit M, Δ and Γ to various deformations. The existence of Δ (§ 3.4) is due to L. AUSLANDER. The deformations of Δ (§ 3.5) are done in sufficient generality for their applications in § 4 as well as in § 3. It is then (§ 3.8) proved that, if $\chi(N) \neq 0$, then $\chi(N) > 0$, $\pi_1(N)$ is finite, and the converse of Theorem 3.2 holds; this is done by combining the retraction and the finite covering. It then suffices to prove that Σ is a 2-group whose elements induce the transformations given; this is done in §§ 3.10–3.11, and is based on a theorem (§ 3.9) that τ^4 has a fixed point if τ is an isometry of an M_i.

3. 4. The free abelian subgroup of finite index in $\pi_1(N)$ will be exhibited as a consequence of a result of L. AUSLANDER ([2], Th. 3) which requires some interpretation. The precise statement, slightly sharpened, is:

Proposition (L. AUSLANDER ([2], Th. 3)). *Let D be a discrete subgroup of a semi-direct product $H \cdot C$, where H is a connected simply connected nilpotent* LIE *group acted upon (by automorphisms, but not necessarily effectively) by a compact* LIE *group C. Then $D^* = D \cap (\overline{DH})_0$ is a subgroup of finite index in D, and $D^* = A \times B$ where A is a finite abelian group and B is isomorphic to a discrete subgroup with compact quotient in some connected subgroup H^* of H.*

Proof. The first two paragraphs of L. AUSLANDER's proof ([2], pp. 279–280) show that, after conjugation by an element of H, $D^* \subset W \cdot T$ where W is a connected subgroup of H and T is a torus in C which centralizes W. For the sharpening, we replace the third paragraph of a slight variant. D^* is finitely generated because it is discrete in the connected solvable group $W \cdot T$ ([5], Th. 1′), so $D^*/[D^*, D^*]$ is a finitely generated abelian group. Thus $D^*/[D^*, D^*] = A' \times B'$ where A' is the torsion subgroup. $[D^*, D^*] \subset W$ because T is abelian and centralizes W; thus the projection

$$f: D^* \to D^*/[D^*, D^*] \text{ maps } A = D \cap T$$

isomorphically onto A'; it follows that $D = A \times B$ where $B = f^{-1}(B')$. Now let $g: W \cdot T \to W$ be the projection and define H^* to be the smallest analytic subgroup of W which contains $g(B)$. g maps B isomorphically onto $g(B)$, $g(B)$ is discrete in H^* because T is compact, and it is standard that $H^*/g(B)$ is compact. *Q.E.D.*

3. 5. The deformation of the free abelian subgroup and the corresponding quotient manifold is given by:

Deformation Theorem. *Let $G = S \cdot C$ be a semidirect product of* Lie *groups, let D be a torsion free subgroup of G with generating set $\{d_1, \ldots, d_n\}$ such that, given $d \in D$, there is a unique set $\{u_i\}$ of integers such that $d = d_1^{u_1} d_2^{u_2} \ldots d_n^{u_n}$; suppose that D acts freely and properly discontinuously on G/C by $d : gC \to dgC$, and assume that the projection of D into C lies in a torus A which centralizes D. Write $d_i = s_i a_i$ with $s_i \in S$ and $a_i \in A$, choose elements X_i in the* Lie *algebra of A such that $a_i = \exp(X_i)$, define $d_i^{(t)} = d_i \cdot \exp(-tX_i) = s_i \cdot \exp((1-t)X_i)$, and let $D^{(t)}$ be the group generated by $\{d_i^{(t)}\}$. Then*

1. *$D^{(0)} = D$ and $D^{(1)} \subset S$.*

2. *If K is a closed subgroup of C, so $P = G/K$ is an analytic manifold on which G acts by $g : hK \to ghK$, then each $D^{(t)}$ acts freely and properly discontinuously on P; in particular, the projections $P \to P/D^{(t)}$ are coverings of analytic manifolds.*

3. *The maps $d_i^{(r)} \to d_i^{(s)}$ define isomorphisms (the "deformation isomorphisms") of $D^{(r)}$ onto $D^{(s)}$, and these isomorphisms induce analytic homeomorphisms of $P/D^{(r)}$ onto $P/D^{(s)}$.*

4. *P/D is analytically homeomorphic to $(S/D^{(1)}) \times (C/K)$.*

Proof. The first statement is obvious. If $\mathcal{R}$ is a group relation and

$$\mathcal{R}(s_1, \ldots, s_n) = 1, \quad \text{then} \quad \mathcal{R}(d_1, \ldots, d_n) \in A$$

because A is abelian, A contains the a_i, and A centralizes the s_i. But

$$D \cap A \subset D \cap C = \{1\}$$

because D acts freely on G/C; thus $\mathcal{R}(d_1, \ldots, d_n) = 1$. This shows that the d_i satisfy every relation satisfied by the s_i; it follows that $d_i^{(t)} \to d_i$ induces a homomorphism of $D^{(t)}$ onto D. Every element of $D^{(t)}$ has some expression $(d_1^{(t)})^{u_1} (d_2^{(t)})^{u_2} \ldots (d_n^{(t)})^{u_n}$ because the s_i satisfy every relation satisfied by the d_i, and every element of D has unique expression $d_1^{u_1} d_2^{u_2} \ldots d_n^{u_n}$; it follows that the epimorphism $D^{(t)} \to D$ is an isomorphism. This gives the deformation isomorphisms.

For the second statement, we note that $D^{(t)} \subset G$ acts freely and properly discontinuously on G/C, if and only if $D^{(t)} \subset S \cdot A$ acts freely and properly discontinuously on $(S \cdot A)/A$. As A is compact, and as $D^{(t)}$ is discrete (because $D^{(1)}$ is discrete in S, consequence of proper discontinuity of D on G/C) and torsionfree (because it is isomorphic to the torsionfree group D), $D^{(t)}$ must be free and properly discontinuous on $(S \cdot A)/A$. This proves the second statement; the third follows because the deformations are along analytic arcs.

For the last statement, view $P/D^{(t)}$ as the double coset space $D^{(t)} \backslash G/K$. Writing $\cong$ for analytic homeomorphism, we then have

$$P/D \cong P/D^{(1)} \cong (D^{(1)} \backslash S) \cdot (C/K) .$$

Now observe that $(s, c) \to sc$ induces $S \times C \cong G \cong S \cdot C$ and $s \to s^{-1}$ induces $D^{(1)} \backslash S \cong S/D^{(1)}$; it follows that $P/D \cong (S/D^{(1)}) \times (C/K)$. *Q.E.D.*

3. 6. The finite covering. Identify $\pi_1(N)$ with the group Γ of deck transformations of the universal RIEMANNian covering $M \to N$. $M = M_0 \times M'$ where M_0 is a EUCLIDean space and M' is a product of irreducible RIEMANNian symmetric spaces, for N is complete, connected and locally symmetric. As N has every sectional curvature $\geqq 0$, the same is true for M'; it follows that M' is compact because a noncompact irreducible RIEMANNian symmetric space has a negative sectional curvature. In particular, the full group of isometries $\mathbf{I}(M')$ is compact.

$\mathbf{I}(M_0)$ is the ordinary EUCLIDean group on $n = \dim. M_0$ variables, and may be viewed as a semidirect product $\mathbf{R}^n \cdot \mathbf{O}(n)$ where $\mathbf{R}^n$ is the vector group and $\mathbf{O}(n)$ is the orthogonal group. This allows us to view $\mathbf{I}(M) = \mathbf{I}(M_0) \times \mathbf{I}(M')$ as a semidirect product $\mathbf{R}^n \cdot C$ where $C = \mathbf{O}(n) \times \mathbf{I}(M')$. As Γ is a discrete subgroup of $\mathbf{I}(M)$, Proposition 3.4 gives a finitely generated free abelian subgroup Δ of finite index in Γ corresponding to the group B there.

By construction, the projection of Δ into C lies in a torus. The condition of Theorem 3.5 for expression of elements in terms of generators is obvious for finitely generated free abelian groups. Δ acts freely and properly discontinuously on $\mathbf{I}(M)/C$ because C is compact and Δ is discrete and torsion free. Now Theorem 3.5 shows that M/Δ is real analytically homeomorphic to $(M_0/\Delta') \times M'$ where Δ' is a discrete group of pure translations of M_0 which is isomorphic to Δ. Define $N' = (M_0/\Delta') \times M'$ and recall that $M/\Delta \to M/\Gamma = \mathrm{N}$ is a finite RIEMANNian covering. This proves the first statement of Theorem 3.1.

3. 7. The deformation retraction of N onto a compact submanifold is accomplished by a deformation of Γ onto another group Γ', followed by a Γ'-equivariant deformation retraction of M. We retain the notation Γ, M, M', M_0 and Δ from § 3.6, except that we may replace Δ by the intersection of its conjugates in Γ, and thus assume Δ normal in Γ.

Every $\gamma \in \Gamma$ is of the form $\gamma_0 \times \gamma'$ where $\gamma_0 \in \mathbf{I}(M_0)$ and $\gamma' \in \mathbf{I}(M')$. For a choice of origin in M_0, γ_0 is further decomposed into (γ_t, γ_r) where $\gamma_t \in M_0$ indicates a translation and γ_r is a rotation. By construction of Δ, we may choose the origin so that $\delta_r : \delta_t \to \delta_t$ for every $\delta \in \Delta$. The origin so chosen, M_0 is identified with the vectorspace $\mathbf{R}^n$, and we have an orthogonal direct sum decomposition $M_0 = U + V$ where V is the subspace spanned by the δ_t. Every γ_r preserves V, and thus preserves U, because Δ is normal in Γ.

Given $\gamma \in \Gamma$, we have $\gamma_t = \gamma_U + \gamma_V$ with $\gamma_U \in U$ and $\gamma_V \in V$. If s is a real number, define $\gamma^{(s)} = (s\gamma_U + \gamma_V, \gamma_r) \times \gamma'$ and let Γ_s be the subgroup of $\mathbf{I}(M)$ generated by the $\gamma^{(s)}$. It is easily checked that $\gamma \to \gamma^{(s)}$ defines an isomorphism of Γ onto Γ_s. Γ_s is discrete in $\mathbf{I}(M)$ because it contains Δ as a subgroup of finite index; thus Γ_s acts properly discontinuously on M. Now if $\gamma^{(s)}$ has a fixed point, it must have finite order, whence γ has finite order; it follows that either $\gamma = 1$ or γ' has no fixed point; $\gamma^{(s)} = 1$ because the latter would prevent $\gamma^{(s)}$ from having a fixed point. Thus Γ_s acts freely on M. We now have a one parameter family of manifolds $N_s = M/\Gamma_s$ which are analytically homeomorphic to $N = N_1$. It will be clear that this isotopy of the metric of N is the identity on a compact totally geodesic submanifold onto which N_0 is retracted. For the proof of Theorem 3.1, then, we may replace N by N_0. In other words, we may assume each $\gamma_t \in V$.

We have M as a RIEMANNian product $U \times V \times M'$ where U and V are EUCLIDean spaces with vectorspace structure, and every $\gamma \in \Gamma$ is of the form $\gamma_1 \times \gamma_2 \times \gamma'$ where γ_1 is a rotation of U, γ_2 is an isometry of V, and γ' is an isometry of M'. Define $f_s: M \to M$ by $f_s(u, v, m') = (su, v, m')$; f_s is Γ-equivariant because each γ_1 is a linear transformation. Thus f_s induces a map $g_s: N \to N$. This gives a deformation retraction of $N = g_1(N)$ onto $g_0(N)$. But $g_0(N) = f_0(M)/\Gamma = (V \times M')/\Gamma$ admits a covering by $(V \times \times M')/\Delta$, and, as in § 3. 7., Theorem 3.5 shows that $(V \times M')/\Delta$ is homeomorphic to $(V/\Delta') \times M'$ where Δ' is the group of translations consisting of the δ_t. V/Δ' is a torus, compact by definition of V; thus $g_0(N)$ is compact.

We have now exhibited a deformation retraction of N onto a compact submanifold. As singular homology and cohomology satisfy the homotopy axiom, the betti numbers of N are finite, and the EULER-POINCARÉ characteristic $\chi(N)$ is a well defined integer, in those theories.

Observe that the deformation of Γ did not move any points of $g_0(N)$. It is now clear that $g_0(N)$ is totally geodesic in N, for it is the image of a totally geodesic submanifold $V \times M'$ of M.

3. 8. Finiteness of the fundamental group. We have seen that the deformation retraction $g_0(N)$ admits a covering of some finite multiplicity r by $T \times M'$, where T is a torus with $\pi_1(T)$ isomorphic to the subgroup Δ of finite index in Γ. As $g_0(N)$, T and M' are compact manifolds, we have

$$\chi(N) = \chi(g_0(N)) = \frac{1}{r}\chi(T \times M') = \frac{1}{r}\chi(T) \cdot \chi(M').$$

Now suppose $\chi(N) \neq 0$. Then $\chi(T) \neq 0 \neq \chi(M')$. $\chi(T) \neq 0$ means that T is a single point, so $\chi(N) = \frac{1}{r}\chi(M')$ and $\Delta = \{1\}$. As Δ has finite index in Γ, $\Gamma = \pi_1(N)$ must be finite. Now $\chi(M') \neq 0$ implies $\chi(M') > 0$ because M' is a quotient space of a compact LIE group $\mathbf{I}(M')$ by a closed subgroup [8]; thus $\chi(N) > 0$.

The second statement, and the finiteness assertion of the third statement, of Theorem 3.1 are now proved.

Suppose again that $\chi(N) \neq 0$. As Γ is finite, the γ_0 of § 3.7 form a finite group; it is classical that some point of M_0 must be fixed under every γ_0. Changing the origin in M_0, Γ is the group of isometries of M consisting of all $f(\sigma) \times \sigma$, as σ runs through a finite group Σ of isometries acting freely on M', where f is a homomorphism $\gamma' \to \gamma_0$ of Σ into the orthogonal group of M_0. Now $\chi(M') \neq 0$, so $\chi(M_i) \neq 0$ $(i > 0)$ where $M' = M_1 \times \ldots \times M_t$ is the decomposition of M' into irreducible factors. It follows that $\chi(M_i) > 0$ [8] and every group of isometries acting freely on M' is finite [11]. This proves the converse and finiteness condition of Theorem 3.2, and that the manifold M/Γ there is a complete connected locally symmetric RIEMANNian manifold with every sectional curvature $\geqq 0$ and $\chi(M/\Gamma) > 0$.

To complete the proofs of Theorems 3.1 and 3.2, now, we need only prove that every element of the group Σ of Theorem 3.2 has some order 2^{u+1} and induces a transformation of the type exhibited there.

3.9. In order to study the elements of Σ, we need some information on fixed points:

Fixed Point Theorem. *Let τ be an isometry of a compact connected simply connected irreducible RIEMANNian symmetric space S with $\chi(S) \neq 0$. If τ^2 has no fixed point, then S is isometric to a real GRASSMANN manifold* $\mathbf{SO}(4k)/\mathbf{SO}(2k) \times \mathbf{SO}(2k)$, $k \geqq 2$, *and τ^4 has a fixed point.*

Proof. Let K be an isotropy subgroup of $G = \mathbf{I}(S)$. The identity component K_0 contains a maximal torus of G_0 because $\chi(G_0/K_0) = \chi(S) \neq 0$, by SAMELSON's theorem [8], so every element of G_0 is conjugate to an element of K_0. In other words, every element of G_0 has a fixed point. Let t be the image of τ in G/G_0; τ^m has a fixed point if $t^m = 1$.

Suppose that τ^2 has no fixed point. Then G/G_0 has an element of order greater than 2. It follows from CARTAN's construction of $\mathbf{I}(S)$ [4] that

$$S = \mathbf{SO}(4k)/\mathbf{SO}(2k) \times \mathbf{SO}(2k)$$

where $k \geqq 2$; if, further, G/G_0 has an element u with $u^4 \neq 1$, then $k = 2$ and u has order 3. But if $t^3 = 1$, then $\tau^3 \in \mathbf{I}_0(S) = G_0$, so τ^3 is homotopic to the identity. It is known that τ must be fixed point in this case ([14], §§ 5.5.9–5.5.10), so τ^2 has a fixed point. This contradicts $t^3 = 1$. The only other possibility is that $t^4 = 1$ and τ^4 has a fixed point. *Q.E.D.*

3.10. 2-groups. We will see that Γ and Σ are 2-groups.

If g is an isometry of $M' = M_1 \times \ldots \times M_t$, then we have decompositions

$$M' = X_1 \times \ldots \times X_u, \; g = g_1 \times \ldots \times g_u$$

where g_i is an isometry of X_i which cyclically permutes its irreducible factors. Thus, under appropriate isometric identifications, we have $X_i = S_i \times \ldots \times S_i$ (v_i factors) with S_i irreducible, and

$$g_i : (s_1, \ldots, s_{vi}) \to (\tau_i s_{vi}, s_1, \ldots, s_{vi-1})$$

gives the action of g_i on X_i, for some isometry τ_i of S_i. Now if g has order m, then each v_i must divide m, say $m = v_i m_i$; g^{vi} induces the transformation $\tau_i \times \ldots \times \tau_i$ on X_i, and $\tau_i^{m_i} = 1$.

Suppose now that g has odd order m, and retain the notation above. Each τ_i must have odd order, so τ_i is a power of τ_i^4. As $\chi(M') \neq 0$, we have $\chi(S_i) \neq 0$, and Theorem 3.9 shows that each τ_i has a fixed point $s_i \in S_i$. Define $x_i = (s_i, s_i, \ldots, s_i) \in X_i$; then $g_i(x_i) = x_i$. It follows that

$$x = (x_1, x_2, \ldots, x_u)$$

is a fixed point for g.

If Γ is not a 2-group, then it has an element γ of odd order $m > 1$. $\gamma = f(\sigma) \times \sigma$ where σ has order m in Σ. The considerations above show that σ has a fixed point, contradicting the hypothesis that Σ act freely on M'.

This proves that Γ and Σ are 2-groups.

3.11. The form of the group elements now comes easily. Let $1 \neq g \in \Sigma$. Then g has some order $m = 2^{u+1}$ $u \geqq 0$. Retain the notation of § 3.10 for the decompositions of M' and g. Then $m_i = 2^{ai}$ and $v_i = 2^{bi}$ where $a_i + b_i = u + 1$. As g^{2^u} has no fixed point, some $g_i^{2^u}$ has no fixed point. For this index i, it is easily seen that $b_i \leqq u$, say $u = b_i + w$, whence $g_i^{2^u} = \tau_i^{2^w} \times \ldots \times \tau_i^{2^w}$. It follows that τ_i^k has a fixed point on S_i if and only if k is a multiple of 2^{w+1}; by Theorem 3.9, $w = 0$ or 1, and S_i is isometric to $\mathbf{SO}(4n)/\mathbf{SO}(2n) \times \mathbf{SO}(2n)$ $(n \geqq 2)$ in case $w = 1$.

This completes the proof of Theorems 3.1 and 3.2. *Q.E.D.*

4. Riemannian nilmanifolds and a structure theorem for locally homogeneous spaces

The proof of some parts of Theorem 3.1 do not make full use of the hypotheses. We will prove the following extension to locally homogeneous spaces.

4.1. Theorem. *Let $M \to N$ be a universal Riemannian covering where $M = M_0 \times M'$, a nilpotent Lie group acts transitively by isometries on M_0, and M' is a compact Riemannian homogeneous manifold. Then there is a real analytic covering $N' \to N$ of some finite multiplicity $r > 0$ where $N' = E \times N'' \times M'$, N'' is a compact coset space of a nilpotent Lie group by a* *discrete subgroup, and* E *is diffeomorphic to a* EUCLIDean *space; if* M_0 *is isometric to a* EUCLIDean *space, then* N'' *is a torus and there is a real analytic deformation retraction of* N *onto a compact totally geodesic submanifold which lifts to one of the deformation retractions of* N' *onto* $N'' \times M'$. *In particular, the* EULER-POINCARÉ *characteristic* $\chi(N')$ *of singular theory is a well defined integer; now* $\chi^*(N) = \frac{1}{r}\chi(N')$, *the so called rational* EULER-POINCARÉ *characteristic of* N, *is a well defined non-negative integer, and* $\chi^*(N) = \chi(N)$ *if* M_0 *is* EUCLIDean. *If* $\chi^*(N) \neq 0$, *then the fundamental group* $\pi_1(N)$ *is finite. If* $\chi^*(N) \neq 0$ *and* M' *is* RIEMANNian *symmetric, then* $\pi_1(N)$ *is a finite 2-group.*

I am indebted to DAVID B. A. EPSTEIN for drawing my attention to C.T.C. WALL's rational EULER characteristic [10] and for suggesting a way of adapting it to this context. § 4.4 is based on conversations with him.

4.2. RIEMANNian nilmanifolds are defined to be RIEMANNian manifolds which admit a transitive nilpotent group of isometries. The structure of M_0 is clarified by:

Theorem. *Let* B *be a positive definite bilinear form on the* LIE *algebra* $\mathfrak{S}$ *of a connected nilpotent* LIE *group* S, *let* K *be the group of all automorphisms of* S *which preserve* B, *and let* X *be* S *with the left invariant* RIEMANNian *metric derived from* B. *Then* X *is a connected* RIEMANNian *nilmanifold,* $\mathbf{I}(X)$ *is the semidirect product* $S \cdot K$ *acting by* $(s, k): x \to s \cdot k(x)$, S *is the nilradical (maximal connected normal nilpotent subgroup) of* $\mathbf{I}_0(X)$, S is a maximal connected nilpotent subgroup of $\mathbf{I}_0(X)$, *and* S *is the only transitive connected nilpotent subgroup of* $\mathbf{I}(X)$. *Conversely, every connected* RIEMANNian *nilmanifold is isometric to one of the manifolds* X *described above.*

Corollary. *If* X *is a connected* RIEMANNian *nilmanifold and* $x \in X$, *then the* RIEMANNian *structure on* X *defines a unique structure of nilpotent* LIE *group in which* $x = 1$.

Corollary. *Let* $\pi: Y \to X$ *be a* RIEMANNian *covering where* X *is a* RIEMANNian *nilmanifold, and let* $y \in Y$. *Then* Y *is a* RIEMANNian *nilmanifold. Endow* Y *(resp.* X*) with its canonical nilpotent* LIE *group structure for which* $y = 1$ *(resp.* $\pi(y) = 1$*). Then* π *is an epimorphism of* LIE *groups, and the deck transformations of* π *are left translations by the elements of the kernel of* π.

Corollary. *Let* Γ *be the group of deck transformations of a universal* RIEMANNian *covering* $X \to Y$ *where* X *is a* RIEMANNian *nilmanifold. Then these are equivalent:*

1. *Y is a* RIEMANNian *nilmanifold.*
2. *Y is a* RIEMANNian *homogeneous manifold.*

3. *Γ consists of isometries of constant displacement.*
4. *Γ consists of isometries of bounded displacement.*

Corollary. *Let $\pi : X \to Z$ be a RIEMANNian covering where Z is compact and X is a RIEMANNian nilmanifold. Then π factors into RIEMANNian coverings $\alpha : X \to Y$ and $\beta : Y \to Z$ where Y is a compact nilmanifold and β is of finite multiplicity; Y is a RIEMANNian nilmanifold if and only if it is isometric to a flat torus.*

We complete § 4.2 by deriving the Corollaries from the Theorem; the Theorem will be proved in § 4.3, and we will then go on to the proof of Theorem 4.1.

The first Corollary is clear because $S \subset \mathbf{I}(X)$ is unique and acts simply transitively on X in the Theorem. For the second, we give X its LIE structure with $\pi(y) = 1$, let $S \subset \mathbf{I}(X)$ denote the left translations, and lift the action of S to Y after backing off to the universal covering group of S.

The third Corollary is a little more complicated. It is clear that (1) implies (2) and that (3) implies (4), and it is known [12] that (2) implies (3); thus we need only prove that (4) implies (1). Choose $x \in X$ and give X the nilpotent LIE group structure S in which $x = 1$. In the notation of the Theorem, we must prove every element of Γ to be central in S; then S induces a transitive nilpotent group of isometries of Y, and (1) is proved.

Let $g \in \mathbf{I}(X)$ be an isometry of bounded displacement, $g = (s, k)$ with $s \in S$ and $k \in K$ in the notation of the Theorem. As K is compact, there is a compact set $C \subset \mathbf{I}(X)$ with $hgh^{-1} \in C$ for every $h \in \mathbf{I}(X)$; $h = (t^{-1}, 1)$ gives $(t^{-1} \cdot s \cdot k(t), k) \in C$, and it follows that S has a compact set which contains $t^{-1} \cdot k(t)$ for every $t \in S$. The exponential map $\exp : \mathfrak{S} \to S$ is a homeomorphism and k is linear on $\mathfrak{S}$; it follows that $k = 1$ because the linear isotropy representation of K is faithful. Now $g = (s, 1)$. Every $(tst^{-1}, 1) \in C$, so the closure of the conjugacy class of s in S is compact. Let T be the centralizer of s in S; now S/T is compact. Let $P \in \mathfrak{S}$ with $\exp(P) = s$, and suppose $Q \in \mathfrak{S}$; it is easily seen that $[P, Q] = 0$ if and only if s commutes with $\exp(Q)$; thus T is connected. It follows that S/T is homeomorphic to a EUCLIDean space. As S/T is compact, we must have $S = T$; thus s is central in S. This completes the proof of the third Corollary.

For the fourth Corollary, let Σ be the group of deck transformations of the universal RIEMANNian covering $\mu : W \to Z$ and let Δ be the deck transformations of $\nu : W \to X$. L. AUSLANDER has proved [1] that $\Gamma = \Sigma \cap S_W$ has finite index in Σ; as $\Delta \subset \Sigma$ and we have just seen $\Delta \subset S_W$ (for Δ is central in S_W), we have $\Delta \subset \Gamma$. Now define $Y = W/\Gamma$, and the existence of α and β is clear. If Y is a RIEMANNian nilmanifold, then $Y = S_W/\Gamma$ is a group, and so S_W/Γ is a torus. This proves the fourth Corollary.

4. 3. Proof of Theorem 4. 2. Let W be a connected RIEMANNian nilmanifold. There is a transitive nilpotent group of isometries of W; its identity component T is transitive. T^* will denote the closure of T in $\mathbf{I}(W)$; T^* is nilpotent and transitive. Write W as a coset space T^*/Z where Z is the isotropy subgroup of T^* at $w \in W$. Z is compact because T^* is closed in $\mathbf{I}(W)$; thus Z is contained in a maximal compact subgroup Z^* of T^*. Z^* is connected because T^* is connected, and a compact connected subgroup of a nilpotent LIE group can be seen to lie in the center by looking at the universal covering group and its exponential mapping; thus Z is central in T^*. T^* acts effectively on W; it follows that $Z = \{1\}$ and T^* is simply transitive. As $T \subset T^*$ and T is transitive, this proves that T is closed in $\mathbf{I}(W)$ and simply transitive on W; it also proves that T is maximal among the connected nilpotent subgroups of $\mathbf{I}_0(W)$.

Suppose that we can prove T to be contained in the nilradical N of $\mathbf{I}_0(W)$. Then $T = N$, so T is normal in $\mathbf{I}(W)$. If H is the isotropy subgroup at $w \in W$, then $H \cap T = \{1\}$ because T is simply transitive, so $\mathbf{I}(W)$ is a semidirect product $T \cdot H$. The representation of H on the LIE algebra $\mathfrak{T}$ is equivalent to the linear isotropy representation of H on the tangentspace W_w, and is thus faithful; now H may be viewed as a group of automorphisms of T. Identify T with W, viewing T as a LIE group with left invariant RIEMANNian metric specified by some positive definite bilinear form A on $\mathfrak{T}$. Then H preserves A, and must contain every automorphism of T which preserves A because it contains every isometry of W which fixes w. Writing $\mathbf{I}(W) = T \cdot H$, now, the action on T is necessarily $(t, h) : v \to t \cdot h(v)$. As the manifold X of Theorem 4.2 is a RIEMANNian nilmanifold under the group S there, this will prove Theorem 4.2.

We now need only prove $T \subset N$ where N is the nilradical of $\mathbf{I}_0(W)$. Let $\pi : W' \to W$ be the universal RIEMANNian covering; we can lift the action of T on W to the action of a covering group T' of T on W', and T' will be transitive on W'. Let Γ be the group of deck transformations of the covering, let N' be the nilradical of $\mathbf{I}_0(W')$, and let P be the normalizer of Γ in $\mathbf{I}(W')$. π induces a homomorphism π^* of P onto $\mathbf{I}(W)$ with kernel Γ, and $T' \subset P$ by construction. If $T' \subset N'$, then $T' \subset P \cap N'$, and the latter lies in the nilradical N'' of P. It is clear that $\pi^*(N'') = N$ and $\pi^*(T') = T$; it will follow that $T \subset N$.

Now we assume W simply connected, and need only prove $T \subset N$. Let R be the radical (maximal connected normal solvable subgroup) of $\mathbf{I}_0(W)$. Then $\mathbf{I}_0(W) = S \cdot R$ where S is a maximal connected semisimple subgroup. Let $\beta : \mathbf{I}_0(W) \to ad(S)$ be the composition of taking quotient by R with the adjoint representation of $S/S \cap R$. Every element $g \in \mathbf{I}(W)$ has unique and continuous decomposition $g = th$, $t \in T$ and $h \in H$ = isotropy at w; thus $\mathbf{I}_0(W)/T$ is compact; it follows that $\mathrm{ad}(S)/\beta(T)$ is compact. As $\beta(T)$ is nilpotent and $\mathrm{ad}(S)$ is a product of centerless simple LIE groups, $\mathrm{ad}(S)$ must be compact. This proves that S is compact.

The identity component H_0 is an isotropy subgroup and a maximal compact subgroup of $\mathbf{I}_0(W)$; thus $H_0 = S \cdot H'$ where $H' = (H \cap R)_0$ is the identity component of the center of H_0. Let $\beta: \mathbf{I}_0(W) \to \mathbf{I}_0(W)/N = U$. $N \cap H$ is a compact subgroup of N and is thus in a maximal compact subgroup of N; this maximal one is central in N, thus unique, and thus central in $\mathbf{I}_0(W)$; it follows that $N \cap H = \{1\}$ so $U = SR'$ where $R' = R/N$. R' is abelian because $[R, R]$ is nilpotent and normal and thus in N; it follows that $R' = H' \times V$ where V is a vector group stable under S. Let $M = \beta^{-1}(V)$. M is a closed normal subgroup of $\mathbf{I}_0(W)$ such that $\mathbf{I}_0(W)/M$ is compact and $\mathbf{I}_0(W)$ is semidirect product $M \cdot H_0$. Thus $\dim. M = \dim. \mathbf{I}_0(W) - \dim. H_0 = \dim. T$. Let

$$\alpha : \mathbf{I}_0(W) \to \mathbf{I}_0(W)/M .$$

TM is closed in $\mathbf{I}_0(W)$ because T and M are closed and M is normal. Thus $\alpha(T) = (TM)/M = T/T \cap M$ is a torus. On the other hand, $T \cap M$ is connected because it is an analytic subgroup of T. As T is connected, simply connected and nilpotent, it follows that $\alpha(T) = T/T \cap M$ is homeomorphic to a EUCLIDean space. Thus $\alpha(T) = \{1\}$. This proves $T \subset M$. As they are connected groups of the same dimension, they must be equal. In particular, T is normal in $\mathbf{I}_0(W)$. This proves $T \subset N$, completing the proof of Theorem 4.2. *Q.E.D.*

Remark. Theorem 4.2 shows that the notion of RIEMANNian nilmanifold is but a mild generalization of the notion of RIEMANNian homogeneous manifold of constant curvature zero. The essential part of the proof was exhibiting of V above. This was essentially done by reducing to the case of constant zero curvature.

Remark. One might define a RIEMANNian solvmanifold to be a RIEMANNian manifold which admits a transitive solvable group of isometries, but the IWASAWA decomposition shows that this notion is not very restrictive. For example, a RIEMANNian symmetric space with every sectional curvature $\leqq 0$ is a RIEMANNian solvmanifold.

4.4. Rational EULER-POINCARÉ characteristic. All spaces are connected, locally arcwise connected, locally simply connected, and with a basepoint which will generally not be mentioned. Let $\mathcal{C}$ be the family of finite CW complexes, $\mathcal{C}'$ the family of spaces homotopy equivalent (respecting basepoints) to an element of $\mathcal{C}$, and $\mathcal{C}^*$ the family of spaces which admit a finite covering by an element of $\mathcal{C}'$. Given $X \in \mathcal{C}'$, we have the EULER-POINCARÉ characteristic (of singular theory) $\chi(X) = \chi(Y)$ where $X \simeq Y \in \mathcal{C}$.

Proposition. *If $Z \in \mathcal{C}^*$, so Z admits a covering of some finite multiplicity $r > 0$ by some $X \in \mathcal{C}'$, then $\chi^*(Z) = \frac{1}{r}\chi(X)$ is a well defined rational number, which we will call the rational EULER-POINCARÉ characteristic of Z. If Z_1 and $Z_2 \in \mathcal{C}^*$, then $\chi^*(Z_1 \times Z_2) = \chi^*(Z_1)\chi^*(Z_2)$. If $Z_1 \in \mathcal{C}^*$ admits a t-fold covering by a space Z_2, then $Z_2 \in \mathcal{C}^*$ and $\chi^*(Z_2) = t\chi^*(Z_1)$.*

The main step in the proof is:

Lemma. *Given a finite covering $g: (U, u) \to (X, x)$ and a homotopy equivalence $h: (X, x) \to (Y, y)$ of spaces with basepoint, let $a: (V, v) \to (Y, y)$ be the covering with $a\pi_1(V, v) = hg\pi_1(U, u)$. Then there is a homotopy equivalence $b: (U, u) \to (V, v)$ which covers h.*

To prove the Lemma, one defines b by $b(u) = v$ and by defining b to cover h along any arc starting at u which is the lift of an arc starting at x; b is well defined because of the condition on fundamental groups. Let $h': (Y, y) \to (X, x$ be a homotopy inverse to h, and let $b': (V, v) \to (U, u)$ be the map covering h', defined from h' as b was defined from h; it is easily seen that b' is a homotopy inverse to b.

Proof of Proposition. To see that $\chi^*(Z)$ is well defined, choose $z \in Z$ and r_i-fold coverings $f_i: (X_i, x_i) \to (Z, z), X_i \in \mathcal{C}'$; we must prove $\frac{1}{r_1}\chi(X_1) = \frac{1}{r_2}\chi(X_2)$. $S_i = f_i\pi_1(X_i, x_i)$ is a subgroup of finite index r_i in $\pi_1(Z, z)$; thus $S = S_1 \cap S_2$ is a subgroup of some finite index $s_1r_1 = s_2r_2$ in $\pi_1(Z, z)$. This gives s_i-fold coverings $g_i: (U_i, u_i) \to (X_i, x_1)$ with $f_ig_i\pi_1(U_i, u_i) = S$. We have homotopy equivalences $h_i: (X_i, x_i) \to (Y_i, y_i)$ with $Y_i \in \mathcal{C}$; if $a_i: (V_i, v_i) \to (Y_i, y_i)$ are the s_i-fold coverings with $a_i\pi_1(V_i, v_i) = h_ig_i\pi_1(U_i, u_i)$, then it is obvious that $V_i \in \mathcal{C}$, and the Lemma gives homotopy equivalences $b_i: (U_i, u_i) \to (V_i, v_i)$. Thus $U_i \in \mathcal{C}'$ and $\chi(U_i) = s_i\chi(X_i)$. Now $f_ig_i: (U_i, u_i) \to (Z, z)$ are coverings with $f_ig_i\pi_1(U_i, u_i) = S$; thus U_1 is homeomorphic to U_2; it follows that $s_1\chi(X_1) = s_2\chi(X_2)$. Dividing by $r_1s_1 = r_2s_2$, we have $\frac{1}{r_1}\chi(X_1) = \frac{1}{r_2}\chi(X_2)$, and $\chi^*(Z)$ is well defined. The other statements follow easily from the corresponding statements in $\mathcal{C}$, but we must use the Lemma to prove $Z_2 \in \mathcal{C}^*$ in the last statement. *Q. E. D.*

4. 5. Proof of Theorem 4. 1. Let Γ be the group of deck transformations of the universal RIEMANNian covering $M = M_0 \times M' \to N$ of Theorem 4.1.

M_0 is a connected simply connected RIEMANNian nilmanifold; the same statement follows for each of its irreducible factors, so these irreducible factors are homeomorphic to EUCLIDean spaces by Theorem 4.2. As M' is compact, none of its irreducible factors can be isometric to an irreducible factor of M_0. Thus $\mathbf{I}(M) = \mathbf{I}(M_0) \times \mathbf{I}(M')$. Theorem 4.2 shows that $\mathbf{I}(M_0)$ is a semidirect product $S \cdot K$ where S is a connected simply connected nilpotent LIE group and K is compact. This allows us to view $\mathbf{I}(M)$ as a semidirect product $S \cdot C$ where $C = K \times \mathbf{I}(M')$ is compact. Proposition 3.4 now provides a torsionfree subgroup $\varDelta$ of finite index in $\varGamma$, an analytic subgroup $S' \subset S$, and a toral subgroup $T \subset C$ which centralizes S', such that $\varDelta \subset S' \cdot T$ and $S'/\varDelta'$ is compact where $\varDelta'$ is the projection of $\varDelta$ on S'.

We now need

Lemma. *Let D be a discrete subgroup of a connected simply connected nilpotent* LIE *group U with U/D compact. Then D is torsionfree and has a generating set $\{d_1, \ldots, d_n\}$ such that, given $d \in D$, there is a unique set $\{v_i\}$ of integers with $d = d_1^{v_1} d_2^{v_2} \ldots d_n^{v_n}$.*

Proof of Lemma. D is torsionfree because U is torsionfree. Let r be the length of the lower central series of U; let Z be the center of U. $D \cap Z$ is the center of D because an automorphism of U is trivial if and only if it is trivial on D. As D is discrete, it follows that DZ is closed, so the image of Z in U/D is closed, whence $Z/(D \cap Z)$ is compact. Let $\{d_1, \ldots, d_a\}$ generate the free abelian group $D \cap Z$. By induction on r, we have a generating set $\{d'_{a+1}, \ldots, d'_n\}$ of the requisite sort for the group $D/(D \cap Z)$ in U/Z. Let d_{a+i} be any element of D mapping onto d'_{a+i}. *Q. E. D.*

The Lemma shows that $\varDelta$, being isomorphic to $\varDelta'$ under the projection of $S' \cdot T$ onto S', satisfies the conditions on generators of the discrete group of Theorem 3.5. The projection of $\varDelta$ on C lies in the torus T, and the action of $\varDelta$ is free and properly discontinuous on $(S \cdot C)/C$ because $\varDelta$ is discrete and torsionfree while C is compact. Thus $M/\varDelta$ is analytically homeomorphic to $M/\varDelta'$ by Theorem 3.5 .This provides the finite real analytic covering

$$N' = M/\varDelta' \to M/\varGamma = N .$$

$N' = (S/\varDelta') \times M'$, and $S/\varDelta'$ is homeomorphic to $E \times (S'/\varDelta')$ where E is homeomorphic to a EUCLIDean space. Let $N'' = S'/\varDelta'$, and the decomposition $N = E \times N'' \times M'$ is exhibited.

Let r be the multiplicity of the covering $N' \to N$. If $\varGamma$ is infinite, then $\varDelta'$ is nontrivial and [6] $\chi(N'') = 0$. Thus $\chi^*(N) = \frac{1}{r}\chi(N') = \frac{1}{r}\chi(N'')\chi(M') = 0$. If $\varGamma$ is finite, then the projection of $\varGamma$ on $\mathbf{I}(M_0)$ must have a stationary point because the maximal compact subgroups of $\mathbf{I}(M_0)$ are isotropy subgroups; thus Γ projects isomorphically onto a subgroup Σ of $\mathbf{I}(M')$ which acts freely on M'. If t is the common order of Γ and Σ, then $M' \to M'/\Sigma$ is a covering of multiplicity t. We have $\chi(M'/\Sigma) = \frac{1}{t}\chi(M')$ because M' is compact, whence t divides $\chi(M')$. Now $\chi^*(N) = \frac{1}{t}\chi(M) = \frac{1}{t}\chi(M_0)\chi(M') = \frac{1}{t}\chi(M')$ is an integer $\geqq 0$.

We have proved that $\chi^*(N)$ is an integer $\geqq 0$ and that $\chi^*(N) \neq 0$ implies finiteness of $\pi_1(N)$. If $\chi^*(N) \neq 0$ and M' is RIEMANNian symmetric, then $\pi_1(N)$ is a finite 2-group as in § 3.10. Similarly, the retraction of N when M_0 is EUCLIDean is exhibited as in § 3. This completes the proof of Theorem 4.1. *Q.E.D.*

5. Classification in the irreducible case

We will classify (up to global isometry) the complete connected locally irreducible locally symmetric RIEMANNian manifolds of nonzero characteristic and all curvatures $\geqq 0$. This is the first step in implementing Theorems 3.1, 3.2 and 4.1.

5.1. The candidates for consideration are not numerous:

Theorem. *Let S be a compact connected simply connected irreducible RIEMANNian symmetric manifold with $\chi(S) \neq 0$, and suppose that S has a fixed point free isometry. Then S is a GRASSMANN manifold,* $\mathbf{SO}(2n)/\mathbf{U}(n)$ *with* $n > 2$, $\mathbf{Sp}(n)/\mathbf{U}(n)$ *with* $n > 1$, $\mathbf{E}_7/A_7$, *or* $\mathbf{E}_7/E_6 \cdot T^1$.

Remark. Here A_7 is a subgroup $\mathbf{SU}(8)/\{\pm I\}$ in the compact simply connected exceptional group $\mathbf{E}_7$, and $E_6 \cdot T^1 = (\mathbf{E}_6 \times \mathbf{T}^1)/\{1, z, z^2\}$ where $\mathbf{T}^1$ is a circle group and $z = (z', z'')$, each component of order 3 and z' central in $\mathbf{E}_6$. *GRASSMANN manifold* means real, complex or quaternion GRASSMANN manifold, and we use oriented subspaces for real GRASSMANN manifolds.

Proof. Let K be an isotropy subgroup of $G = \mathbf{I}(S)$. Both groups are compact, and rank. K = rank. G because $\chi(S) \neq 0$. In particular, every element of G_0 has a fixed point on S. Thus we need only examine the cases where $G \neq G_0$. According to CARTAN [4], these are, besides the ones mentioned in the statement of the Theorem, only $\mathbf{E}_6/\{\mathbf{SU}(6) \times \mathbf{SU}(2)/\text{discrete}\}$ and $\mathbf{E}_6/\{\mathbf{SO}(10) \times \mathbf{SO}(2)/\text{discrete}\}$. We will check that, for both of these spaces, every isometry has a fixed point. Theorem 5.1 will then be proven.

5.2. Let M be a symmetric space $\mathbf{E}_6/\{\mathbf{SU}(6) \times \mathbf{SU}(2)/\text{discrete}\}$ or $\mathbf{E}_6/\{\mathbf{SO}(10) \times \mathbf{SO}(2)/\text{discrete}\}$, and let K be an isotropy subgroup of $G = \mathbf{I}(M)$.

Then $K = K_0 \cup \alpha K_0$ and $G = G_0 \cup \alpha G_0$ where conjugation by α induces outer automorphisms both on K_0 and G_0. For conjugation by α is outer on K_0 by construction of $\mathbf{I}(M)$ [4]. Now let $\pi : \mathbf{E}_6 \to G_0$ be the projection; the kernel D of π is the center of $\mathbf{E}_6$, cyclic of order 3, and $\pi^{-1}(K_0)$ is the centralizer of an element $s \in \mathbf{E}_6$ with $s^2 \in D$. As D has odd order, we may assume $s^2 = 1$. It follows that K_0 is its own normalizer in G_0. If conjugation by α were inner on G_0, it would be inner on K_0; this it not the case.

Now let A and B be the centralizers of α in G_0 and K_0, respectively. Checking both cases, we see that both A and B have rank 4. It follows that B contains a maximal torus T of A.

Let $g \in G$. If $g \in G_0$, then we know that g has a fixed point because rank.K= =rank. G. If $g \notin G_0$, then $g \in \alpha G_0$. Then, if V is a maximal torus of A, $hgh^{-1} \in \alpha V$ for some $h \in G_0$ ([9], Th. on p. 57). Let V be the maximal torus T above. Then $V \subset K_0$, so $hgh^{-1} \in \alpha K$. This shows that g has a fixed point, proving Theorem 5.1. *Q.E.D.*

5. 3. Space forms of Grassmann manifolds. Theorem 5.1 tells us which spaces should be studied in order to find the groups Δ of isometries acting freely on a compact irreducible simply connected symmetric space S with $\chi(S) \neq 0$. Classification of these groups Δ up to conjugacy in $\mathbf{I}(S)$ is the same as classification of the space forms S/Δ of S up to isometry. In [13] we solved the complicated case—the case where S is a Grassmann manifold. For the convenience of the reader, we will recall the results.

Let $\mathbf{F}$ be a field $\mathbf{R}$ (real), $\mathbf{C}$ (complex) or $\mathbf{H}$ (quaternion), and let $\mathbf{F}^n$ denote a left positive definite hermitian vectorspace of dimension n over $\mathbf{F}$. If $0<q<n$, then the unitary group $\mathbf{U}(n, \mathbf{F})$ of $\mathbf{F}^n$ acts transitively on the set $\mathbf{G}_{q,n}(\mathbf{F})$ of q-dimensional subspaces (oriented if $\mathbf{F} = \mathbf{R}$) of $\mathbf{F}^n$. We exclude $\mathbf{G}_{1,2}(\mathbf{R})$ and $\mathbf{G}_{2,4}(\mathbf{R})$; then $\mathbf{G}_{q,n}(\mathbf{F})$ has a unique (up to a scalar multiple) $\mathbf{U}(n, \mathbf{F})$-invariant Riemannian metric, and is always envisaged with that metric; it is simply connected and Riemannian symmetric, and has topological dimension $q(n - q)r$ where r is the dimension of $\mathbf{F}$ over $\mathbf{R}$. The characteristic $\chi(\mathbf{G}_{q,n}(\mathbf{F})) \neq 0$ except when $\mathbf{F} = \mathbf{R}$ and $q(n - q)$ is odd.

$\mathbf{I}_0(\mathbf{G}_{q,n}(\mathbf{F}))$ is the group of motions induced by $\mathbf{U}(n, \mathbf{F})_0$ (which is $\mathbf{SO}(n)$, $\mathbf{U}(n)$ or $\mathbf{Sp}(n)$). If $q = n - q$, we have an isometry β given by orthogonal complementation (and consistent with orientation if $\mathbf{F} = \mathbf{R}$). In any case, we use β to assume q even if $q(n - q)$ is even and $\mathbf{F} = \mathbf{R}$. If $\mathbf{F} = \mathbf{C}$, we have an isometry α induced by conjugation of $\mathbf{C}$ over $\mathbf{R}$. If $\mathbf{F} = \mathbf{R}$, we have an isometry ω given by reversal of orientation. Let $g_v (0 \leqq v \leqq n)$ be the isometry induced by $\begin{pmatrix} I_{n-v} & O \\ O & -I_v \end{pmatrix} \in \mathbf{U}(n, \mathbf{F})$. If $n = 2m$, let k be the isometry induced by $\begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix} \in \mathbf{U}(n, \mathbf{F})$. If $\mathbf{F} = \mathbf{C}$, let $h_v (0 \leqq v \leqq n)$ be the isometry induced by $\begin{pmatrix} aI_{n-v} & O \\ O & -aI_v \end{pmatrix} \in \mathbf{U}(n, \mathbf{C})$ where $a = \exp(\pi\sqrt{-1}/n)$. Let $\mathbf{Z}_m$ denote the cyclic group of order m. Now the space forms of GRASSMANN manifolds of nonzero characteristic are classified ([13], Theorems 1, 2, 3) by:

Theorem. *Let Δ be a group of isometries acting freely on $\mathbf{G}_{q,n}(\mathbf{F})$, where $q(n-q)$ is even (so we may apply β and assume q even) if $\mathbf{F} = \mathbf{R}$. If $\Delta \neq \{1\}$, then Δ is conjugate in $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{F}))$ to one of the groups:*

F	group	isomorphic to	conditions
H	$\{1, \beta g_v\}$	$\mathbf{Z}_2$	$2q = n, 0 \leqq v < q$
C	$\{1, \alpha k\}$	$\mathbf{Z}_2$	q and $n-q$ odd
C	$\{1, \beta g_{2v}\}$	$\mathbf{Z}_2$	$2q=n, 0 \leqq 2v<q$
C	$\{1, \beta h_{2v-1}\}$	$\mathbf{Z}_2$	$2q=n, 1 \leqq 2v-1<q$
R	$\{1, \omega\}$	$\mathbf{Z}_2$	none
R	$\{1, \omega k\}$	$\mathbf{Z}_2$	n even
R	$\{1, \beta g_{2v}\}$	$\mathbf{Z}_2$	$2q=n, 0 \leqq 2v<q$
R	$\{1, \beta g_{2v}, \omega, \omega\beta g_{2v}\}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$2q=n, 0 \leqq 2v<q$
R	$\{1, \beta g_{2v-1}, \omega, \omega\beta g_{2v-1}\}$	$\mathbf{Z}_4$	$2q=n, 1 \leqq 2v-1<q$

Each of these groups acts freely on $\mathbf{G}_{q,n}(\mathbf{F})$, and any two distinct ones are not conjugate in $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{F}))$.

5. 4. The space forms of $\mathbf{SO}(4n)/\mathbf{U}(2n)$ are given by:

Theorem. *Let M be the RIEMANNian symmetric manifold $\mathbf{SO}(4n)/\mathbf{U}(2n)$, $n > 1$, and let g_v and $k_1 \in \mathbf{I}_0(M)$ be the respective isometries induced by the elements $\begin{pmatrix} I_{4n-2v} & O \\ O & -I_{2v} \end{pmatrix}$ and* diag. $\{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right); \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\}$ *of* $\mathbf{SO}(4n)$. *We have $\mathbf{I}(M) = \mathbf{I}_0(M) \cup \tau \cdot \mathbf{I}_0(M)$ where τ is central, $\tau^2 = 1$ and $\tau \notin \mathbf{I}_0(M)$. Let Δ be a nontrivial group of isometries acting freely on M. Then Δ is conjugate in $\mathbf{I}(M)$ to one of the n groups $\{1, \tau g_u\}$, $0 \leqq u < n$, or to $\{1, \tau k_1\}$. Conversely, these groups act freely on M and are mutually non-conjugate in $\mathbf{I}(M)$.*

Proof. Let $G = \mathbf{I}(M)$. We have a point $p \in M$ at which the symmetry is given by $s = \pm$ diag. $\{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$. Let K be the isotropy subgroup of G at p. Then $G_0 = \mathbf{SO}(4n)/\{\pm I\}$, $K_0 = \mathbf{U}(2n)/\{\pm I\}$, $G = G_0 \cup \alpha \cdot G_0$

and $K = K_0 \cup \alpha \cdot K_0$ where conjugation of G_0 by α is the same as conjugation by $a = \pm \operatorname{diag.} \{1, -1; \ldots; 1, -1\}$. Observe that $a \in G_0$, define $\tau = \alpha a$, and note that the first statement is proved.

Let $h \in G_0$. Then τh has a fixed point on M if and only if $u\tau h u^{-1} = \alpha k$ for some $u \in G_0$ and $k \in K_0$. As $u\tau h u^{-1} = \tau u h u^{-1} = \alpha a u h u^{-1}$, this is equivalent to $auhu^{-1} = k$, i.e., to h being G_0-conjugate to an element of aK_0. If primes denote representing matrices, we observe that a' anticommutes with s' and that $\mathbf{U}(2n)$ is the full centralizer of s' in $\mathbf{SO}(4n)$. Thus τh has a fixed point if and only if some $\mathbf{SO}(4n)$-conjugate of h' anticommutes with s'.

Suppose further that $h^2 = 1$. Then h' has square $\pm I$. Suppose first that $h'^2 = I$; then h' is conjugate to some g'_v, and we may assume $v \leqq n$ because h' may be replaced by its negative. It τh has a fixed point, then s' must exchange the eigenspaces of $+1$ and of -1 for some conjugate of h', and it follows that $v = 2n$. On the other hand, if $v = 2n$, then h' is conjugate to a' and it follows that τh has a fixed point.

Now suppose $h'^2 = -I$. Thus h' is $\mathbf{SO}(4n)$-conjugate to k'_1 or to s'. Observe that k'_1 and s' are not conjugate in $\mathbf{SO}(4n)$, even though they are conjugate in $\mathbf{O}(4n)$. If τh has a fixed point, then we may conjugate and assume that h' anticommutes with s'. Now s' and h' generate a quaternion algebra, and it is easily seen that they are $\mathbf{SO}(4n)$-conjugate. On the other hand, if h' is conjugate to s', then we may assume that they generate a quaternion algebra; this done, they anticommute and τh has a fixed point. Thus τh is fixed point free if and only if h' is $\mathbf{SO}(4n)$-conjugate to k'_1.

Δ has at most one element in each component of $\mathbf{I}(M)$. As $\Delta \neq \{1\}$, it follows that $\Delta = \{1, \tau h\}$ where $(\tau h)^2 = \tau h \tau h = \tau^2 h^2 = h^2 = 1$, $h \in G_0$. The Theorem now follows. *Q.E.D.*

5.5. The space forms of $\mathbf{SO}(4n+2)/\mathbf{U}(2n+1)$ are given by:

Theorem. *Let M be the RIEMANNian symmetric manifold*

$$\mathbf{SO}(4n+2)/\mathbf{U}(2n+1),\ n \geqq 1 .$$

Then $\mathbf{I}(M) = \mathbf{O}(4n+2)/\{\pm I\}$ and we have isometries $h_v = \pm \begin{pmatrix} I_{4n+2-v} & O \\ O & -I_v \end{pmatrix}$ of M. Every nontrivial group of isometries acting freely on M is conjugate in $\mathbf{I}(M)$ to one of the n groups $\{1, h_{2u+1}\}$, $0 \leqq u < n$. Conversely, these groups act freely on M and are mutually non-conjugate in $\mathbf{I}(M)$.

Proof. M has a point p at which the symmetry is given by

$$s = \pm \operatorname{diag.} \{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\};$$

let K be the isotropy subgroup of $G = \mathbf{I}(M)$ at p. Then $G_0 = \mathbf{SO}(2m)/\{\pm I\}$

and $K_0 = \mathbf{U}(m)/\{\pm I\}$ where we define $m = 2n+1$. $G = G_0 \cup \alpha\cdot G_0$ and $K = K_0 \cup \alpha \cdot K_0$ where conjugation of G_0 by α is the same as conjugation by $a = \pm\,\text{diag.}\{1, -1; \ldots ; 1, -1\}$. As this conjugation is an outer automorphism of G_0 (because m is odd) we may identify α with a, viewing G as $\mathbf{O}(2m)/\{\pm I\}$ and K as $\{\mathbf{U}(m) \cup a\cdot \mathbf{U}(m)\}/\{\pm I\}$. This proves the first statement.

Given $g \in \mathbf{I}(M)$, g' will denote one of the two matrices in $\mathbf{O}(2m)$ representing g. If h_v (v odd) has a fixed point on M, then h'_v is conjugate in $\mathbf{O}(2m)$ to an element $h''_v = a'k'$ for some $k' \in \mathbf{U}(m)$, whence $s'h''_v s'^{-1} = -h''_v$. This shows that s' exchanges the eigenspaces of $+1$ and of -1 for h''_v, proving that $v = m$. It follows that the groups $\{1, h_{2u+1}\}$ $(0 \leqq u < n)$ act freely on M. As they are obviously mutually nonconjugate, the converse of the second statement is proven.

Let Δ be a nontrivial group of isometries acting freely on M. As every element of G_0 has a fixed point, $\Delta = \{1, g\}$ with $\det. g' = -1$. $g^2 = 1$ implies $g'^2 = \pm I$, whence $g'^2 = +I$ because $\det. g' = -1$, so g is conjugate to some h_v (v odd). We may take $v \leqq m$ because h_v is conjugate to h_{2m-v}, and then $v < m$ because g is not conjugate to a. The second statement follows. *Q. E. D.*

5. 6. The space forms of $\mathbf{Sp}(n)/\mathbf{U}(n)$ are given by:

Theorem. *Let* M *be the* RIEMANN*ian symmetric manifold* $\mathbf{Sp}(n)/\mathbf{U}(n)$, $n > 1$, *let* $\mathbf{Sp}(n)$ *be viewed as the group of all* $g \in \mathbf{U}(2n)$ *such that* $gJ{}^tg = J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, *and let* $g_v \in \mathbf{I}_0(M)$ *be the isometry induced by* $\text{diag.}\{I_{n-v}, -I_v, I_{n-v}, -I_v\} \in \mathbf{Sp}(n)$. *We have* $\mathbf{I}(M) = \mathbf{I}_0(M) \cup \tau\cdot \mathbf{I}_0(M)$ *where* τ *is central,* $\tau^2 = 1$ *and* $\tau \notin \mathbf{I}_0(M)$. *Let* Δ *be a nontrivial group of isometries acting freely on* M. *Then* Δ *is conjugate in* $\mathbf{I}(M)$ *to one of the* $\left[\frac{n+1}{2}\right]$ *groups* $\{1, \tau g_v\}$, $0 \leqq v < \frac{n}{2}$. *Conversely, these groups act freely on* M *and are mutually non-conjugate in* $\mathbf{I}(M)$.

Proof. Let $G = \mathbf{I}(M)$. Then $G_0 = \mathbf{Sp}(n)/\{\pm I\}$ and

$$s = \pm\begin{pmatrix} \sqrt{-1}\,I_n & 0 \\ 0 & -\sqrt{-1}\,I_n \end{pmatrix} \in G_0$$

is the symmetry at some $p \in M$. Let K be the isotropy subgroup of G at p. Then $K_0 = \mathbf{U}(n)/\{\pm I\}$ where $\mathbf{U}(n)$ consists of all $\begin{pmatrix} b & 0 \\ 0 & {}^tb^{-1} \end{pmatrix}$ for which b is an $n\times n$ unitary matrix, $K = K_0 \cup \alpha\cdot K_0$ and $G = G_0 \cup \alpha \cdot G_0$, where conjugation of G_0 by α is the same as conjugation by $\pm J$. As $\pm J \in G_0$, the first statement is proved by setting $\tau = \alpha\cdot(\pm J)$.

Let $h \in G_0$. As in § 5.4, τh has a fixed point on M if and only if h is G_0-conjugate to an element of $(\pm J) \cdot K_0$. Suppose that $h^2 = 1$, and let primes denote representing matrices. If $h'^2 = -I$, then h' is $\mathbf{Sp}(n)$-conjugate to J, whence τh has a fixed point. Now suppose $h'^2 = I$. Then h is conjugate to some g_v. If g'_v is conjugate to $Jk', k' = \begin{pmatrix} b & 0 \\ 0 & {}^t b^{-1} \end{pmatrix}$, then $I = (Jk')^2 = \begin{pmatrix} -{}^t b^{-1} \cdot b & 0 \\ 0 & -b \cdot {}^t b^{-1} \end{pmatrix}$ shows ${}^t b = -b$, whence $Jk' = \begin{pmatrix} 0 & -b^{-1} \\ -b & 0 \end{pmatrix}$. This last is conjugate by $\begin{pmatrix} I_n & 0 \\ 0 & -b^{-1} \end{pmatrix} \in \mathbf{U}(2n)$ to $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$; it follows that $v = \frac{n}{2}$ by counting eigenvalues. On the other hand, if $v = \frac{n}{2}$, then it is not difficult to see, using the WEYL group, that h is G_0-conjugate to $(\pm J) \cdot k$ for every $k \in K_0$ such that $k' = \begin{pmatrix} b & 0 \\ 0 & {}^t b^{-1} \end{pmatrix}$ and ${}^t b = -b$.

The Theorem now follows. *Q. E. D.*

5. 7. The space forms of $E_7/(A_7$ or $\mathbf{E}_6 \cdot T^1)$ can be described, as in §§ 5.4 – 5.6, in terms of the elements of square 1 in the group $\mathrm{ad}(\mathbf{E}_7) = \mathbf{E}_7/C$ where C is the center of $\mathbf{E}_7$. These elements are known:

Lemma (É. CARTAN [3]). *The group* $\mathrm{ad}(\mathbf{E}_7)$ *has elements* $1 = s_{E_7}$, s_{A_7}, $s_{E_6 \times T^1}$ *and* $s_{D_6 \times A_1}$ *of square* 1 *where the centralizer of* s_H *in* $\mathrm{ad}(\mathbf{E}_7)$ *is of* CARTAN *classification type* H; *these four elements are mutually non-conjugate in* $\mathrm{ad}(\mathbf{E}_7)$ *and any element of square* 1 *in* $\mathrm{ad}(\mathbf{E}_7)$ *is conjugate to one of them.*

Complement to Lemma. *Let* $\pi : \mathbf{E}_7 \to \mathrm{ad}(\mathbf{E}_7)$ *be the projection and let* $s'_H \in \pi^{-1}(s_H)$. *Recall that* $C = \mathrm{Ker.}\,\pi = \{1\ z\}$ *cyclic order two. Then* $(s'_{E_7})^2 = (s'_{D_6 \times A_1})^2 = 1$ *and* $(s'_{A_7})^2 = (s'_{E_6 \times T^1})^2 = z$.

Proof. The Lemma is CARTAN's classification of RIEMANNian symmetric spaces M with $\mathbf{I}_0(M) = \mathrm{ad}(\mathbf{E}_7)$.

Let Z be the identify component of the centralizer of s_H in $\mathrm{ad}(\mathbf{E}_7)$, observe that $Z' = \pi^{-1}(Z)$ is connected because Z contains a maximal torus; let S and S' be the respective centers of Z and Z', and note that $\pi : S' \to S$ is 2–to–1 sending z to 1 and s'_H to s_H.

If $H = E_7$ then S' has order two, so $(s'_H)^2 = 1$.

If $H = D_6 \times A_1$, then the universal covering group of Z' is $\mathbf{Spin}(12) \times \mathbf{SU}(2)$. That group has center isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, and S' is a quotient of its center. Thus $(s'_H)^2 = 1$.

In the other cases, we look at the linear isotropy representation on the RIEMANNian symmetric space $\mathrm{ad}(\mathbf{E}_7)/Z$. As this space is irreducible, it follows that $S = \{1, s_H\}$ if $H = A_7$ and S is a circle group if $H = E_6 \times T^1$. Looking at $\pi: S' \to S$, it is easily seen that $(s'_{A_7})^2 = z$.

Let $H = E_6 \times T^1$ and let Z'' be the group $\mathbf{E}_6 \times \mathbf{T}^1$. Z'' has center S'' isomorphic to $\mathbf{Z}_3 \times T^1$; we represent the elements of S'' by pairs (u^a, v) where u generates the center of $\mathbf{E}_6$ and v is a unimodular complex number. We have coverings $Z'' \overset{\beta}{\to} Z' \overset{\pi}{\to} Z$, and $S = \pi\beta(S'')$ is a circle group. Thus Ker. $(\pi\beta) = L_1 \times L_2$ where L_2 is a finite cyclic subgroup of T^1 and L_1 is cyclic order 3 with a generator (u, w) where $w^3 = 1$. Now Ker. $\beta = L_1 \times L_3$ where L_3 has index 2 in L_2; thus we may choose β such that Ker. $\beta = L_1$ and L_2 is generated by $(1, -1)$. It follows that $z = \beta((1, -1))$ and $s_H = \pi\beta((1, \sqrt{-1}))$. This shows s'_H to be $\beta((1, \pm\sqrt{-1}))$, whence $(s'_H)^2 = z$. *Q.E.D.*

We can now enumerate the space forms of $\mathbf{E}_7/A_7$ and of $\mathbf{E}_7/E_6 \cdot T^1$:

Theorem. *Let M be one of the* RIEMANN*ian symmetric manifolds $\mathbf{E}_7/A_7$ or $\mathbf{E}_7/E_6 \cdot T^1$. We have $\mathbf{I}(M) = \mathbf{I}_0(M) \cup \tau \cdot \mathbf{I}_0(M)$ where τ is central, $\tau^2 = 1$ and $\tau \notin \mathbf{I}_0(M)$. Let Δ be a nontrivial group of isometries acting freely on M. Then either $\Delta = \{1, \tau\}$* or *Δ is conjugate in $\mathbf{I}(M)$ to $\{1, \tau s_{D_6 \times A_1}\}$. These two groups act freely on M and are not conjugate in $\mathbf{I}(M)$.*

Proof. The first statement is known [4], τ being central because $\mathbf{E}_7$ admits no outer automorphism. Let K be an isotropy subgroup of $G = \mathbf{I}(M)$. Then $G_0 = \mathrm{ad}(\mathbf{E}_7)$, $K = K_0 \cup \alpha \cdot K_0$ and $G = G_0 \cup \alpha \cdot G_0$ where $\alpha^2 = 1$ and conjugation by α is the same as conjugation by $a \in G_0$; $\tau = \alpha a$. Altering a by an element of K_0 if necessary, we may assume that a is conjugate to s_{A_7}.

As before, let $C = \{1, z\}$ be the kernel of the projection $\pi: \mathbf{E}_7 \to \mathrm{ad}(\mathbf{E}_7)$ and let primes denote representing elements in $\mathbf{E}_7$. Let A be the centralizer of a' in $\mathbf{E}_7$; $A \cong \mathbf{SU}(8)/\{\pm I\}$ as seen in the proof of the complement to the Lemma, z is represented by $\pm\sqrt{-1} \cdot I_8$, and a' is represented by

$$\pm \exp(2\pi\sqrt{-1}/8) \cdot I_8 .$$

Let $h \in G_0$, $h'^2 = z$. Replacing h by a conjugate, $h' \in A$ and h' is represented by $\pm \begin{pmatrix} \exp(2\pi\sqrt{-1}/8) I_p & 0 \\ 0 & \exp(2\pi\sqrt{-1}\ 5/8) I_q \end{pmatrix}$ where $p + q = 8$. That matrix must have determinant $+1$; it follows that p and q are even, $p = 2u$ and $q = 2v$. Again replacing h by a conjugate, $h' \in A$ is represented by $\pm\mathrm{diag}.\{\varepsilon I_u, \varepsilon^5 I_v, \varepsilon I_u, \varepsilon^5 I_v\}$ where $\varepsilon = \exp(2\pi\sqrt{-1}/8)$.

Let s be the symmetry to M at the point at which K is isotropy subgroup of G. Although s commutes with a because it commutes with α, s' cannot commute with a' because conjugation by a induces an outer automorphism of K_0. Thus the commutator $[s', a'] = z$. It follows that s' normalizes A and that conjugation of A by s' is an involutive outer automorphism. Thus we may assume [3] $s'gs'^{-1} = {}^tg^{-1} = \bar{g}$ for every $g \in A$, or that $s'gs'^{-1} = \overline{JgJ^{-1}}$ $\left(J = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}\right)$ for every $g \in A$. It follows that $s'h's'^{-1} = h'z$. Now let $h' = h''a'$. As $s'a's'^{-1} = a'z, h''$ must commute with s'. This implies $\pi(h'') \in K_0$ because $\pi^{-1}(K_0)$ is connected and is the centralizer of s' in $\mathbf{E}_7$.

We have now proved, given $h \in G_0$ with $h'^2 = z$, that $uhu^{-1} = ak$ for some $k \in K_0$, $u \in G_0$. This gives $auhu^{-1} = k$, i.e., $\alpha auhu^{-1} = \alpha k$, i.e., $\tau uhu^{-1} \epsilon \alpha K_0$, i.e., τh conjugate to an element of αK_0. Thus τh has a fixed point on M.

Now let $h \in G_0$, $h'^2 = 1$. We will see that τh has no fixed point on M. For if it had a fixed point, we would have $u\tau hu^{-1} = \alpha k$ with $u \in G_0$ and $k \in K_0$. Then h would be conjugate to $ak \in aK_0$. Replacing h by that conjugate, $h' = a'k'$ with k' in the centralizer $K' = \pi^{-1}(K_0)$ of s' in $\mathbf{E}_7$. Now $B = A \cap K'$ is both the centralizer of s' in A and the centralizer of a' in K'. Every element of $a'K'$ is conjugate to an element of $a'B$. For $K_0 \cup aK_0$ is the centralizer of s in $\mathrm{ad}(\mathbf{E}_7)$; if T is a maximal torus of the centralizer of a in K_0, then a result of DE SIEBENTHAL ([9], Th. on p. 57) shows that every element of aK_0 is K_0-conjugate to an element of aT; thus every element of $a'K'$ is conjugate to an element of $a' \cdot \pi^{-1}(T) \subset a'B$. Now we conjugate h and assume $h' = a'k'$ where k' commutes with both s' and a'. Thus we have $k' \in A$. Let double primes denote elements of $\mathbf{SU}(8)$ representing elements of $A = \mathbf{SU}(8)/\{\pm I\}$. $h'' = a''k''$ is conjugate (by s'') to $h''z''$; thus $-I = h''^2$, and it is conjugate in $\mathbf{SU}(8)$ to $(h''z'')^2 = h''^2z''^2 = (-I)(-I) = I$. This being impossible, τh cannot have a fixed point.

Our group $\Delta = \{1, \tau h\}$ where $1 = (\tau h)^2 = \tau^2 h^2 = h^2$. Thus, by the Lemma, h is conjugate to 1, $s_{D_6 \times A_1}$, s_{A_7} or $s_{E_6 \times T^1}$. But $h'^2 = 1$, as we have just seen, because τh has no fixed point; the Complement to the Lemma now shows h conjugate to 1 or $s_{D_6 \times A_1}$. On the other hand, the Complement and the preceding paragraph show that $\{1, \tau\}$ and $\{1, \tau s_{D_6 \times A_1}\}$ act freely on M.

5. 8. Combining Theorems 5.1, 5.3, 5.4, 5.5, 5.6 and 5.7, one has a global classification for the space forms of compact connected simply connected irreducible RIEMANNian symmetric manifolds of nonzero characteristic.

6. Reducibility and commutativity

6. 1. Order. Let N be an irreducible compact connected simply connected RIEMANNian symmetric manifold of nonzero characteristic. We have just seen that a group of isometries acting freely on N must be of order 1, 2 or 4. We now define the *order of* N, written order. N, to be the maximal of the orders of the groups of isometries acting freely on N. This concept is useful for:

6.2. Commutativity Theorem. *Let* M' *be a compact connected simply connected* RIEMANN*ian symmetric manifold of nonzero characteristic. Then these are equivalent*:

1. *A group of isometries acting freely on* M' *is necessarily abelian.*

2. *Any group of isometries acting freely on* M' *is a direct product of some number* $m \geqq 0$ *of groups* $\mathbf{Z}_2$, *or is cyclic of order* 4.

3. *If one of the irreducible factors of* M' *has order* 4, *then all the others have order* 1. *If* M' *has two isometric irreducible factors of order* 2, *then all the others have order* 1.

Complement to the Commutativity Theorem. *Let* N *be an irreducible compact connected* RIEMANN*ian symmetric manifold of nonzero characteristic.*

1. *These are equivalent*:

(a) N *has order* 2.

(b) $\mathbf{Z}_2$ *acts freely by isometries on* N, *but* $\mathbf{Z}_4$ *does not.*

(c) $\mathbf{Z}_2$ *acts freely by isometries on* N, *but* $\mathbf{Z}_2 \times \mathbf{Z}_2$ *does not.*

(d) N *is isometric to* $\mathbf{G}_{q,n}(\mathbf{R})$ *where* $n \neq 2q$ *and* $q(n-q)$ *is even, or to* $\mathbf{G}_{q,n}(\mathbf{C})$ *where either* $2q = n$ *or* $q(n-q)$ *is odd, or to* $\mathbf{G}_{q,2q}(\mathbf{H})$, *or to* $\mathbf{SO}(2n)/\mathbf{U}(n)$ *where* $n > 2$, *or to* $\mathbf{Sp}(n)/\mathbf{U}(n)$ *where* $n \geqq 1$, *or to* $\mathbf{E}_7/A_7$, *or to* $\mathbf{E}_7/E_6 \cdot T^1$.

2. *These are equivalent*:

(a) N *has order* 4.

(b) $\mathbf{Z}_4$ *acts freely by isometries on* N.

(c) $\mathbf{Z}_2 \times \mathbf{Z}_2$ *acts freely by isometries on* N.

(d) N *is isometric to* $\mathbf{G}_{2n,4n}(\mathbf{R})$ *where* $n > 1$.

Here $\mathbf{Z}_m$ denotes the cyclic group of order m.

The Complement follows trivially from the results of § 5. The remainder of § 6 is devoted to the proof of the Commutativity Theorem. As (2) obviously implies (1) there, we need only prove that (3) implies (2) and that (1) implies (3).

6.3. The proof that (3) implies (2) is based on Theorem 3.2 and on

Lemma. *Let* Δ *be a nontrivial group of isometries acting freely on* $N \times N$ *where* N *is a complete connected simply connected irreducible* RIEMANN*ian symmetric manifold of nonzero characteristic and order* 2. *Then* Δ *is isomorphic to* $\mathbf{Z}_2$, $\mathbf{Z}_2 \times \mathbf{Z}_2$ *or* $\mathbf{Z}_4$.

Proof. $\Delta \frown \{\mathbf{I}(N) \times \mathbf{I}(N)\}$ has order $\leqq 4$ and has index $\leqq 2$ in Δ, by Theorem 3.2; it suffices to prove that Δ is not a nonabelian group of order 8.

Suppose Δ nonabelian of order 8. Then Δ is generated by an element γ of order 4 and an element δ of order 2 or 4, where $\delta\gamma\delta^{-1} = \gamma^{-1}$. By Theorem 3.2, we may assume γ to be given by $(x, y) \to (\tau y, x)$ where τ is a fixed point free isometry of order 2 on N. If $\delta^2 = 1$, then $\delta(x, y) = (\delta_1 x, \delta_2 y)$ where δ_i is an isometry of square 1 on N. Then $\delta\gamma\delta = \gamma^{-1}$ implies $\delta_1 = \tau\delta_2$; it follows that $\gamma\delta(x, y) = (\delta_1 y, \delta_1 x)$; thus $(x, \delta_1 x)$ is a fixed point for $\gamma\delta$. This proves $\delta^2 \neq 1$.

Now δ must have order 4, and is thus given by $(x, y) \to (\delta_1 y, \delta_2 x)$ where δ_i are isometries of N. Thus $\sigma = \delta\gamma$ is given by $(x, y) \to (\sigma_1 x, \sigma_2 y)$ where σ_i are isometries of N. By Theorem 3.2 we have $\sigma^2 = 1$. But $\sigma^2 \neq 1$ because Δ is the quaternion group. The Lemma follows. *Q.E.D.*

We will prove that (3) implies (2) in Theorem 6.2. Assume (3) and let Γ be a group of isometries acting freely on M'. If $\gamma \in \Gamma$, then $\gamma^4 = 1$ by Theorem 3.2. If Γ has no element of order 4, it must be a product of groups $\mathbf{Z}_2$, and we are done. Now suppose that Γ has an element of order 4. By our assumption (3) and by Theorem 3.2, there is a RIEMANNian product decomposition $M' = S \times X$ where X is a product of irreducible manifolds of order 1 and either S is irreducible with order. $S = 4$ or $S = S_1 \times S_2$, S_1 isometric to S_2, with order. $S_i = 2$. Let Δ be the restriction of Γ to S. The restriction $\Gamma \to \Delta$ is an isomorphism; thus it suffices to prove Δ isomorphic to $\mathbf{Z}_4$.

Observe that Δ acts freely on S. If S is irreducible of order 4, then $\Delta \cong \mathbf{Z}_4$ by the Complement, by Theorem 5.3, and because it contains an element of order 4. If S is reducible, then $\Delta \cong \mathbf{Z}_4$ by the Lemma above.

6.4. To prove that (1) implies (3) it suffices to exhibit a noncommutative group of isometries acting freely on a direct factor of M', in case the conditions of (3) do not hold. For this noncommutative group will then act freely by isometries on M'. Thus we need only take compact connected simply connected irreducible RIEMANNian symmetric manifolds N and L, order. $L > 1$, and prove:

If order. $N = 4$, then there is a noncommutative group of isometries acting freely on $N \times L$. If order. $N = 2$, then there is a noncommutative group of isometries acting freely on $N \times N \times L$.

We will construct examples of such groups which are dihedral groups of order 8.

Suppose that N has order 4. Then $N = \mathbf{G}_{2n,4n}(\mathbf{R})$, $n \geqq 2$, and (Theorem 5.3) $\beta g_{2v-1} = \nu$ generates a cyclic group of order 4 of isometries acting freely on N. Let $\gamma = \nu \times 1 \in \mathbf{I}(N \times L)$. Choose a fixed point free isometry τ of order 2 on L and define $\delta = g_{2v-1} \times \tau$. Then γ has order 4, δ has order 2, and $\delta\gamma\delta = \gamma^{-1}$ because $g_{2v-1}\beta g_{2v-1} = \omega\beta$. Now

$$\Gamma = \{1, \gamma, \gamma^2, \gamma^3;\ \delta, \delta\gamma, \delta\gamma^2, \delta\gamma^3\}$$

is the (dihedral) group generated by γ and δ. The powers of γ act freely on the

N-component, and the last four elements move every L-coordinate. Thus Γ is a noncommutative group of isometries acting freely on $N \times L$.

Suppose that N has order 2. Let ν and τ be involutive fixed point free isometries of N and L, respectively. We define elements γ and δ of $\mathbf{I}(N \times N \times L)$ by $\gamma(x, y, z) = (\nu y, x, z)$ and $\delta(x, y, z) = (\nu x, y, \tau z)$. γ has order 4 and its powers act freely. δ has order 2 and any $\delta\gamma^a$ moves the L-coordinate. $\delta\gamma\delta = \gamma^{-1}$ is easily checked. Thus the group Γ generated by γ and δ is a noncommutative group acting freely by isometries on $N \times N \times L$.

Theorem 6.2 is now proven.

Remark. The other noncommutative group of order 8, the quaternion group, can act freely by isometries on $N \times N \times N \times N$ where N is as above with order. $N > 1$.

6. 5. Corollary. *Let $M \to N$ be the universal Riemannian covering of a complete connected locally symmetric Riemannian manifold N with every sectional curvature $\geqq 0$ and characteristic $\chi(N) \neq 0$. Suppose, if one of the compact irreducible factors of M has order 4, that all the others have order 1; suppose, if M has a pair of isometric compact irreducible factors of order 2, that all the others have order 1. Then the fundamental group $\pi_1(N)$ is a finite direct product of groups $\mathbf{Z}_2$, or is cyclic of order 4.*

This follows immediately from Theorems 3.2 and 6.2.

We can give a good description of the manifold N of the Corollary. One has

$$M = M_0 \times M_1 \times \ldots \times M_t$$

where M_0 is a Euclidean space $\mathbf{R}^m$ and each M_i $(i > 0)$ is compact and irreducible with $\chi(M_i) > 0$. If $\pi_1(N) \cong \mathbf{Z}_4$, there are two sorts of possibilities: some M_i has order 4 or two isometric M_i have order 2. We permute the M_i and obtain $M = M_0 \times S \times X$ where S is irreducible of order 4 or the product of two isometric irreducible manifolds S_i of order 2. $N = M/\Gamma$ where Γ is generated by an element $\gamma = \gamma_0 \times \gamma_S \times \gamma_X$, $\gamma_0^4 = 1$, $\gamma_X^4 = 1$, and γ_S is given by: If S is irreducible, $S = \mathbf{G}_{2n,4n}(\mathbf{R})$ with $n \geqq 2$, then γ_S is conjugate to an isometry βg_{2v-1} of S. In the other case, γ_S is conjugate to an isometry $(s_1, s_2) \to (\tau s_2, s_1)$ of $S = S_1 \times S_2$ where τ is a fixed point free involutive isometry of $S_1 = S_2$.

Suppose $\pi_1(N) \neq \mathbf{Z}_4$; then $\pi_1(N)$ is a product of some number $k \geqq 0$ of groups $\mathbf{Z}_2$, and $N = M/\Gamma$ for a group Γ isomorphic to $\pi_1(N)$ and given as follows. Γ has generators $\{\gamma_1, \ldots, \gamma_k\}$. Suppose first that some M_i (say S) is of order 4, or that two isometric M_i (say S_1 and S_2; let $S = S_1 \times S_2$) are of order two. Then $M = M_0 \times S \times X$, where X is a product of irreducible manifolds of order 1, $k \leqq 2$, and each $\gamma_i = \gamma_{i,0} \times \gamma_{i,S} \times \gamma_{i,X}$; $\gamma_i \to \gamma_{i,S}$

is an isomorphism of Γ onto a group Σ of isometries acting freely on S and each $\gamma_{i,S}$ preserves each S if S_j is a product, so the possibilites for Σ are given in § 5; the $\gamma_{i,0}$ commute and have square 1, as do the $\gamma_{i,X}$. We now consider the other possibility—the case where no M_i has order 4 and no two M_i of order 2 are isometric. Re-ordering the M_i, we may assume that $M_1, \ldots, M_k$ each has order 2 and is preserved by each γ_j, and that γ_j induces a fixed point free involutive isometry of M_j. Then $M = M_0 \times M_1 \times \ldots \times M_k \times X$, $\gamma_j = \gamma_{j,0} \times \ldots \times \gamma_{j,k} \times \gamma_{j,X}$, $\gamma_j \to \gamma_{j,\varepsilon}$ $(\varepsilon = 0, 1, \ldots, k, X)$ is a homomorphism of Γ, and $\gamma_{j,j}$ has no fixed point.

The Institute for Advanced Study

REFERENCES

[1] L. Auslander, Bieberbach's *theorem on space groups and discrete uniform subgroups of* Lie *groups*, Annals of Math. 71 (1960), 579–590.

[2] L. Auslander, Bieberbach's *theorem on space groups and discrete uniform subgroups of* Lie *groups, II*, American J. Math. 83 (1961), 276–280.

[3] É. Cartan, *Sur une classe remarquable d'espaces de* Riemann, Bull. Soc. Math. de France 54 (1926), 214–264; 55 (1927), 114–134. Also Oeuvres Complètes, part. 1, vol. 2, 587–660.

[4] É. Cartan, *Sur certaines formes* Riemann*iennes remarquables des géométries à groupe fondamental simple*, Ann. Sci. Ec. Norm. Super. 44 (1927), 345–467. Also, Oeuvres Complètes, part. 1, vol. 2, 867–990.

[5] G. D. Mostow, *On the fundamental group of a homogeneous space*, Annals of Math. 66 (1957), 249–255.

[6] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent* Lie *groups*, Annals of Math. 59 (1954), 531–538.

[7] G. de Rham, *Sur la réductibilité d'un espace de* Riemann, Comm. Math. Helv. *26* (1952), 328–344.

[8] H. Samelson, *On curvature and characteristic of homogeneous spaces*, Michigan Math. J. 5 (1958), 13–18.

[9] J. de Siebenthal, *Sur les groupes de* Lie *non connexes*. Comm. Math. Helv. *31* (1956), 41–89.

[10] C. T. C. Wall, *Rational* Euler *characteristics*, Proc. Cambridge Phil. Soc. 57 (1961), 182–184.

[11] J. A. Wolf, *The manifolds covered by a* Riemann*ian homogeneous manifold*, American J. Math. 82 (1960), 661–688.

[12] J. A. Wolf, *Sur la classification des variétés* Riemann*iennes homogènes à courbure constante*, C. r. Acad. Sci. Paris 250 (1960), 3443–3445.

[13] J. A. Wolf, *Space forms of* Grassmann *manifolds*, Canadian J. Math. 15 (1963), 193–205.

[14] J. A. Wolf, *Locally symmetric homogeneous spaces*, Comm. Math. Helv. *37* (1962), 65–101.

(Received May 24, 1962)