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# On Locally Symmetric Spaces of Non-negative Curvature and certain other Locally Homogeneous Spaces

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To Professor Georges de Rham on his sixtieth birthday

## 1. Introduction and summary

This paper is a study of the global structure of the complete connected locally symmetric RIEMANNIAN manifolds  $N$  in which every sectional curvature is non-negative. Our main result is that the fundamental group  $\pi_1(N)$  is a finite 2-group if the EULER-POINCARÉ characteristic (singular theory)  $\chi(N) \neq 0$ . In fact, that result is proved under slightly weaker conditions on  $N$ .

The first principle result (Theorem 3.1) states that there is a real analytic covering  $N' \rightarrow N$  of finite multiplicity and a real analytic deformation retraction of  $N$  onto a compact totally geodesic submanifold, such that  $N' = E \times T \times M'$  where  $E$  is a EUCLIDEAN space,  $T$  is a torus,  $M'$  is a compact simply connected RIEMANNIAN symmetric space, and the deformation retraction of  $N$  lifts to a deformation retraction of  $N'$  onto  $T \times M'$ . In particular, the betti numbers (singular theory) of  $N$  are finite and the EULER-POINCARÉ characteristic  $\chi(N)$  is defined. Theorem 3.1 then states that  $\chi(N) \geq 0$ , and that the fundamental group  $\pi_1(N)$  is a finite 2-group if  $\chi(N) \neq 0$ .

The second principle result (Theorem 3.2) gives a general method of constructing all manifolds  $N$  with  $\chi(N) \neq 0$ . Application of this method is a combinatorial problem which requires a classification (up to global isometry) of the space forms of the irreducible compact simply connected RIEMANNIAN symmetric manifolds  $S$  with  $\chi(S) \neq 0$ . That classification problem is solved in § 5. We first prove (Theorem 5.1) that  $S$  is equal to any of its space forms unless  $S$  is a GRASSMANN manifold,  $\mathbf{SO}(2n)/\mathbf{U}(n)$ ,  $\mathbf{Sp}(n)/\mathbf{U}(n)$ ,  $\mathbf{E}_7/A_7$  or  $\mathbf{E}_7/E_6 \cdot T^1$ . We have already classified the space forms of GRASSMANN manifolds of nonzero characteristic [13]; the result is recalled as Theorem 5.3. We then (Theorems 5.4–5.7) classify the space forms of the other possibilities of  $S$ . From these classification theorems we are able (Theorem 6.2) to give a necessary and sufficient condition on the set of factors of a product  $M'$

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of manifolds  $S$ , that every space form of  $M'$  have abelian fundamental group. If  $M'$  is irreducible, the condition is automatically fulfilled. Finally, in § 6.5, we give a good description of the possibilities for a manifold  $N$  with  $\chi(N) \neq 0$  when, in the universal RIEMANNIAN covering manifold  $M_0 \times M'$  ( $M_0$  EUCLIDEAN and  $M'$  compact; this is the  $M'$  which occurs in  $N'$ ),  $M'$  satisfies the commutativity conditions of Theorem 6.2.

Some parts of Theorem 3.1 do not fully use the hypotheses on  $N$ . This leads us to define a *RIEMANNIAN nilmanifold* to be a RIEMANNIAN manifold which admits a transitive nilpotent group of isometries. We prove (Theorem 4.2) that a connected RIEMANNIAN nilmanifold is isometric to a connected nilpotent Lie group in a left invariant RIEMANNIAN metric, that the nilradical of its connected group of isometries is the only connected transitive nilpotent group of isometries, and that its full group of isometries is the semidirect product of this nilradical with an isotropy group. Now let  $N$  be a RIEMANNIAN manifold with universal RIEMANNIAN covering manifold of the form  $M_0 \times M'$  where  $M_0$  is a RIEMANNIAN nilmanifold and  $M'$  is a compact RIEMANNIAN homogeneous manifold. Theorem 4.1 provides a real analytic covering

$$N' = E \times N'' \times M' \rightarrow N$$

of some finite multiplicity  $r > 0$ , where  $E$  is a EUCLIDEAN space and  $N''$  is a compact nilmanifold. While I am unfortunately unable to retract  $N$  onto a compact submanifold unless  $M_0$  is a EUCLIDEAN space (and so cannot prove the betti numbers of  $N$  to be finite, and so cannot assert that  $\chi(N)$  is defined) it is shown (Proposition 4.4) that  $\chi^*(N) = \frac{1}{r} \chi(N')$  is a topological invariant of  $N$ . Theorem 4.1 then states that  $\chi^*(N)$  is an integer, that  $\chi^*(N) \geq 0$ , that  $\chi^*(N) = \chi(N)$  if  $M_0$  is a EUCLIDEAN space, that  $\pi_1(N)$  is finite if  $\chi^*(N) \neq 0$ , and that  $\pi_1(N)$  is a finite 2-group if  $\chi^*(N) \neq 0$  and  $M'$  is RIEMANNIAN symmetric.

The ‘‘rational EULER-POINCARÉ characteristic’’  $\chi^*$  was invented by C.T.C. WALL [10] in another context. D.B.A. EPSTEIN suggested that I use it here, and gave valuable suggestions for adapting it to noncompact spaces and then proving it well defined.

By Theorem 3.1, we mean the theorem in § 3.1. Similarly, Theorem 3.9 is the theorem in § 3.9, etc.

*Added in proof:* By different methods, J.C. SANWAL has obtained the flat case of the fourth corollary of § 4.2 and has shown that the fundamental group of a complete flat RIEMANNIAN manifold is isomorphic to that of a compact flat RIEMANNIAN manifold, special case of our Theorem 3.1.

## 2. Preliminaries and notation

We will assume familiarity with LIE groups and discrete subgroups, RIEMANNIAN manifolds, and covering spaces.

**2.1. LIE groups and algebras.** If  $G$  is a LIE group, then  $G_0$  will denote its identity component,  $\mathfrak{G}$  will denote its LIE algebra,  $\exp: \mathfrak{G} \rightarrow G_0$  will be the exponential mapping, and adjoint representation of  $G$  on  $\mathfrak{G}$  will be denoted by “ad”. If  $H$  is a LIE subgroup of  $G$ , then  $\mathfrak{H}$  is viewed as a subalgebra of  $\mathfrak{G}$ . If  $\mathfrak{H}$  is a subalgebra of  $\mathfrak{G}$ , then the *corresponding analytic subgroup of  $G$*  is the analytic (= connected LIE) subgroup generated by the image of  $\mathfrak{H}$  under the exponential mapping of  $G$ .

If  $G$  and  $H$  are LIE groups and  $\beta: H \rightarrow \text{Aut}(G)$  is a continuous homomorphism of  $H$  into the group of automorphisms of  $G$ , then the *semidirect product  $G \cdot_\beta H$*  (denoted  $G \cdot H$  when there is no possibility of confusion) is the manifold  $G \times H$  with group structure  $(g_1, h_1)(g_2, h_2) = (g_1 \cdot \beta(h_1)g_2, h_1 h_2)$ .  $G \cdot H$  is a LIE group,  $G$  and  $H$  are closed subgroups under identifications  $g \rightarrow (g, 1)$  and  $h \rightarrow (1, h)$  (we always use 1 to denote the group identity), and  $G$  is a normal subgroup. The two extreme cases are when  $\beta$  is trivial, so  $G \cdot H$  is the direct product  $G \times H$ , and when  $\beta$  is faithful (trivial kernel), so  $H$  may be viewed as a group of automorphisms of  $G$  if  $\beta(H)$  is closed in  $\text{Aut}(G)$ .

The compact classical groups are the orthogonal groups  $\mathbf{O}(n)$  in  $n$  real variables, the identity components,  $\mathbf{SO}(n)$ , the special (= determinant 1) orthogonal groups, the unitary groups  $\mathbf{U}(n)$  in  $n$  complex variables and the special unitary groups  $\mathbf{SU}(n)$ , the symplectic groups  $\mathbf{Sp}(n)$  which are the unitary groups in  $n$  quaternion variables, and the universal covering groups  $\mathbf{Spin}(n)$  of  $\mathbf{SO}(n)$ .  $T^m$  will denote an  $m$ -torus.  $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7$  and  $E_8$  will refer both to the CARTAN classification types and to compact connected groups of those types. In boldface, these letters will denote the compact simply connected groups. For example,  $\mathbf{A}_n = \mathbf{SU}(n+1)$ ,  $\mathbf{B}_n = \mathbf{Spin}(2n+1)$ ,  $\mathbf{C}_n = \mathbf{Sp}(n)$ ,  $\mathbf{D}_n = \mathbf{Spin}(2n)$ , and  $\mathbf{F}_4$  is the group of isometries of the CAYLEY elliptic plane.

**2.2. Discrete groups.** A subgroup  $\Gamma$  of a topological group  $G$  is called *discrete* if it is a discrete subset, i.e., if there is a neighborhood  $U$  of  $1 \in G$  such that  $\Gamma \cap U = \{1\}$ .  $\Gamma$  is called *uniform in  $G$*  if ( $\bar{\Gamma}$  denotes the topological closure)  $G/\bar{\Gamma}$  is compact.

Let  $\Gamma$  be a topological group and let  $X$  be a topological space. An *action* of  $\Gamma$  on  $X$  is a homomorphism of  $\Gamma$  into the group of homeomorphisms of  $X$  such that the associated map  $\Gamma \times X \rightarrow X$  is continuous. We write  $\gamma(x)$  for the image of  $(\gamma, x)$ . The action is *effective* if  $1 \neq \gamma \in \Gamma$  implies  $\gamma(x) \neq x$  for

some element  $x \in X$ ; the action is *free* if  $1 \neq \gamma \in \Gamma$  implies  $\gamma(x) \neq x$  for every element  $x \in X$ ; the action is *properly discontinuous* if every  $x \in X$  has a neighborhood which meets its transforms by only a finite number of elements of  $\Gamma$ .

Let  $\Gamma$  and  $K$  be subgroups of  $G$ ,  $K$  closed in  $G$ . Then there is a natural action  $\gamma: gK \rightarrow \gamma gK$  of  $\Gamma$  on the coset space  $G/K$ . If  $G$  is a LIE group, or even locally compact with only finitely many components, and if  $K$  is compact, then the action is properly discontinuous if and only if  $\Gamma$  is discrete in  $G$ . In any case, the action is free if and only if  $1$  is the only element of  $\Gamma$  conjugate (in  $G$ ) to an element of  $K$ , and the identification space of  $G/K$  under  $\Gamma$  is the double coset space  $\Gamma \backslash G/K$ .

**2. 3. Isometries and product structure.** An *isometry* of a RIEMANNIAN manifold is an automorphism of the RIEMANNIAN structure. If  $M$  is a RIEMANNIAN manifold, then its *full group of* (= group of all) *isometries* is a LIE group denoted  $\mathbf{I}(M)$ ; the *connected group of isometries* is the identity component  $\mathbf{I}(M)_0$ ; following tradition, we write  $\mathbf{I}_0(M)$  for  $\mathbf{I}(M)_0$ .  $M$  is *homogeneous* if  $\mathbf{I}(M)$  is transitive on the points of  $M$ . If  $M$  is homogeneous and connected, and if  $x \in M$ , then  $g \rightarrow g(x)$  induces differentiable homeomorphisms of  $M$  with the coset spaces  $\mathbf{I}(M)/K$  and  $\mathbf{I}_0(M)/(\mathbf{I}_0(M) \cap K)$  where  $K = \{g \in \mathbf{I}(M) : g(x) = x\}$  is the *isotropy subgroup of  $\mathbf{I}(M)$  at  $x$* ;  $K$  is compact. If  $s \in \mathbf{I}(M)$  has square  $1$  and has  $x \in M$  as an isolated fixed point, then  $s$  is a *symmetry to  $M$  at  $x$* ; if  $M$  is connected,  $s$  is unique because it induces  $-I$  ( $I$  = identity) on the tangentspace  $M_x$ .  $M$  is *symmetric* if it has a symmetry at each of its points. If  $M$  is connected and symmetric, then any two points  $x$  and  $y$  can be joined by a broken geodesic, and the product of the symmetries to  $M$  at the midpoints of the geodesic segments will send  $x$  to  $y$ ; thus  $M$  is homogeneous.

Let  $M$  be complete and simply connected. Then [7]  $M$  is isometric to a product  $M_0 \times M_1 \times \dots \times M_t$  where  $M_0$  is a EUCLIDEAN space (the *EUCLIDEAN factor of  $M$* ) and the other  $M_i$ , the *irreducible factors of  $M$* , are irreducible, i.e., are non-EUCLIDEAN and not locally products of lower dimensional manifolds. This *DE RHAM decomposition* is unique up to the order of the factors.  $M$  is homogeneous (resp. symmetric) if and only if each of the  $M_i$  is homogeneous (resp. symmetric). Identifying  $M$  with  $M_0 \times \dots \times M_t$  and letting  $\mathbf{I}(M_i)$  act on  $M$  by acting on  $M_i$  in the usual way and by acting trivially on the other  $M_j$ ,  $\mathbf{I}(M)$  is generated by the  $\mathbf{I}(M_i)$  and by all permutations on sets of mutually isometric factors  $M_i$ . In particular,  $\mathbf{I}_0(M) = \mathbf{I}_0(M_0) \times \mathbf{I}_0(M_1) \times \dots \times \mathbf{I}_0(M_t)$ .

**2. 4. Curvature, characteristic and submanifolds.** If  $S$  is a two dimensional subspace of a tangentspace  $M_x$  to a RIEMANNIAN manifold  $M$ , then in a neigh-

neighborhood of  $x$  the geodesics of  $M$  through  $x$  tangent to  $S$  form a surface; the *sectional curvature of  $M$  at  $(S, x)$*  is the GAUSSIAN curvature of that surface at  $x$ . In a EUCLIDEAN space, every sectional curvature is zero. In a compact RIEMANNIAN symmetric space, every sectional curvature is  $\geq 0$ . In a non-compact irreducible RIEMANNIAN symmetric space, every sectional curvature is  $\leq 0$  and some are  $< 0$ . In particular, if  $M$  is a complete simply connected RIEMANNIAN symmetric space, then  $M$  has every sectional curvature  $\geq 0$  if and only if every irreducible factor of  $M$  is compact.

A submanifold of  $M$  is *totally geodesic* if and only if every geodesic of the submanifold is a geodesic of  $M$ , i.e., if and only if the submanifold contains every geodesic of  $M$  to which it is tangent. If  $X$  is a totally geodesic submanifold of  $M$ ,  $x \in X$  and  $S$  is a two dimensional subspace of  $X_x$ , then it is clear that  $M$  and  $X$  have the same sectional curvature at  $(S, x)$ . In particular, the sectional curvatures of  $X$  satisfy any bounds satisfied by those of  $M$ .

The *rank* of a compact LIE group is the common dimension of its maximal toral subgroups. If  $K$  is a closed subgroup of a compact LIE group  $G$ , then [8] the EULER-POINCARÉ characteristic (in any homology or cohomology theory)  $\chi(G/K) \geq 0$ , and  $\chi(G/K) > 0$  if and only if  $\text{rank. } G = \text{rank. } K$ .

**2. 5. RIEMANNIAN coverings and locally symmetric spaces.** A *RIEMANNIAN covering* is a covering  $\pi: M \rightarrow N$  of *connected* RIEMANNIAN manifolds where  $\pi$  is a local isometry. It is then easily seen that the group  $\Gamma$  of deck transformations of  $\pi$  (homeomorphisms  $\gamma: M \rightarrow M$  with  $\pi = \pi \cdot \gamma$ ) is a discrete subgroup of  $I(M)$  acting freely and properly discontinuously on  $M$ . If  $M$  is simply connected, then  $\Gamma$  is identified with the fundamental group  $\pi_1(N)$  and  $N$  is identified with the quotient space  $M/\Gamma$ . Conversely, if  $M$  is a connected RIEMANNIAN manifold and  $\Gamma$  is a subgroup of  $I(M)$  acting freely and properly discontinuously, then  $M/\Gamma$  admits a unique RIEMANNIAN structure such that the projection  $M \rightarrow M/\Gamma$  is a RIEMANNIAN covering.

A RIEMANNIAN manifold  $M$  is *locally symmetric* if every  $x \in M$  has an open neighborhood which, in the induced RIEMANNIAN structure, admits a symmetry at  $x$ . This is the case if  $M$  is symmetric, if  $M$  is a RIEMANNIAN covering manifold of a locally symmetric RIEMANNIAN manifold, or if  $M$  admits a RIEMANNIAN covering by a locally symmetric RIEMANNIAN manifold.  $M$  is complete, connected and locally symmetric, if and only if its universal RIEMANNIAN covering manifold is symmetric. In particular,  *$M$  is a complete connected locally symmetric RIEMANNIAN manifold with every sectional curvature  $\geq 0$ , if and only if the universal RIEMANNIAN covering manifold of  $M$  is the product of a EUCLIDEAN space and a compact simply connected RIEMANNIAN symmetric space.* This is the sort of manifold with which we shall concern ourselves here.

### 3. The structure theorems for locally symmetric spaces

Our main results on the structure of locally symmetric spaces of non-negative curvature are:

**3. 1. Topological Structure Theorem.** *Let  $N$  be a complete connected locally symmetric RIEMANNIAN manifold with every sectional curvature  $\geq 0$ . Then:*

1. *There is a real analytic covering  $N' \rightarrow N$  of finite multiplicity where  $N'$  is the product of a EUCLIDEAN space, a torus, and a compact simply connected RIEMANNIAN symmetric space. This covering need not be RIEMANNIAN. In particular, the fundamental group  $\pi_1(N)$  has a free abelian subgroup of finite index.*

2. *There is a real analytic deformation retraction of  $N$  onto a compact totally geodesic submanifold which lifts to a deformation retraction of  $N'$  onto the product of its toral and compact simply connected factors. In particular the betti numbers of  $N$  are finite for singular homology and cohomology, and the EULER-POINCARÉ characteristic  $\chi(N)$ , alternating sum of the betti numbers, is a well defined integer. We have  $\chi(N) \geq 0$ .*

3. *If  $\chi(N) \neq 0$ , then  $\pi_1(N)$  is a finite 2-group (finite of some order  $2^a$ ).*

Given the first and second statements above, it is easily seen that  $\pi_1(N)$  must be finite when  $\chi(N) \neq 0$ , but it is a bit surprising that  $\pi_1(N)$  must be a 2-group. This comes from an examination of the universal covering of  $N$  and the form of the elements of  $\pi_1(N)$ , and from É. CARTAN's determination [4] of the full groups of isometries of symmetric spaces:

**3. 2. Geometric Structure Theorem.** *Let  $M = M_0 \times M_1 \times \dots \times M_t$  where  $M_0$  is a EUCLIDEAN space and each  $M_i (i > 0)$  is a compact connected simply connected irreducible RIEMANNIAN symmetric space with  $\chi(M_i) > 0$ . Let  $\Sigma$  be a group of isometries acting freely on  $M_1 \times \dots \times M_t$ , let  $f$  be a homomorphism of  $\Sigma$  into the orthogonal group of  $M_0$ , and let  $\Gamma$  be the group of isometries of  $M$  consisting of all  $\gamma = f(\sigma) \times \sigma$ . Then  $\Gamma$  is isomorphic to  $\Sigma$  and is a finite 2-group, and  $M/\Gamma$  is a complete connected locally symmetric RIEMANNIAN manifold with every sectional curvature  $\geq 0$  and EULER-POINCARÉ characteristic  $\chi(M/\Gamma) > 0$ . If an element of  $\Sigma$  has order  $2^{u+1}$ , then it induces a transformation*

$$(x_1, \dots, x_m) \rightarrow (\tau x_m, x_1, \dots, x_{m-1})$$

*on a product of  $m$  distinct mutually isometric factors  $M_i$  of  $M$ , where either  $m = 2^u$  and  $\tau$  is a fixed point free involutive isometry, or  $m = 2^{u-1}$  and (for some  $n \geq 2$ ) each of these  $M_i$  is isometric to the oriented real GRASSMANN manifold  $\text{SO}(4n)/\text{SO}(2n) \times \text{SO}(2n)$ , and  $\tau^2$  is a fixed point free involutive isometry.*

*Conversely, every complete connected locally symmetric RIEMANNIAN manifold, with all curvatures  $\geq 0$  and nonzero characteristic, is isometric to a manifold  $M/\Gamma$  described above.*



**3. 3. Outline of proof.** The remainder of § 3 is devoted to proving Theorems 3.1 and 3.2.

We identify  $\pi_1(N)$  with the group  $\Gamma$  of deck transformations of the universal RIEMANNIAN covering  $M \rightarrow N$ . To obtain the finite covering and the retraction of  $N$ , we find a free abelian subgroup  $\Delta$  of finite index in  $\Gamma$  and submit  $M$ ,  $\Delta$  and  $\Gamma$  to various deformations. The existence of  $\Delta$  (§ 3.4) is due to L. AUSLANDER. The deformations of  $\Delta$  (§ 3.5) are done in sufficient generality for their applications in § 4 as well as in § 3. It is then (§ 3.8) proved that, if  $\chi(N) \neq 0$ , then  $\chi(N) > 0$ ,  $\pi_1(N)$  is finite, and the converse of Theorem 3.2 holds; this is done by combining the retraction and the finite covering. It then suffices to prove that  $\Sigma$  is a 2-group whose elements induce the transformations given; this is done in §§ 3.10–3.11, and is based on a theorem (§ 3.9) that  $\tau^4$  has a fixed point if  $\tau$  is an isometry of an  $M_i$ .

**3. 4. The free abelian subgroup** of finite index in  $\pi_1(N)$  will be exhibited as a consequence of a result of L. AUSLANDER ([2], Th. 3) which requires some interpretation. The precise statement, slightly sharpened, is:

**Proposition** (L. AUSLANDER ([2], Th. 3)). *Let  $D$  be a discrete subgroup of a semi-direct product  $H \cdot C$ , where  $H$  is a connected simply connected nilpotent LIE group acted upon (by automorphisms, but not necessarily effectively) by a compact LIE group  $C$ . Then  $D^* = D \cap (\overline{DH})_0$  is a subgroup of finite index in  $D$ , and  $D^* = A \times B$  where  $A$  is a finite abelian group and  $B$  is isomorphic to a discrete subgroup with compact quotient in some connected subgroup  $H^*$  of  $H$ .*

*Proof.* The first two paragraphs of L. AUSLANDER's proof ([2], pp. 279–280) show that, after conjugation by an element of  $H$ ,  $D^* \subset W \cdot T$  where  $W$  is a connected subgroup of  $H$  and  $T$  is a torus in  $C$  which centralizes  $W$ . For the sharpening, we replace the third paragraph of a slight variant.  $D^*$  is finitely generated because it is discrete in the connected solvable group  $W \cdot T$  ([5], Th. 1'), so  $D^*/[D^*, D^*]$  is a finitely generated abelian group. Thus  $D^*/[D^*, D^*] = A' \times B'$  where  $A'$  is the torsion subgroup.  $[D^*, D^*] \subset W$  because  $T$  is abelian and centralizes  $W$ ; thus the projection

$$f: D^* \rightarrow D^*/[D^*, D^*] \text{ maps } A = D \cap T$$

isomorphically onto  $A'$ ; it follows that  $D = A \times B$  where  $B = f^{-1}(B')$ . Now let  $g: W \cdot T \rightarrow W$  be the projection and define  $H^*$  to be the smallest analytic subgroup of  $W$  which contains  $g(B)$ .  $g$  maps  $B$  isomorphically onto  $g(B)$ ,  $g(B)$  is discrete in  $H^*$  because  $T$  is compact, and it is standard that  $H^*/g(B)$  is compact. Q.E.D.

**3. 5. The deformation** of the free abelian subgroup and the corresponding quotient manifold is given by:

**Deformation Theorem.** Let  $G = S \cdot C$  be a semidirect product of LIE groups, let  $D$  be a torsion free subgroup of  $G$  with generating set  $\{d_1, \dots, d_n\}$  such that, given  $d \in D$ , there is a unique set  $\{u_i\}$  of integers such that  $d = d_1^{u_1} d_2^{u_2} \dots d_n^{u_n}$ ; suppose that  $D$  acts freely and properly discontinuously on  $G/C$  by  $d: gC \rightarrow dgC$ , and assume that the projection of  $D$  into  $C$  lies in a torus  $A$  which centralizes  $D$ . Write  $d_i = s_i a_i$  with  $s_i \in S$  and  $a_i \in A$ , choose elements  $X_i$  in the LIE algebra of  $A$  such that  $a_i = \exp(X_i)$ , define  $d_i^{(t)} = d_i \cdot \exp(-tX_i) = s_i \cdot \exp((1-t)X_i)$ , and let  $D^{(t)}$  be the group generated by  $\{d_i^{(t)}\}$ . Then

1.  $D^{(0)} = D$  and  $D^{(1)} \subset S$ .

2. If  $K$  is a closed subgroup of  $C$ , so  $P = G/K$  is an analytic manifold on which  $G$  acts by  $g: hK \rightarrow ghK$ , then each  $D^{(t)}$  acts freely and properly discontinuously on  $P$ ; in particular, the projections  $P \rightarrow P/D^{(t)}$  are coverings of analytic manifolds.

3. The maps  $d_i^{(r)} \rightarrow d_i^{(s)}$  define isomorphisms (the "deformation isomorphisms") of  $D^{(r)}$  onto  $D^{(s)}$ , and these isomorphisms induce analytic homeomorphisms of  $P/D^{(r)}$  onto  $P/D^{(s)}$ .

4.  $P/D$  is analytically homeomorphic to  $(S/D^{(1)}) \times (C/K)$ .

*Proof.* The first statement is obvious. If  $\mathcal{R}$  is a group relation and

$$\mathcal{R}(s_1, \dots, s_n) = 1, \text{ then } \mathcal{R}(d_1, \dots, d_n) \in A$$

because  $A$  is abelian,  $A$  contains the  $a_i$ , and  $A$  centralizes the  $s_i$ . But

$$D \cap A \subset D \cap C = \{1\}$$

because  $D$  acts freely on  $G/C$ ; thus  $\mathcal{R}(d_1, \dots, d_n) = 1$ . This shows that the  $d_i$  satisfy every relation satisfied by the  $s_i$ ; it follows that  $d_i^{(t)} \rightarrow d_i$  induces a homomorphism of  $D^{(t)}$  onto  $D$ . Every element of  $D^{(t)}$  has some expression  $(d_1^{(t)})^{u_1} (d_2^{(t)})^{u_2} \dots (d_n^{(t)})^{u_n}$  because the  $s_i$  satisfy every relation satisfied by the  $d_i$ , and every element of  $D$  has unique expression  $d_1^{u_1} d_2^{u_2} \dots d_n^{u_n}$ ; it follows that the epimorphism  $D^{(t)} \rightarrow D$  is an isomorphism. This gives the deformation isomorphisms.

For the second statement, we note that  $D^{(t)} \subset G$  acts freely and properly discontinuously on  $G/C$ , if and only if  $D^{(t)} \subset S \cdot A$  acts freely and properly discontinuously on  $(S \cdot A)/A$ . As  $A$  is compact, and as  $D^{(t)}$  is discrete (because  $D^{(1)}$  is discrete in  $S$ , consequence of proper discontinuity of  $D$  on  $G/C$ ) and torsionfree (because it is isomorphic to the torsionfree group  $D$ ),  $D^{(t)}$  must be free and properly discontinuous on  $(S \cdot A)/A$ . This proves the second statement; the third follows because the deformations are along analytic arcs.

For the last statement, view  $P/D^{(t)}$  as the double coset space  $D^{(t)} \backslash G/K$ . Writing  $\cong$  for analytic homeomorphism, we then have

$$P/D \cong P/D^{(1)} \cong (D^{(1)} \backslash S) \cdot (C/K).$$

Now observe that  $(s, c) \rightarrow sc$  induces  $S \times C \cong G \cong S \cdot C$  and  $s \rightarrow s^{-1}$  induces  $D^{(1)} \setminus S \cong S/D^{(1)}$ ; it follows that  $P/D \cong (S/D^{(1)}) \times (C/K)$ . *Q.E.D.*

**3. 6. The finite covering.** Identify  $\pi_1(N)$  with the group  $\Gamma$  of deck transformations of the universal RIEMANNIAN covering  $M \rightarrow N$ .  $M = M_0 \times M'$  where  $M_0$  is a EUCLIDEAN space and  $M'$  is a product of irreducible RIEMANNIAN symmetric spaces, for  $N$  is complete, connected and locally symmetric. As  $N$  has every sectional curvature  $\geq 0$ , the same is true for  $M'$ ; it follows that  $M'$  is compact because a noncompact irreducible RIEMANNIAN symmetric space has a negative sectional curvature. In particular, the full group of isometries  $I(M')$  is compact.

$I(M_0)$  is the ordinary EUCLIDEAN group on  $n = \dim. M_0$  variables, and may be viewed as a semidirect product  $\mathbf{R}^n \cdot \mathbf{O}(n)$  where  $\mathbf{R}^n$  is the vector group and  $\mathbf{O}(n)$  is the orthogonal group. This allows us to view  $I(M) = I(M_0) \times I(M')$  as a semidirect product  $\mathbf{R}^n \cdot C$  where  $C = \mathbf{O}(n) \times I(M')$ . As  $\Gamma$  is a discrete subgroup of  $I(M)$ , Proposition 3.4 gives a finitely generated free abelian subgroup  $\Delta$  of finite index in  $\Gamma$  corresponding to the group  $B$  there.

By construction, the projection of  $\Delta$  into  $C$  lies in a torus. The condition of Theorem 3.5 for expression of elements in terms of generators is obvious for finitely generated free abelian groups.  $\Delta$  acts freely and properly discontinuously on  $I(M)/C$  because  $C$  is compact and  $\Delta$  is discrete and torsion free. Now Theorem 3.5 shows that  $M/\Delta$  is real analytically homeomorphic to  $(M_0/\Delta') \times M'$  where  $\Delta'$  is a discrete group of pure translations of  $M_0$  which is isomorphic to  $\Delta$ . Define  $N' = (M_0/\Delta') \times M'$  and recall that  $M/\Delta \rightarrow M/\Gamma = N$  is a finite RIEMANNIAN covering. This proves the first statement of Theorem 3.1.

**3. 7. The deformation retraction of  $N$  onto a compact submanifold** is accomplished by a deformation of  $\Gamma$  onto another group  $\Gamma'$ , followed by a  $\Gamma'$ -equivariant deformation retraction of  $M$ . We retain the notation  $\Gamma, M, M', M_0$  and  $\Delta$  from § 3.6, except that we may replace  $\Delta$  by the intersection of its conjugates in  $\Gamma$ , and thus assume  $\Delta$  normal in  $\Gamma$ .

Every  $\gamma \in \Gamma$  is of the form  $\gamma_0 \times \gamma'$  where  $\gamma_0 \in I(M_0)$  and  $\gamma' \in I(M')$ . For a choice of origin in  $M_0$ ,  $\gamma_0$  is further decomposed into  $(\gamma_t, \gamma_r)$  where  $\gamma_t \in M_0$  indicates a translation and  $\gamma_r$  is a rotation. By construction of  $\Delta$ , we may choose the origin so that  $\delta_r: \delta_t \rightarrow \delta_t$  for every  $\delta \in \Delta$ . The origin so chosen,  $M_0$  is identified with the vectorspace  $\mathbf{R}^n$ , and we have an orthogonal direct sum decomposition  $M_0 = U + V$  where  $V$  is the subspace spanned by the  $\delta_t$ . Every  $\gamma_r$  preserves  $V$ , and thus preserves  $U$ , because  $\Delta$  is normal in  $\Gamma$ .

Given  $\gamma \in \Gamma$ , we have  $\gamma_t = \gamma_U + \gamma_V$  with  $\gamma_U \in U$  and  $\gamma_V \in V$ . If  $s$  is a real number, define  $\gamma^{(s)} = (s\gamma_U + \gamma_V, \gamma_r) \times \gamma'$  and let  $\Gamma_s$  be the subgroup of  $I(M)$  generated by the  $\gamma^{(s)}$ . It is easily checked that  $\gamma \rightarrow \gamma^{(s)}$  defines an



isomorphism of  $\Gamma$  onto  $\Gamma_s$ .  $\Gamma_s$  is discrete in  $\mathbf{I}(M)$  because it contains  $\Delta$  as a subgroup of finite index; thus  $\Gamma_s$  acts properly discontinuously on  $M$ . Now if  $\gamma^{(s)}$  has a fixed point, it must have finite order, whence  $\gamma$  has finite order; it follows that either  $\gamma = 1$  or  $\gamma'$  has no fixed point;  $\gamma^{(s)} = 1$  because the latter would prevent  $\gamma^{(s)}$  from having a fixed point. Thus  $\Gamma_s$  acts freely on  $M$ . We now have a one parameter family of manifolds  $N_s = M/\Gamma_s$  which are analytically homeomorphic to  $N = N_1$ . It will be clear that this isotopy of the metric of  $N$  is the identity on a compact totally geodesic submanifold onto which  $N_0$  is retracted. For the proof of Theorem 3.1, then, we may replace  $N$  by  $N_0$ . In other words, we may assume each  $\gamma_t \in V$ .

We have  $M$  as a RIEMANNIAN product  $U \times V \times M'$  where  $U$  and  $V$  are EUCLIDEAN spaces with vectorspace structure, and every  $\gamma \in \Gamma$  is of the form  $\gamma_1 \times \gamma_2 \times \gamma'$  where  $\gamma_1$  is a rotation of  $U$ ,  $\gamma_2$  is an isometry of  $V$ , and  $\gamma'$  is an isometry of  $M'$ . Define  $f_s: M \rightarrow M$  by  $f_s(u, v, m') = (su, v, m')$ ;  $f_s$  is  $\Gamma$ -equivariant because each  $\gamma_1$  is a linear transformation. Thus  $f_s$  induces a map  $g_s: N \rightarrow N$ . This gives a deformation retraction of  $N = g_1(N)$  onto  $g_0(N)$ . But  $g_0(N) = f_0(M)/\Gamma = (V \times M')/\Gamma$  admits a covering by  $(V \times M')/\Delta$ , and, as in § 3.7., Theorem 3.5 shows that  $(V \times M')/\Delta$  is homeomorphic to  $(V/\Delta') \times M'$  where  $\Delta'$  is the group of translations consisting of the  $\delta_t$ .  $V/\Delta'$  is a torus, compact by definition of  $V$ ; thus  $g_0(N)$  is compact.

We have now exhibited a deformation retraction of  $N$  onto a compact submanifold. As singular homology and cohomology satisfy the homotopy axiom, the betti numbers of  $N$  are finite, and the EULER-POINCARÉ characteristic  $\chi(N)$  is a well defined integer, in those theories.

Observe that the deformation of  $\Gamma$  did not move any points of  $g_0(N)$ . It is now clear that  $g_0(N)$  is totally geodesic in  $N$ , for it is the image of a totally geodesic submanifold  $V \times M'$  of  $M$ .

**3.8. Finiteness of the fundamental group.** We have seen that the deformation retraction  $g_0(N)$  admits a covering of some finite multiplicity  $r$  by  $T \times M'$ , where  $T$  is a torus with  $\pi_1(T)$  isomorphic to the subgroup  $\Delta$  of finite index in  $\Gamma$ . As  $g_0(N)$ ,  $T$  and  $M'$  are compact manifolds, we have

$$\chi(N) = \chi(g_0(N)) = \frac{1}{r} \chi(T \times M') = \frac{1}{r} \chi(T) \cdot \chi(M').$$

Now suppose  $\chi(N) \neq 0$ . Then  $\chi(T) \neq 0 \neq \chi(M')$ .  $\chi(T) \neq 0$  means that  $T$  is a single point, so  $\chi(N) = \frac{1}{r} \chi(M')$  and  $\Delta = \{1\}$ . As  $\Delta$  has finite index in  $\Gamma$ ,  $\Gamma = \pi_1(N)$  must be finite. Now  $\chi(M') \neq 0$  implies  $\chi(M') > 0$  because  $M'$  is a quotient space of a compact LIE group  $\mathbf{I}(M')$  by a closed subgroup [8]; thus  $\chi(N) > 0$ .

The second statement, and the finiteness assertion of the third statement, of Theorem 3.1 are now proved.

Suppose again that  $\chi(N) \neq 0$ . As  $\Gamma$  is finite, the  $\gamma_0$  of § 3.7 form a finite group; it is classical that some point of  $M_0$  must be fixed under every  $\gamma_0$ . Changing the origin in  $M_0$ ,  $\Gamma$  is the group of isometries of  $M$  consisting of all  $f(\sigma) \times \sigma$ , as  $\sigma$  runs through a finite group  $\Sigma$  of isometries acting freely on  $M'$ , where  $f$  is a homomorphism  $\gamma' \rightarrow \gamma_0$  of  $\Sigma$  into the orthogonal group of  $M_0$ . Now  $\chi(M') \neq 0$ , so  $\chi(M_i) \neq 0$  ( $i > 0$ ) where  $M' = M_1 \times \dots \times M_t$  is the decomposition of  $M'$  into irreducible factors. It follows that  $\chi(M_i) > 0$  [8] and every group of isometries acting freely on  $M'$  is finite [11]. This proves the converse and finiteness condition of Theorem 3.2, and that the manifold  $M/\Gamma$  there is a complete connected locally symmetric RIEMANNIAN manifold with every sectional curvature  $\geq 0$  and  $\chi(M/\Gamma) > 0$ .

To complete the proofs of Theorems 3.1 and 3.2, now, we need only prove that every element of the group  $\Sigma$  of Theorem 3.2 has some order  $2^{u+1}$  and induces a transformation of the type exhibited there.

**3.9.** In order to study the elements of  $\Sigma$ , we need some information on fixed points:

**Fixed Point Theorem.** *Let  $\tau$  be an isometry of a compact connected simply connected irreducible RIEMANNIAN symmetric space  $S$  with  $\chi(S) \neq 0$ . If  $\tau^2$  has no fixed point, then  $S$  is isometric to a real GRASSMANN manifold  $\mathbf{SO}(4k)/\mathbf{SO}(2k) \times \mathbf{SO}(2k)$ ,  $k \geq 2$ , and  $\tau^4$  has a fixed point.*

*Proof.* Let  $K$  be an isotropy subgroup of  $G = \mathbf{I}(S)$ . The identity component  $K_0$  contains a maximal torus of  $G_0$  because  $\chi(G_0/K_0) = \chi(S) \neq 0$ , by SAMUELSON'S theorem [8], so every element of  $G_0$  is conjugate to an element of  $K_0$ . In other words, every element of  $G_0$  has a fixed point. Let  $t$  be the image of  $\tau$  in  $G/G_0$ ;  $\tau^m$  has a fixed point if  $t^m = 1$ .

Suppose that  $\tau^2$  has no fixed point. Then  $G/G_0$  has an element of order greater than 2. It follows from CARTAN'S construction of  $\mathbf{I}(S)$  [4] that

$$S = \mathbf{SO}(4k)/\mathbf{SO}(2k) \times \mathbf{SO}(2k)$$

where  $k \geq 2$ ; if, further,  $G/G_0$  has an element  $u$  with  $u^4 \neq 1$ , then  $k = 2$  and  $u$  has order 3. But if  $t^3 = 1$ , then  $\tau^3 \in \mathbf{I}_0(S) = G_0$ , so  $\tau^3$  is homotopic to the identity. It is known that  $\tau$  must be fixed point in this case ([14], §§ 5.5.9–5.5.10), so  $\tau^2$  has a fixed point. This contradicts  $t^3 = 1$ . The only other possibility is that  $t^4 = 1$  and  $\tau^4$  has a fixed point. Q.E.D.

**3.10. 2-groups.** We will see that  $\Gamma$  and  $\Sigma$  are 2-groups.

If  $g$  is an isometry of  $M' = M_1 \times \dots \times M_t$ , then we have decompositions

$$M' = X_1 \times \dots \times X_u, \quad g = g_1 \times \dots \times g_u$$

where  $g_i$  is an isometry of  $X_i$  which cyclically permutes its irreducible factors. Thus, under appropriate isometric identifications, we have  $X_i = S_i \times \dots \times S_i$  ( $v_i$  factors) with  $S_i$  irreducible, and

$$g_i: (s_1, \dots, s_{v_i}) \rightarrow (\tau_i s_{v_i}, s_1, \dots, s_{v_i-1})$$

gives the action of  $g_i$  on  $X_i$ , for some isometry  $\tau_i$  of  $S_i$ . Now if  $g$  has order  $m$ , then each  $v_i$  must divide  $m$ , say  $m = v_i m_i$ ;  $g^{v_i}$  induces the transformation  $\tau_i \times \dots \times \tau_i$  on  $X_i$ , and  $\tau_i^{m_i} = 1$ .

Suppose now that  $g$  has odd order  $m$ , and retain the notation above. Each  $\tau_i$  must have odd order, so  $\tau_i$  is a power of  $\tau_i^4$ . As  $\chi(M') \neq 0$ , we have  $\chi(S_i) \neq 0$ , and Theorem 3.9 shows that each  $\tau_i$  has a fixed point  $s_i \in S_i$ . Define  $x_i = (s_i, s_i, \dots, s_i) \in X_i$ ; then  $g_i(x_i) = x_i$ . It follows that

$$x = (x_1, x_2, \dots, x_u)$$

is a fixed point for  $g$ .

If  $\Gamma$  is not a 2-group, then it has an element  $\gamma$  of odd order  $m > 1$ .  $\gamma = f(\sigma) \times \sigma$  where  $\sigma$  has order  $m$  in  $\Sigma$ . The considerations above show that  $\sigma$  has a fixed point, contradicting the hypothesis that  $\Sigma$  act freely on  $M'$ .

This proves that  $\Gamma$  and  $\Sigma$  are 2-groups.

**3.11. The form of the group elements** now comes easily. Let  $1 \neq g \in \Sigma$ . Then  $g$  has some order  $m = 2^{u+1}$   $u \geq 0$ . Retain the notation of § 3.10 for the decompositions of  $M'$  and  $g$ . Then  $m_i = 2^{a_i}$  and  $v_i = 2^{b_i}$  where  $a_i + b_i = u + 1$ . As  $g^{2^u}$  has no fixed point, some  $g_i^{2^u}$  has no fixed point. For this index  $i$ , it is easily seen that  $b_i \leq u$ , say  $u = b_i + w$ , whence  $g_i^{2^u} = \tau_i^{2^w} \times \dots \times \tau_i^{2^w}$ . It follows that  $\tau_i^k$  has a fixed point on  $S_i$  if and only if  $k$  is a multiple of  $2^{w+1}$ ; by Theorem 3.9,  $w = 0$  or  $1$ , and  $S_i$  is isometric to  $\mathbf{SO}(4n)/\mathbf{SO}(2n) \times \mathbf{SO}(2n)$  ( $n \geq 2$ ) in case  $w = 1$ .

This completes the proof of Theorems 3.1 and 3.2.

*Q.E.D.*

#### 4. RIEMANNIAN nilmanifolds

##### and a structure theorem for locally homogeneous spaces

The proof of some parts of Theorem 3.1 do not make full use of the hypotheses. We will prove the following extension to locally homogeneous spaces.

**4.1. Theorem.** *Let  $M \rightarrow N$  be a universal RIEMANNIAN covering where  $M = M_0 \times M'$ , a nilpotent LIE group acts transitively by isometries on  $M_0$ , and  $M'$  is a compact RIEMANNIAN homogeneous manifold. Then there is a real analytic covering  $N' \rightarrow N$  of some finite multiplicity  $r > 0$  where  $N' = E \times N'' \times M'$ ,  $N''$  is a compact coset space of a nilpotent LIE group by a*

discrete subgroup, and  $E$  is diffeomorphic to a EUCLIDEAN space; if  $M_0$  is isometric to a EUCLIDEAN space, then  $N''$  is a torus and there is a real analytic deformation retraction of  $N$  onto a compact totally geodesic submanifold which lifts to one of the deformation retractions of  $N'$  onto  $N'' \times M'$ . In particular, the EULER-POINCARÉ characteristic  $\chi(N')$  of singular theory is a well defined integer; now  $\chi^*(N) = \frac{1}{r} \chi(N')$ , the so called rational EULER-POINCARÉ characteristic of  $N$ , is a well defined non-negative integer, and  $\chi^*(N) = \chi(N)$  if  $M_0$  is EUCLIDEAN. If  $\chi^*(N) \neq 0$ , then the fundamental group  $\pi_1(N)$  is finite. If  $\chi^*(N) \neq 0$  and  $M'$  is RIEMANNIAN symmetric, then  $\pi_1(N)$  is a finite 2-group.

I am indebted to DAVID B. A. EPSTEIN for drawing my attention to C.T.C. WALL's rational EULER characteristic [10] and for suggesting a way of adapting it to this context. § 4.4 is based on conversations with him.

**4.2. RIEMANNIAN nilmanifolds** are defined to be RIEMANNIAN manifolds which admit a transitive nilpotent group of isometries. The structure of  $M_0$  is clarified by:

**Theorem.** Let  $B$  be a positive definite bilinear form on the LIE algebra  $\mathfrak{S}$  of a connected nilpotent LIE group  $S$ , let  $K$  be the group of all automorphisms of  $S$  which preserve  $B$ , and let  $X$  be  $S$  with the left invariant RIEMANNIAN metric derived from  $B$ . Then  $X$  is a connected RIEMANNIAN nilmanifold,  $I(X)$  is the semidirect product  $S \cdot K$  acting by  $(s, k): x \rightarrow s \cdot k(x)$ ,  $S$  is the nilradical (maximal connected normal nilpotent subgroup) of  $I_0(X)$ ,  $S$  is a maximal connected nilpotent subgroup of  $I_0(X)$ , and  $S$  is the only transitive connected nilpotent subgroup of  $I(X)$ . Conversely, every connected RIEMANNIAN nilmanifold is isometric to one of the manifolds  $X$  described above.

**Corollary.** If  $X$  is a connected RIEMANNIAN nilmanifold and  $x \in X$ , then the RIEMANNIAN structure on  $X$  defines a unique structure of nilpotent LIE group in which  $x = 1$ .

**Corollary.** Let  $\pi: Y \rightarrow X$  be a RIEMANNIAN covering where  $X$  is a RIEMANNIAN nilmanifold, and let  $y \in Y$ . Then  $Y$  is a RIEMANNIAN nilmanifold. Endow  $Y$  (resp.  $X$ ) with its canonical nilpotent LIE group structure for which  $y = 1$  (resp.  $\pi(y) = 1$ ). Then  $\pi$  is an epimorphism of LIE groups, and the deck transformations of  $\pi$  are left translations by the elements of the kernel of  $\pi$ .

**Corollary.** Let  $\Gamma$  be the group of deck transformations of a universal RIEMANNIAN covering  $X \rightarrow Y$  where  $X$  is a RIEMANNIAN nilmanifold. Then these are equivalent:

1.  $Y$  is a RIEMANNIAN nilmanifold.
2.  $Y$  is a RIEMANNIAN homogeneous manifold.

3.  $\Gamma$  consists of isometries of constant displacement.

4.  $\Gamma$  consists of isometries of bounded displacement.

**Corollary.** *Let  $\pi: X \rightarrow Z$  be a RIEMANNIAN covering where  $Z$  is compact and  $X$  is a RIEMANNIAN nilmanifold. Then  $\pi$  factors into RIEMANNIAN coverings  $\alpha: X \rightarrow Y$  and  $\beta: Y \rightarrow Z$  where  $Y$  is a compact nilmanifold and  $\beta$  is of finite multiplicity;  $Y$  is a RIEMANNIAN nilmanifold if and only if it is isometric to a flat torus.*

We complete § 4.2 by deriving the Corollaries from the Theorem; the Theorem will be proved in § 4.3, and we will then go on to the proof of Theorem 4.1.

The first Corollary is clear because  $S \subset \mathbf{I}(X)$  is unique and acts simply transitively on  $X$  in the Theorem. For the second, we give  $X$  its LIE structure with  $\pi(y) = 1$ , let  $S \subset \mathbf{I}(X)$  denote the left translations, and lift the action of  $S$  to  $Y$  after backing off to the universal covering group of  $S$ .

The third Corollary is a little more complicated. It is clear that (1) implies (2) and that (3) implies (4), and it is known [12] that (2) implies (3); thus we need only prove that (4) implies (1). Choose  $x \in X$  and give  $X$  the nilpotent LIE group structure  $S$  in which  $x = 1$ . In the notation of the Theorem, we must prove every element of  $\Gamma$  to be central in  $S$ ; then  $S$  induces a transitive nilpotent group of isometries of  $Y$ , and (1) is proved.

Let  $g \in \mathbf{I}(X)$  be an isometry of bounded displacement,  $g = (s, k)$  with  $s \in S$  and  $k \in K$  in the notation of the Theorem. As  $K$  is compact, there is a compact set  $C \subset \mathbf{I}(X)$  with  $hgh^{-1} \in C$  for every  $h \in \mathbf{I}(X)$ ;  $h = (t^{-1}, 1)$  gives  $(t^{-1} \cdot s \cdot k(t), k) \in C$ , and it follows that  $S$  has a compact set which contains  $t^{-1} \cdot k(t)$  for every  $t \in S$ . The exponential map  $\exp: \mathfrak{S} \rightarrow S$  is a homeomorphism and  $k$  is linear on  $\mathfrak{S}$ ; it follows that  $k = 1$  because the linear isotropy representation of  $K$  is faithful. Now  $g = (s, 1)$ . Every  $(tst^{-1}, 1) \in C$ , so the closure of the conjugacy class of  $s$  in  $S$  is compact. Let  $T$  be the centralizer of  $s$  in  $S$ ; now  $S/T$  is compact. Let  $P \in \mathfrak{S}$  with  $\exp(P) = s$ , and suppose  $Q \in \mathfrak{S}$ ; it is easily seen that  $[P, Q] = 0$  if and only if  $s$  commutes with  $\exp(Q)$ ; thus  $T$  is connected. It follows that  $S/T$  is homeomorphic to a EUCLIDEAN space. As  $S/T$  is compact, we must have  $S = T$ ; thus  $s$  is central in  $S$ . This completes the proof of the third Corollary.

For the fourth Corollary, let  $\Sigma$  be the group of deck transformations of the universal RIEMANNIAN covering  $\mu: W \rightarrow Z$  and let  $\Delta$  be the deck transformations of  $\nu: W \rightarrow X$ . L. AUSLANDER has proved [1] that  $\Gamma = \Sigma \cap S_W$  has finite index in  $\Sigma$ ; as  $\Delta \subset \Sigma$  and we have just seen  $\Delta \subset S_W$  (for  $\Delta$  is central in  $S_W$ ), we have  $\Delta \subset \Gamma$ . Now define  $Y = W/\Gamma$ , and the existence of  $\alpha$  and  $\beta$  is clear. If  $Y$  is a RIEMANNIAN nilmanifold, then  $Y = S_W/\Gamma$  is a group, and so  $S_W/\Gamma$  is a torus. This proves the fourth Corollary.

**4.3. Proof of Theorem 4.2.** Let  $W$  be a connected RIEMANNIAN nilmanifold. There is a transitive nilpotent group of isometries of  $W$ ; its identity component  $T$  is transitive.  $T^*$  will denote the closure of  $T$  in  $\mathbf{I}(W)$ ;  $T^*$  is nilpotent and transitive. Write  $W$  as a coset space  $T^*/Z$  where  $Z$  is the isotropy subgroup of  $T^*$  at  $w \in W$ .  $Z$  is compact because  $T^*$  is closed in  $\mathbf{I}(W)$ ; thus  $Z$  is contained in a maximal compact subgroup  $Z^*$  of  $T^*$ .  $Z^*$  is connected because  $T^*$  is connected, and a compact connected subgroup of a nilpotent LIE group can be seen to lie in the center by looking at the universal covering group and its exponential mapping; thus  $Z$  is central in  $T^*$ .  $T^*$  acts effectively on  $W$ ; it follows that  $Z = \{1\}$  and  $T^*$  is simply transitive. As  $T \subset T^*$  and  $T$  is transitive, this proves that  $T$  is closed in  $\mathbf{I}(W)$  and simply transitive on  $W$ ; it also proves that  $T$  is maximal among the connected nilpotent subgroups of  $\mathbf{I}_0(W)$ .

Suppose that we can prove  $T$  to be contained in the nilradical  $N$  of  $\mathbf{I}_0(W)$ . Then  $T = N$ , so  $T$  is normal in  $\mathbf{I}(W)$ . If  $H$  is the isotropy subgroup at  $w \in W$ , then  $H \cap T = \{1\}$  because  $T$  is simply transitive, so  $\mathbf{I}(W)$  is a semi-direct product  $T \cdot H$ . The representation of  $H$  on the LIE algebra  $\mathfrak{L}$  is equivalent to the linear isotropy representation of  $H$  on the tangentspace  $W_w$ , and is thus faithful; now  $H$  may be viewed as a group of automorphisms of  $T$ . Identify  $T$  with  $W$ , viewing  $T$  as a LIE group with left invariant RIEMANNIAN metric specified by some positive definite bilinear form  $A$  on  $\mathfrak{L}$ . Then  $H$  preserves  $A$ , and must contain every automorphism of  $T$  which preserves  $A$  because it contains every isometry of  $W$  which fixes  $w$ . Writing  $\mathbf{I}(W) = T \cdot H$ , now, the action on  $T$  is necessarily  $(t, h): v \rightarrow t \cdot h(v)$ . As the manifold  $X$  of Theorem 4.2 is a RIEMANNIAN nilmanifold under the group  $S$  there, this will prove Theorem 4.2.

We now need only prove  $T \subset N$  where  $N$  is the nilradical of  $\mathbf{I}_0(W)$ . Let  $\pi: W' \rightarrow W$  be the universal RIEMANNIAN covering; we can lift the action of  $T$  on  $W$  to the action of a covering group  $T'$  of  $T$  on  $W'$ , and  $T'$  will be transitive on  $W'$ . Let  $\Gamma$  be the group of deck transformations of the covering, let  $N'$  be the nilradical of  $\mathbf{I}_0(W')$ , and let  $P$  be the normalizer of  $\Gamma$  in  $\mathbf{I}(W')$ .  $\pi$  induces a homomorphism  $\pi^*$  of  $P$  onto  $\mathbf{I}(W)$  with kernel  $\Gamma$ , and  $T' \subset P$  by construction. If  $T' \subset N'$ , then  $T' \subset P \cap N'$ , and the latter lies in the nilradical  $N''$  of  $P$ . It is clear that  $\pi^*(N'') = N$  and  $\pi^*(T') = T$ ; it will follow that  $T \subset N$ .

Now we assume  $W$  simply connected, and need only prove  $T \subset N$ . Let  $R$  be the radical (maximal connected normal solvable subgroup) of  $\mathbf{I}_0(W)$ . Then  $\mathbf{I}_0(W) = S \cdot R$  where  $S$  is a maximal connected semisimple subgroup. Let  $\beta: \mathbf{I}_0(W) \rightarrow \text{ad}(S)$  be the composition of taking quotient by  $R$  with the adjoint representation of  $S/S \cap R$ . Every element  $g \in \mathbf{I}(W)$  has unique and



continuous decomposition  $g = th$ ,  $t \in T$  and  $h \in H = \text{isotropy at } w$ ; thus  $\mathbf{I}_0(W)/T$  is compact; it follows that  $\text{ad}(S)/\beta(T)$  is compact. As  $\beta(T)$  is nilpotent and  $\text{ad}(S)$  is a product of centerless simple LIE groups,  $\text{ad}(S)$  must be compact. This proves that  $S$  is compact.

The identity component  $H_0$  is an isotropy subgroup and a maximal compact subgroup of  $\mathbf{I}_0(W)$ ; thus  $H_0 = S \cdot H'$  where  $H' = (H \cap R)_0$  is the identity component of the center of  $H_0$ . Let  $\beta: \mathbf{I}_0(W) \rightarrow \mathbf{I}_0(W)/N = U$ .  $N \cap H$  is a compact subgroup of  $N$  and is thus in a maximal compact subgroup of  $N$ ; this maximal one is central in  $N$ , thus unique, and thus central in  $\mathbf{I}_0(W)$ ; it follows that  $N \cap H = \{1\}$  so  $U = SR'$  where  $R' = R/N$ .  $R'$  is abelian because  $[R, R]$  is nilpotent and normal and thus in  $N$ ; it follows that  $R' = H' \times V$  where  $V$  is a vector group stable under  $S$ . Let  $M = \beta^{-1}(V)$ .  $M$  is a closed normal subgroup of  $\mathbf{I}_0(W)$  such that  $\mathbf{I}_0(W)/M$  is compact and  $\mathbf{I}_0(W)$  is semidirect product  $M \cdot H_0$ . Thus  $\dim. M = \dim. \mathbf{I}_0(W) - \dim. H_0 = \dim. T$ . Let

$$\alpha: \mathbf{I}_0(W) \rightarrow \mathbf{I}_0(W)/M.$$

$TM$  is closed in  $\mathbf{I}_0(W)$  because  $T$  and  $M$  are closed and  $M$  is normal. Thus  $\alpha(T) = (TM)/M = T/T \cap M$  is a torus. On the other hand,  $T \cap M$  is connected because it is an analytic subgroup of  $T$ . As  $T$  is connected, simply connected and nilpotent, it follows that  $\alpha(T) = T/T \cap M$  is homeomorphic to a EUCLIDEAN space. Thus  $\alpha(T) = \{1\}$ . This proves  $T \subset M$ . As they are connected groups of the same dimension, they must be equal. In particular,  $T$  is normal in  $\mathbf{I}_0(W)$ . This proves  $T \subset N$ , completing the proof of Theorem 4.2. Q.E.D.

*Remark.* Theorem 4.2 shows that the notion of RIEMANNIAN nilmanifold is but a mild generalization of the notion of RIEMANNIAN homogeneous manifold of constant curvature zero. The essential part of the proof was exhibiting of  $V$  above. This was essentially done by reducing to the case of constant zero curvature.

*Remark.* One might define a RIEMANNIAN solvmanifold to be a RIEMANNIAN manifold which admits a transitive solvable group of isometries, but the IWASAWA decomposition shows that this notion is not very restrictive. For example, a RIEMANNIAN symmetric space with every sectional curvature  $\leq 0$  is a RIEMANNIAN solvmanifold.

**4.4. Rational EULER-POINCARÉ characteristic.** All spaces are connected, locally arcwise connected, locally simply connected, and with a basepoint which will generally not be mentioned. Let  $\mathcal{C}$  be the family of finite CW complexes,  $\mathcal{C}'$  the family of spaces homotopy equivalent (respecting basepoints)

to an element of  $\mathcal{C}$ , and  $\mathcal{C}^*$  the family of spaces which admit a finite covering by an element of  $\mathcal{C}$ . Given  $X \in \mathcal{C}$ , we have the EULER-POINCARÉ characteristic (of singular theory)  $\chi(X) = \chi(Y)$  where  $X \simeq Y \in \mathcal{C}$ .

**Proposition.** *If  $Z \in \mathcal{C}^*$ , so  $Z$  admits a covering of some finite multiplicity  $r > 0$  by some  $X \in \mathcal{C}$ , then  $\chi^*(Z) = \frac{1}{r} \chi(X)$  is a well defined rational number, which we will call the rational EULER-POINCARÉ characteristic of  $Z$ . If  $Z_1$  and  $Z_2 \in \mathcal{C}^*$ , then  $\chi^*(Z_1 \times Z_2) = \chi^*(Z_1) \chi^*(Z_2)$ . If  $Z_1 \in \mathcal{C}^*$  admits a  $t$ -fold covering by a space  $Z_2$ , then  $Z_2 \in \mathcal{C}^*$  and  $\chi^*(Z_2) = t \chi^*(Z_1)$ .*

The main step in the proof is:

**Lemma.** *Given a finite covering  $g: (U, u) \rightarrow (X, x)$  and a homotopy equivalence  $h: (X, x) \rightarrow (Y, y)$  of spaces with basepoint, let  $a: (V, v) \rightarrow (Y, y)$  be the covering with  $a\pi_1(V, v) = hg\pi_1(U, u)$ . Then there is a homotopy equivalence  $b: (U, u) \rightarrow (V, v)$  which covers  $h$ .*

To prove the Lemma, one defines  $b$  by  $b(u) = v$  and by defining  $b$  to cover  $h$  along any arc starting at  $u$  which is the lift of an arc starting at  $x$ ;  $b$  is well defined because of the condition on fundamental groups. Let  $h': (Y, y) \rightarrow (X, x)$  be a homotopy inverse to  $h$ , and let  $b': (V, v) \rightarrow (U, u)$  be the map covering  $h'$ , defined from  $h'$  as  $b$  was defined from  $h$ ; it is easily seen that  $b'$  is a homotopy inverse to  $b$ .

*Proof of Proposition.* To see that  $\chi^*(Z)$  is well defined, choose  $z \in Z$  and  $r_i$ -fold coverings  $f_i: (X_i, x_i) \rightarrow (Z, z)$ ,  $X_i \in \mathcal{C}$ ; we must prove  $\frac{1}{r_1} \chi(X_1) = \frac{1}{r_2} \chi(X_2)$ .  $S_i = f_i\pi_1(X_i, x_i)$  is a subgroup of finite index  $r_i$  in  $\pi_1(Z, z)$ ; thus  $S = S_1 \cap S_2$  is a subgroup of some finite index  $s_1 r_1 = s_2 r_2$  in  $\pi_1(Z, z)$ . This gives  $s_i$ -fold coverings  $g_i: (U_i, u_i) \rightarrow (X_i, x_i)$  with  $f_i g_i \pi_1(U_i, u_i) = S$ . We have homotopy equivalences  $h_i: (X_i, x_i) \rightarrow (Y_i, y_i)$  with  $Y_i \in \mathcal{C}$ ; if  $a_i: (V_i, v_i) \rightarrow (Y_i, y_i)$  are the  $s_i$ -fold coverings with  $a_i \pi_1(V_i, v_i) = h_i g_i \pi_1(U_i, u_i)$ , then it is obvious that  $V_i \in \mathcal{C}$ , and the Lemma gives homotopy equivalences  $b_i: (U_i, u_i) \rightarrow (V_i, v_i)$ . Thus  $U_i \in \mathcal{C}$  and  $\chi(U_i) = s_i \chi(X_i)$ . Now  $f_i g_i: (U_i, u_i) \rightarrow (Z, z)$  are coverings with  $f_i g_i \pi_1(U_i, u_i) = S$ ; thus  $U_1$  is homeomorphic to  $U_2$ ; it follows that  $s_1 \chi(X_1) = s_2 \chi(X_2)$ . Dividing by  $r_1 s_1 = r_2 s_2$ , we have  $\frac{1}{r_1} \chi(X_1) = \frac{1}{r_2} \chi(X_2)$ , and  $\chi^*(Z)$  is well defined. The other statements follow easily from the corresponding statements in  $\mathcal{C}$ , but we must use the Lemma to prove  $Z_2 \in \mathcal{C}^*$  in the last statement. Q. E. D.

**4. 5. Proof of Theorem 4. 1.** Let  $\Gamma$  be the group of deck transformations of the universal RIEMANNIAN covering  $M = M_0 \times M' \rightarrow N$  of Theorem 4. 1.



$M_0$  is a connected simply connected RIEMANNIAN nilmanifold; the same statement follows for each of its irreducible factors, so these irreducible factors are homeomorphic to EUCLIDEAN spaces by Theorem 4.2. As  $M'$  is compact, none of its irreducible factors can be isometric to an irreducible factor of  $M_0$ . Thus  $I(M) = I(M_0) \times I(M')$ . Theorem 4.2 shows that  $I(M_0)$  is a semidirect product  $S \cdot K$  where  $S$  is a connected simply connected nilpotent LIE group and  $K$  is compact. This allows us to view  $I(M)$  as a semidirect product  $S \cdot C$  where  $C = K \times I(M')$  is compact. Proposition 3.4 now provides a torsionfree subgroup  $\Delta$  of finite index in  $\Gamma$ , an analytic subgroup  $S' \subset S$ , and a toral subgroup  $T \subset C$  which centralizes  $S'$ , such that  $\Delta \subset S' \cdot T$  and  $S'/\Delta'$  is compact where  $\Delta'$  is the projection of  $\Delta$  on  $S'$ .

We now need

**Lemma.** *Let  $D$  be a discrete subgroup of a connected simply connected nilpotent LIE group  $U$  with  $U/D$  compact. Then  $D$  is torsionfree and has a generating set  $\{d_1, \dots, d_n\}$  such that, given  $d \in D$ , there is a unique set  $\{v_i\}$  of integers with  $d = d_1^{v_1} d_2^{v_2} \dots d_n^{v_n}$ .*

*Proof of Lemma.*  $D$  is torsionfree because  $U$  is torsionfree. Let  $r$  be the length of the lower central series of  $U$ ; let  $Z$  be the center of  $U$ .  $D \cap Z$  is the center of  $D$  because an automorphism of  $U$  is trivial if and only if it is trivial on  $D$ . As  $D$  is discrete, it follows that  $DZ$  is closed, so the image of  $Z$  in  $U/D$  is closed, whence  $Z/(D \cap Z)$  is compact. Let  $\{d_1, \dots, d_a\}$  generate the free abelian group  $D \cap Z$ . By induction on  $r$ , we have a generating set  $\{d'_{a+1}, \dots, d'_n\}$  of the requisite sort for the group  $D/(D \cap Z)$  in  $U/Z$ . Let  $d_{a+i}$  be any element of  $D$  mapping onto  $d'_{a+i}$ . Q. E. D.

The Lemma shows that  $\Delta$ , being isomorphic to  $\Delta'$  under the projection of  $S' \cdot T$  onto  $S'$ , satisfies the conditions on generators of the discrete group of Theorem 3.5. The projection of  $\Delta$  on  $C$  lies in the torus  $T$ , and the action of  $\Delta$  is free and properly discontinuous on  $(S \cdot C)/C$  because  $\Delta$  is discrete and torsionfree while  $C$  is compact. Thus  $M/\Delta$  is analytically homeomorphic to  $M/\Delta'$  by Theorem 3.5. This provides the finite real analytic covering

$$N' = M/\Delta' \rightarrow M/\Gamma = N.$$

$N' = (S/\Delta') \times M'$ , and  $S/\Delta'$  is homeomorphic to  $E \times (S'/\Delta')$  where  $E$  is homeomorphic to a EUCLIDEAN space. Let  $N'' = S'/\Delta'$ , and the decomposition  $N = E \times N'' \times M'$  is exhibited.

Let  $r$  be the multiplicity of the covering  $N' \rightarrow N$ . If  $\Gamma$  is infinite, then  $\Delta'$  is nontrivial and [6]  $\chi(N'') = 0$ . Thus  $\chi^*(N) = \frac{1}{r} \chi(N') = \frac{1}{r} \chi(N'') \chi(M') = 0$ . If  $\Gamma$  is finite, then the projection of  $\Gamma$  on  $I(M_0)$  must have a stationary point

because the maximal compact subgroups of  $\mathbf{I}(M_0)$  are isotropy subgroups; thus  $\Gamma$  projects isomorphically onto a subgroup  $\Sigma$  of  $\mathbf{I}(M')$  which acts freely on  $M'$ . If  $t$  is the common order of  $\Gamma$  and  $\Sigma$ , then  $M' \rightarrow M'/\Sigma$  is a covering of multiplicity  $t$ . We have  $\chi(M'/\Sigma) = \frac{1}{t} \chi(M')$  because  $M'$  is compact, whence  $t$  divides  $\chi(M')$ . Now  $\chi^*(N) = \frac{1}{t} \chi(M) = \frac{1}{t} \chi(M_0) \chi(M') = \frac{1}{t} \chi(M')$  is an integer  $\geq 0$ .

We have proved that  $\chi^*(N)$  is an integer  $\geq 0$  and that  $\chi^*(N) \neq 0$  implies finiteness of  $\pi_1(N)$ . If  $\chi^*(N) \neq 0$  and  $M'$  is RIEMANNIAN symmetric, then  $\pi_1(N)$  is a finite 2-group as in § 3.10. Similarly, the retraction of  $N$  when  $M_0$  is EUCLIDEAN is exhibited as in § 3. This completes the proof of Theorem 4.1. Q. E. D.

## 5. Classification in the irreducible case

We will classify (up to global isometry) the complete connected locally irreducible locally symmetric RIEMANNIAN manifolds of nonzero characteristic and all curvatures  $\geq 0$ . This is the first step in implementing Theorems 3.1, 3.2 and 4.1.

### 5.1. The candidates for consideration are not numerous:

**Theorem.** *Let  $S$  be a compact connected simply connected irreducible RIEMANNIAN symmetric manifold with  $\chi(S) \neq 0$ , and suppose that  $S$  has a fixed point free isometry. Then  $S$  is a GRASSMANN manifold,  $\mathbf{SO}(2n)/\mathbf{U}(n)$  with  $n > 2$ ,  $\mathbf{Sp}(n)/\mathbf{U}(n)$  with  $n > 1$ ,  $\mathbf{E}_7/A_7$ , or  $\mathbf{E}_7/E_6 \cdot T^1$ .*

*Remark.* Here  $A_7$  is a subgroup  $\mathbf{SU}(8)/\{\pm I\}$  in the compact simply connected exceptional group  $\mathbf{E}_7$ , and  $E_6 \cdot T^1 = (\mathbf{E}_6 \times \mathbf{T}^1)/\{1, z, z^2\}$  where  $\mathbf{T}^1$  is a circle group and  $z = (z', z'')$ , each component of order 3 and  $z'$  central in  $\mathbf{E}_6$ . GRASSMANN manifold means real, complex or quaternion GRASSMANN manifold, and we use oriented subspaces for real GRASSMANN manifolds.

*Proof.* Let  $K$  be an isotropy subgroup of  $G = \mathbf{I}(S)$ . Both groups are compact, and  $\text{rank } K = \text{rank } G$  because  $\chi(S) \neq 0$ . In particular, every element of  $G_0$  has a fixed point on  $S$ . Thus we need only examine the cases where  $G \neq G_0$ . According to CARTAN [4], these are, besides the ones mentioned in the statement of the Theorem, only  $\mathbf{E}_6/\{\mathbf{SU}(6) \times \mathbf{SU}(2)/\text{discrete}\}$  and  $\mathbf{E}_6/\{\mathbf{SO}(10) \times \mathbf{SO}(2)/\text{discrete}\}$ . We will check that, for both of these spaces, every isometry has a fixed point. Theorem 5.1 will then be proven.

**5.2.** Let  $M$  be a symmetric space  $\mathbf{E}_6/\{\mathbf{SU}(6) \times \mathbf{SU}(2)/\text{discrete}\}$  or  $\mathbf{E}_6/\{\mathbf{SO}(10) \times \mathbf{SO}(2)/\text{discrete}\}$ , and let  $K$  be an isotropy subgroup of  $G = \mathbf{I}(M)$ .

Then  $K = K_0 \cup \alpha K_0$  and  $G = G_0 \cup \alpha G_0$  where conjugation by  $\alpha$  induces outer automorphisms both on  $K_0$  and  $G_0$ . For conjugation by  $\alpha$  is outer on  $K_0$  by construction of  $\mathbf{I}(M)$  [4]. Now let  $\pi: \mathbf{E}_6 \rightarrow G_0$  be the projection; the kernel  $D$  of  $\pi$  is the center of  $\mathbf{E}_6$ , cyclic of order 3, and  $\pi^{-1}(K_0)$  is the centralizer of an element  $s \in \mathbf{E}_6$  with  $s^2 \in D$ . As  $D$  has odd order, we may assume  $s^2 = 1$ . It follows that  $K_0$  is its own normalizer in  $G_0$ . If conjugation by  $\alpha$  were inner on  $G_0$ , it would be inner on  $K_0$ ; this is not the case.

Now let  $A$  and  $B$  be the centralizers of  $\alpha$  in  $G_0$  and  $K_0$ , respectively. Checking both cases, we see that both  $A$  and  $B$  have rank 4. It follows that  $B$  contains a maximal torus  $T$  of  $A$ .

Let  $g \in G$ . If  $g \in G_0$ , then we know that  $g$  has a fixed point because  $\text{rank } K = \text{rank } G$ . If  $g \notin G_0$ , then  $g \in \alpha G_0$ . Then, if  $V$  is a maximal torus of  $A$ ,  $hgh^{-1} \in \alpha V$  for some  $h \in G_0$  ([9], Th. on p. 57). Let  $V$  be the maximal torus  $T$  above. Then  $V \subset K_0$ , so  $hgh^{-1} \in \alpha K$ . This shows that  $g$  has a fixed point, proving Theorem 5.1. Q.E.D.

**5.3. Space forms of GRASSMANN manifolds.** Theorem 5.1 tells us which spaces should be studied in order to find the groups  $\Delta$  of isometries acting freely on a compact irreducible simply connected symmetric space  $S$  with  $\chi(S) \neq 0$ . Classification of these groups  $\Delta$  up to conjugacy in  $\mathbf{I}(S)$  is the same as classification of the space forms  $S/\Delta$  of  $S$  up to isometry. In [13] we solved the complicated case—the case where  $S$  is a GRASSMANN manifold. For the convenience of the reader, we will recall the results.

Let  $\mathbf{F}$  be a field  $\mathbf{R}$  (real),  $\mathbf{C}$  (complex) or  $\mathbf{H}$  (quaternion), and let  $\mathbf{F}^n$  denote a left positive definite hermitian vectorspace of dimension  $n$  over  $\mathbf{F}$ . If  $0 < q < n$ , then the unitary group  $\mathbf{U}(n, \mathbf{F})$  of  $\mathbf{F}^n$  acts transitively on the set  $\mathbf{G}_{q,n}(\mathbf{F})$  of  $q$ -dimensional subspaces (oriented if  $\mathbf{F} = \mathbf{R}$ ) of  $\mathbf{F}^n$ . We exclude  $\mathbf{G}_{1,2}(\mathbf{R})$  and  $\mathbf{G}_{2,4}(\mathbf{R})$ ; then  $\mathbf{G}_{q,n}(\mathbf{F})$  has a unique (up to a scalar multiple)  $\mathbf{U}(n, \mathbf{F})$ -invariant RIEMANNIAN metric, and is always envisaged with that metric; it is simply connected and RIEMANNIAN symmetric, and has topological dimension  $q(n-q)r$  where  $r$  is the dimension of  $\mathbf{F}$  over  $\mathbf{R}$ . The characteristic  $\chi(\mathbf{G}_{q,n}(\mathbf{F})) \neq 0$  except when  $\mathbf{F} = \mathbf{R}$  and  $q(n-q)$  is odd.

$\mathbf{I}_0(\mathbf{G}_{q,n}(\mathbf{F}))$  is the group of motions induced by  $\mathbf{U}(n, \mathbf{F})_0$  (which is  $\mathbf{SO}(n)$ ,  $\mathbf{U}(n)$  or  $\mathbf{Sp}(n)$ ). If  $q = n - q$ , we have an isometry  $\beta$  given by orthogonal complementation (and consistent with orientation if  $\mathbf{F} = \mathbf{R}$ ). In any case, we use  $\beta$  to assume  $q$  even if  $q(n-q)$  is even and  $\mathbf{F} = \mathbf{R}$ . If  $\mathbf{F} = \mathbf{C}$ , we have an isometry  $\alpha$  induced by conjugation of  $\mathbf{C}$  over  $\mathbf{R}$ . If  $\mathbf{F} = \mathbf{R}$ , we have an isometry  $\omega$  given by reversal of orientation. Let  $g_v (0 \leq v \leq n)$  be the isometry induced by  $\begin{pmatrix} I_{n-v} & 0 \\ 0 & -I_v \end{pmatrix} \in \mathbf{U}(n, \mathbf{F})$ . If  $n = 2m$ , let  $k$  be the isometry

induced by  $\begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix} \in \mathbf{U}(n, \mathbf{F})$ . If  $\mathbf{F} = \mathbf{C}$ , let  $h_v (0 \leq v \leq n)$  be the isometry induced by  $\begin{pmatrix} aI_{n-v} & O \\ O & -aI_v \end{pmatrix} \in \mathbf{U}(n, \mathbf{C})$  where  $a = \exp(\pi \sqrt{-1}/n)$ . Let  $\mathbf{Z}_m$  denote the cyclic group of order  $m$ . Now the space forms of GRASSMANN manifolds of nonzero characteristic are classified ([13], Theorems 1, 2, 3) by:

**Theorem.** *Let  $\Delta$  be a group of isometries acting freely on  $\mathbf{G}_{q,n}(\mathbf{F})$ , where  $q(n-q)$  is even (so we may apply  $\beta$  and assume  $q$  even) if  $\mathbf{F} = \mathbf{R}$ . If  $\Delta \neq \{1\}$ , then  $\Delta$  is conjugate in  $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{F}))$  to one of the groups:*

$\mathbf{F}$	group	isomorphic to	conditions
$\mathbf{H}$	$\{1, \beta g_v\}$	$\mathbf{Z}_2$	$2q = n, 0 \leq v < q$
$\mathbf{C}$	$\{1, \alpha k\}$	$\mathbf{Z}_2$	$q$ and $n-q$ odd
$\mathbf{C}$	$\{1, \beta g_{2v}\}$	$\mathbf{Z}_2$	$2q = n, 0 \leq 2v < q$
$\mathbf{C}$	$\{1, \beta h_{2v-1}\}$	$\mathbf{Z}_2$	$2q = n, 1 \leq 2v-1 < q$
$\mathbf{R}$	$\{1, \omega\}$	$\mathbf{Z}_2$	none
$\mathbf{R}$	$\{1, \omega k\}$	$\mathbf{Z}_2$	$n$ even
$\mathbf{R}$	$\{1, \beta g_{2v}\}$	$\mathbf{Z}_2$	$2q = n, 0 \leq 2v < q$
$\mathbf{R}$	$\{1, \beta g_{2v}, \omega, \omega \beta g_{2v}\}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$2q = n, 0 \leq 2v < q$
$\mathbf{R}$	$\{1, \beta g_{2v-1}, \omega, \omega \beta g_{2v-1}\}$	$\mathbf{Z}_4$	$2q = n, 1 \leq 2v-1 < q$

Each of these groups acts freely on  $\mathbf{G}_{q,n}(\mathbf{F})$ , and any two distinct ones are not conjugate in  $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{F}))$ .

#### 5.4. The space forms of $\mathbf{SO}(4n)/\mathbf{U}(2n)$ are given by:

**Theorem.** *Let  $M$  be the RIEMANNIAN symmetric manifold  $\mathbf{SO}(4n)/\mathbf{U}(2n)$ ,  $n > 1$ , and let  $g_v$  and  $k_1 \in \mathbf{I}_0(M)$  be the respective isometries induced by the elements  $\begin{pmatrix} I_{4n-2v} & O \\ O & -I_{2v} \end{pmatrix}$  and  $\text{diag. } \{(-1 \ 0), \dots, (-1 \ 0); (1 \ -1)\}$  of  $\mathbf{SO}(4n)$ . We have  $\mathbf{I}(M) = \mathbf{I}_0(M) \cup \tau \cdot \mathbf{I}_0(M)$  where  $\tau$  is central,  $\tau^2 = 1$  and  $\tau \notin \mathbf{I}_0(M)$ . Let  $\Delta$  be a nontrivial group of isometries acting freely on  $M$ . Then  $\Delta$  is conjugate in  $\mathbf{I}(M)$  to one of the  $n$  groups  $\{1, \tau g_u\}$ ,  $0 \leq u < n$ , or to  $\{1, \tau k_1\}$ . Conversely, these groups act freely on  $M$  and are mutually non-conjugate in  $\mathbf{I}(M)$ .*

*Proof.* Let  $G = \mathbf{I}(M)$ . We have a point  $p \in M$  at which the symmetry is given by  $s = \pm \text{diag. } \{(-1 \ 0), \dots, (-1 \ 0)\}$ . Let  $K$  be the isotropy subgroup of  $G$  at  $p$ . Then  $G_0 = \mathbf{SO}(4n)/\{\pm I\}$ ,  $K_0 = \mathbf{U}(2n)/\{\pm I\}$ ,  $G = G_0 \cup \alpha \cdot G_0$

and  $K = K_0 \cup \alpha \cdot K_0$  where conjugation of  $G_0$  by  $\alpha$  is the same as conjugation by  $a = \pm \text{diag. } \{1, -1; \dots; 1, -1\}$ . Observe that  $a \in G_0$ , define  $\tau = \alpha a$ , and note that the first statement is proved.

Let  $h \in G_0$ . Then  $\tau h$  has a fixed point on  $M$  if and only if  $u\tau h u^{-1} = \alpha k$  for some  $u \in G_0$  and  $k \in K_0$ . As  $u\tau h u^{-1} = \tau u h u^{-1} = \alpha a u h u^{-1}$ , this is equivalent to  $a u h u^{-1} = k$ , i.e., to  $h$  being  $G_0$ -conjugate to an element of  $\alpha K_0$ . If primes denote representing matrices, we observe that  $a'$  anticommutes with  $s'$  and that  $\mathbf{U}(2n)$  is the full centralizer of  $s'$  in  $\mathbf{SO}(4n)$ . Thus  $\tau h$  has a fixed point if and only if some  $\mathbf{SO}(4n)$ -conjugate of  $h'$  anticommutes with  $s'$ .

Suppose further that  $h^2 = 1$ . Then  $h'$  has square  $\pm I$ . Suppose first that  $h'^2 = I$ ; then  $h'$  is conjugate to some  $g'_v$ , and we may assume  $v \leq n$  because  $h'$  may be replaced by its negative. If  $\tau h$  has a fixed point, then  $s'$  must exchange the eigenspaces of  $+1$  and of  $-1$  for some conjugate of  $h'$ , and it follows that  $v = 2n$ . On the other hand, if  $v = 2n$ , then  $h'$  is conjugate to  $a'$  and it follows that  $\tau h$  has a fixed point.

Now suppose  $h'^2 = -I$ . Thus  $h'$  is  $\mathbf{SO}(4n)$ -conjugate to  $k'_1$  or to  $s'$ . Observe that  $k'_1$  and  $s'$  are not conjugate in  $\mathbf{SO}(4n)$ , even though they are conjugate in  $\mathbf{O}(4n)$ . If  $\tau h$  has a fixed point, then we may conjugate and assume that  $h'$  anticommutes with  $s'$ . Now  $s'$  and  $h'$  generate a quaternion algebra, and it is easily seen that they are  $\mathbf{SO}(4n)$ -conjugate. On the other hand, if  $h'$  is conjugate to  $s'$ , then we may assume that they generate a quaternion algebra; this done, they anticommute and  $\tau h$  has a fixed point. Thus  $\tau h$  is fixed point free if and only if  $h'$  is  $\mathbf{SO}(4n)$ -conjugate to  $k'_1$ .

$\Delta$  has at most one element in each component of  $\mathbf{I}(M)$ . As  $\Delta \neq \{1\}$ , it follows that  $\Delta = \{1, \tau h\}$  where  $(\tau h)^2 = \tau h \tau h = \tau^2 h^2 = h^2 = 1$ ,  $h \in G_0$ . The Theorem now follows. Q. E. D.

**5.5. The space forms of  $\mathbf{SO}(4n+2)/\mathbf{U}(2n+1)$  are given by:**

**Theorem.** *Let  $M$  be the RIEMANNIAN symmetric manifold*

$$\mathbf{SO}(4n+2)/\mathbf{U}(2n+1), \quad n \geq 1.$$

*Then  $\mathbf{I}(M) = \mathbf{O}(4n+2)/\{\pm I\}$  and we have isometries  $h_v = \pm \begin{pmatrix} I_{4n+2-v} & O \\ O & -I_v \end{pmatrix}$  of  $M$ . Every nontrivial group of isometries acting freely on  $M$  is conjugate in  $\mathbf{I}(M)$  to one of the  $n$  groups  $\{1, h_{2u+1}\}$ ,  $0 \leq u < n$ . Conversely, these groups act freely on  $M$  and are mutually non-conjugate in  $\mathbf{I}(M)$ .*

*Proof.*  $M$  has a point  $p$  at which the symmetry is given by

$$s = \pm \text{diag. } \{(-1 \ 0), \dots, (-1 \ 0)\};$$

let  $K$  be the isotropy subgroup of  $G = \mathbf{I}(M)$  at  $p$ . Then  $G_0 = \mathbf{SO}(2m)/\{\pm I\}$

and  $K_0 = \mathbf{U}(m)/\{\pm I\}$  where we define  $m = 2n + 1$ .  $G = G_0 \cup \alpha \cdot G_0$  and  $K = K_0 \cup \alpha \cdot K_0$  where conjugation of  $G_0$  by  $\alpha$  is the same as conjugation by  $a = \pm \text{diag}\{1, -1; \dots; 1, -1\}$ . As this conjugation is an outer automorphism of  $G_0$  (because  $m$  is odd) we may identify  $\alpha$  with  $a$ , viewing  $G$  as  $\mathbf{O}(2m)/\{\pm I\}$  and  $K$  as  $\{\mathbf{U}(m) \cup a \cdot \mathbf{U}(m)\}/\{\pm I\}$ . This proves the first statement.

Given  $g \in \mathbf{I}(M)$ ,  $g'$  will denote one of the two matrices in  $\mathbf{O}(2m)$  representing  $g$ . If  $h_v$  ( $v$  odd) has a fixed point on  $M$ , then  $h'_v$  is conjugate in  $\mathbf{O}(2m)$  to an element  $h''_v = a'k'$  for some  $k' \in \mathbf{U}(m)$ , whence  $s'h''_v s'^{-1} = -h''_v$ . This shows that  $s'$  exchanges the eigenspaces of  $+1$  and of  $-1$  for  $h''_v$ , proving that  $v = m$ . It follows that the groups  $\{1, h_{2u+1}\}$  ( $0 \leq u < n$ ) act freely on  $M$ . As they are obviously mutually nonconjugate, the converse of the second statement is proven.

Let  $\Delta$  be a nontrivial group of isometries acting freely on  $M$ . As every element of  $G_0$  has a fixed point,  $\Delta = \{1, g\}$  with  $\det. g' = -1$ .  $g^2 = 1$  implies  $g'^2 = \pm I$ , whence  $g'^2 = +I$  because  $\det. g' = -1$ , so  $g$  is conjugate to some  $h_v$  ( $v$  odd). We may take  $v \leq m$  because  $h_v$  is conjugate to  $h_{2m-v}$ , and then  $v < m$  because  $g$  is not conjugate to  $a$ . The second statement follows. Q. E. D.

### 5.6. The space forms of $\mathbf{Sp}(n)/\mathbf{U}(n)$ are given by:

**Theorem.** Let  $M$  be the RIEMANNIAN symmetric manifold  $\mathbf{Sp}(n)/\mathbf{U}(n)$ ,  $n > 1$ , let  $\mathbf{Sp}(n)$  be viewed as the group of all  $g \in \mathbf{U}(2n)$  such that  $gJ^t g = J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ , and let  $g_v \in \mathbf{I}_0(M)$  be the isometry induced by  $\text{diag}\{I_{n-v}, -I_v, I_{n-v}, -I_v\} \in \mathbf{Sp}(n)$ . We have  $\mathbf{I}(M) = \mathbf{I}_0(M) \cup \tau \cdot \mathbf{I}_0(M)$  where  $\tau$  is central,  $\tau^2 = 1$  and  $\tau \notin \mathbf{I}_0(M)$ . Let  $\Delta$  be a nontrivial group of isometries acting freely on  $M$ . Then  $\Delta$  is conjugate in  $\mathbf{I}(M)$  to one of the  $\left\lfloor \frac{n+1}{2} \right\rfloor$  groups  $\{1, \tau g_v\}$ ,  $0 \leq v < \frac{n}{2}$ . Conversely, these groups act freely on  $M$  and are mutually non-conjugate in  $\mathbf{I}(M)$ .

*Proof.* Let  $G = \mathbf{I}(M)$ . Then  $G_0 = \mathbf{Sp}(n)/\{\pm I\}$  and

$$s = \pm \begin{pmatrix} \sqrt{-1} I_n & 0 \\ 0 & -\sqrt{-1} I_n \end{pmatrix} \in G_0$$

is the symmetry at some  $p \in M$ . Let  $K$  be the isotropy subgroup of  $G$  at  $p$ . Then  $K_0 = \mathbf{U}(n)/\{\pm I\}$  where  $\mathbf{U}(n)$  consists of all  $\begin{pmatrix} b & 0 \\ 0 & {}^t b^{-1} \end{pmatrix}$  for which  $b$  is an  $n \times n$  unitary matrix,  $K = K_0 \cup \alpha \cdot K_0$  and  $G = G_0 \cup \alpha \cdot G_0$ , where conjugation of  $G_0$  by  $\alpha$  is the same as conjugation by  $\pm J$ . As  $\pm J \in G_0$ , the first statement is proved by setting  $\tau = \alpha \cdot (\pm J)$ .



Let  $h \in G_0$ . As in § 5.4,  $\tau h$  has a fixed point on  $M$  if and only if  $h$  is  $G_0$ -conjugate to an element of  $(\pm J) \cdot K_0$ . Suppose that  $h^2 = 1$ , and let primes denote representing matrices. If  $h'^2 = -I$ , then  $h'$  is  $\mathrm{Sp}(n)$ -conjugate to  $J$ , whence  $\tau h$  has a fixed point. Now suppose  $h'^2 = I$ . Then  $h$  is conjugate to some  $g_v$ . If  $g'_v$  is conjugate to  $Jk'$ ,  $k' = \begin{pmatrix} b & 0 \\ 0 & {}^t b^{-1} \end{pmatrix}$ , then  $I = (Jk')^2 = \begin{pmatrix} -{}^t b^{-1} \cdot b & 0 \\ 0 & -b \cdot {}^t b^{-1} \end{pmatrix}$  shows  ${}^t b = -b$ , whence  $Jk' = \begin{pmatrix} 0 & -b^{-1} \\ -b & 0 \end{pmatrix}$ . This last is conjugate by  $\begin{pmatrix} I_n & 0 \\ 0 & -b^{-1} \end{pmatrix} \in \mathrm{U}(2n)$  to  $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ ; it follows that  $v = \frac{n}{2}$  by counting eigenvalues. On the other hand, if  $v = \frac{n}{2}$ , then it is not difficult to see, using the WEYL group, that  $h$  is  $G_0$ -conjugate to  $(\pm J) \cdot k$  for every  $k \in K_0$  such that  $k' = \begin{pmatrix} b & 0 \\ 0 & {}^t b^{-1} \end{pmatrix}$  and  ${}^t b = -b$ .

The Theorem now follows.

*Q. E. D.*

**5.7. The space forms of  $E_7/(A_7$  or  $E_6 \cdot T^1)$**  can be described, as in §§ 5.4–5.6, in terms of the elements of square 1 in the group  $\mathrm{ad}(\mathbf{E}_7) = \mathbf{E}_7/C$  where  $C$  is the center of  $\mathbf{E}_7$ . These elements are known:

**Lemma** (É. CARTAN [3]). *The group  $\mathrm{ad}(\mathbf{E}_7)$  has elements  $1 = s_{E_7}$ ,  $s_{A_7}$ ,  $s_{E_6} \times {}_{T^1}1$  and  $s_{D_6} \times {}_{A_1}1$  of square 1 where the centralizer of  $s_H$  in  $\mathrm{ad}(\mathbf{E}_7)$  is of CARTAN classification type  $H$ ; these four elements are mutually non-conjugate in  $\mathrm{ad}(\mathbf{E}_7)$  and any element of square 1 in  $\mathrm{ad}(\mathbf{E}_7)$  is conjugate to one of them.*

**Complement to Lemma.** Let  $\pi: \mathbf{E}_7 \rightarrow \mathrm{ad}(\mathbf{E}_7)$  be the projection and let  $s'_H \in \pi^{-1}(s_H)$ . Recall that  $C = \mathrm{Ker.} \pi = \{1 \ z\}$  cyclic order two. Then  $(s'_{E_7})^2 = (s'_{D_6} \times {}_{A_1}1)^2 = 1$  and  $(s'_{A_7})^2 = (s'_{E_6} \times {}_{T^1}1)^2 = z$ .

*Proof.* The Lemma is CARTAN's classification of RIEMANNIAN symmetric spaces  $M$  with  $\mathbf{I}_0(M) = \mathrm{ad}(\mathbf{E}_7)$ .

Let  $Z$  be the identity component of the centralizer of  $s_H$  in  $\mathrm{ad}(\mathbf{E}_7)$ , observe that  $Z' = \pi^{-1}(Z)$  is connected because  $Z$  contains a maximal torus; let  $S$  and  $S'$  be the respective centers of  $Z$  and  $Z'$ , and note that  $\pi: S' \rightarrow S$  is 2-to-1 sending  $z$  to 1 and  $s'_H$  to  $s_H$ .

If  $H = E_7$  then  $S'$  has order two, so  $(s'_H)^2 = 1$ .

If  $H = D_6 \times A_1$ , then the universal covering group of  $Z'$  is  $\mathrm{Spin}(12) \times \mathrm{SU}(2)$ . That group has center isomorphic to  $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ , and  $S'$  is a quotient of its center. Thus  $(s'_H)^2 = 1$ .

In the other cases, we look at the linear isotropy representation on the RIEMANNIAN symmetric space  $\mathrm{ad}(\mathbf{E}_7)/Z$ . As this space is irreducible, it follows

that  $S = \{1, s_H\}$  if  $H = A_7$  and  $S$  is a circle group if  $H = E_6 \times T^1$ . Looking at  $\pi: S' \rightarrow S$ , it is easily seen that  $(s'_{A_7})^2 = z$ .

Let  $H = E_6 \times T^1$  and let  $Z''$  be the group  $E_6 \times T^1$ .  $Z''$  has center  $S''$  isomorphic to  $\mathbf{Z}_3 \times T^1$ ; we represent the elements of  $S''$  by pairs  $(u^a, v)$  where  $u$  generates the center of  $E_6$  and  $v$  is a unimodular complex number. We have coverings  $Z'' \xrightarrow{\beta} Z' \xrightarrow{\pi} Z$ , and  $S = \pi\beta(S'')$  is a circle group. Thus  $\text{Ker.}(\pi\beta) = L_1 \times L_2$  where  $L_2$  is a finite cyclic subgroup of  $T^1$  and  $L_1$  is cyclic order 3 with a generator  $(u, w)$  where  $w^3 = 1$ . Now  $\text{Ker.} \beta = L_1 \times L_3$  where  $L_3$  has index 2 in  $L_2$ ; thus we may choose  $\beta$  such that  $\text{Ker.} \beta = L_1$  and  $L_2$  is generated by  $(1, -1)$ . It follows that  $z = \beta((1, -1))$  and  $s_H = \pi\beta((1, \sqrt{-1}))$ . This shows  $s'_H$  to be  $\beta((1, \pm \sqrt{-1}))$ , whence  $(s'_H)^2 = z$ . *Q.E.D.*

We can now enumerate the space forms of  $E_7/A_7$  and of  $E_7/E_6 \cdot T^1$ :

**Theorem.** *Let  $M$  be one of the RIEMANNIAN symmetric manifolds  $E_7/A_7$  or  $E_7/E_6 \cdot T^1$ . We have  $\mathbf{I}(M) = \mathbf{I}_0(M) \cup \tau \cdot \mathbf{I}_0(M)$  where  $\tau$  is central,  $\tau^2 = 1$  and  $\tau \notin \mathbf{I}_0(M)$ . Let  $\Delta$  be a nontrivial group of isometries acting freely on  $M$ . Then either  $\Delta = \{1, \tau\}$  or  $\Delta$  is conjugate in  $\mathbf{I}(M)$  to  $\{1, \tau s_{D_6 \times A_1}\}$ . These two groups act freely on  $M$  and are not conjugate in  $\mathbf{I}(M)$ .*

*Proof.* The first statement is known [4],  $\tau$  being central because  $E_7$  admits no outer automorphism. Let  $K$  be an isotropy subgroup of  $G = \mathbf{I}(M)$ . Then  $G_0 = \text{ad}(E_7)$ ,  $K = K_0 \cup \alpha \cdot K_0$  and  $G = G_0 \cup \alpha \cdot G_0$  where  $\alpha^2 = 1$  and conjugation by  $\alpha$  is the same as conjugation by  $a \in G_0$ ;  $\tau = \alpha a$ . Altering  $a$  by an element of  $K_0$  if necessary, we may assume that  $a$  is conjugate to  $s_{A_7}$ .

As before, let  $C = \{1, z\}$  be the kernel of the projection  $\pi: E_7 \rightarrow \text{ad}(E_7)$  and let primes denote representing elements in  $E_7$ . Let  $A$  be the centralizer of  $a'$  in  $E_7$ ;  $A \cong \text{SU}(8)/\{\pm I\}$  as seen in the proof of the complement to the Lemma,  $z$  is represented by  $\pm \sqrt{-1} \cdot I_8$ , and  $a'$  is represented by

$$\pm \exp(2\pi\sqrt{-1}/8) \cdot I_8.$$

Let  $h \in G_0$ ,  $h'^2 = z$ . Replacing  $h$  by a conjugate,  $h' \in A$  and  $h'$  is represented by  $\pm \begin{pmatrix} \exp(2\pi\sqrt{-1}/8)I_p & 0 \\ 0 & \exp(2\pi\sqrt{-1}5/8)I_q \end{pmatrix}$  where  $p + q = 8$ .

That matrix must have determinant  $+1$ ; it follows that  $p$  and  $q$  are even,  $p = 2u$  and  $q = 2v$ . Again replacing  $h$  by a conjugate,  $h' \in A$  is represented by  $\pm \text{diag.} \{\varepsilon I_u, \varepsilon^5 I_v, \varepsilon I_u, \varepsilon^5 I_v\}$  where  $\varepsilon = \exp(2\pi\sqrt{-1}/8)$ .

Let  $s$  be the symmetry to  $M$  at the point at which  $K$  is isotropy subgroup of  $G$ . Although  $s$  commutes with  $a$  because it commutes with  $\alpha$ ,  $s'$  cannot commute with  $a'$  because conjugation by  $a$  induces an outer automorphism of  $K_0$ . Thus the commutator  $[s', a'] = z$ . It follows that  $s'$  normalizes  $A$  and



that conjugation of  $A$  by  $s'$  is an involutive outer automorphism. Thus we may assume [3]  $s'gs'^{-1} = {}^tg^{-1} = \bar{g}$  for every  $g \in A$ , or that  $s'gs'^{-1} = \overline{JgJ^{-1}}$  ( $J = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}$ ) for every  $g \in A$ . It follows that  $s'h's'^{-1} = h'z$ . Now let  $h' = h''a'$ . As  $s'a's'^{-1} = a'z$ ,  $h''$  must commute with  $s'$ . This implies  $\pi(h'') \in K_0$  because  $\pi^{-1}(K_0)$  is connected and is the centralizer of  $s'$  in  $E_7$ .

We have now proved, given  $h \in G_0$  with  $h'^2 = z$ , that  $uhu^{-1} = ak$  for some  $k \in K_0$ ,  $u \in G_0$ . This gives  $auhu^{-1} = k$ , i.e.,  $\alpha auhu^{-1} = \alpha k$ , i.e.,  $\tau uhu^{-1} \in \alpha K_0$ , i.e.,  $\tau h$  conjugate to an element of  $\alpha K_0$ . Thus  $\tau h$  has a fixed point on  $M$ .

Now let  $h \in G_0$ ,  $h'^2 = 1$ . We will see that  $\tau h$  has no fixed point on  $M$ . For if it had a fixed point, we would have  $u\tau hu^{-1} = \alpha k$  with  $u \in G_0$  and  $k \in K_0$ . Then  $h$  would be conjugate to  $ak \in \alpha K_0$ . Replacing  $h$  by that conjugate,  $h' = a'k'$  with  $k'$  in the centralizer  $K' = \pi^{-1}(K_0)$  of  $s'$  in  $E_7$ . Now  $B = A \cap K'$  is both the centralizer of  $s'$  in  $A$  and the centralizer of  $a'$  in  $K'$ . Every element of  $a'K'$  is conjugate to an element of  $a'B$ . For  $K_0 \cup \alpha K_0$  is the centralizer of  $s$  in  $\text{ad}(E_7)$ ; if  $T$  is a maximal torus of the centralizer of  $a$  in  $K_0$ , then a result of DESIEBENTHAL ([9], Th. on p. 57) shows that every element of  $\alpha K_0$  is  $K_0$ -conjugate to an element of  $aT$ ; thus every element of  $a'K'$  is conjugate to an element of  $a' \cdot \pi^{-1}(T) \subset a'B$ . Now we conjugate  $h$  and assume  $h' = a'k'$  where  $k'$  commutes with both  $s'$  and  $a'$ . Thus we have  $k' \in A$ . Let double primes denote elements of  $\text{SU}(8)$  representing elements of  $A = \text{SU}(8)/\{\pm I\}$ .  $h'' = a''k''$  is conjugate (by  $s''$ ) to  $h''z''$ ; thus  $-I = h''^2$ , and it is conjugate in  $\text{SU}(8)$  to  $(h''z'')^2 = h''^2z''^2 = (-I)(-I) = I$ . This being impossible,  $\tau h$  cannot have a fixed point.

Our group  $\Delta = \{1, \tau h\}$  where  $1 = (\tau h)^2 = \tau^2 h^2 = h^2$ . Thus, by the Lemma,  $h$  is conjugate to  $1$ ,  $s_{D_6 \times A_1}$ ,  $s_{A_7}$  or  $s_{E_6 \times T^1}$ . But  $h'^2 = 1$ , as we have just seen, because  $\tau h$  has no fixed point; the Complement to the Lemma now shows  $h$  conjugate to  $1$  or  $s_{D_6 \times A_1}$ . On the other hand, the Complement and the preceding paragraph show that  $\{1, \tau\}$  and  $\{1, \tau s_{D_6 \times A_1}\}$  act freely on  $M$ .

**5.8. Combining Theorems 5.1, 5.3, 5.4, 5.5, 5.6 and 5.7**, one has a global classification for the space forms of compact connected simply connected irreducible RIEMANNIAN symmetric manifolds of nonzero characteristic.

## 6. Reducibility and commutativity

**6.1. Order.** Let  $N$  be an irreducible compact connected simply connected RIEMANNIAN symmetric manifold of nonzero characteristic. We have just seen that a group of isometries acting freely on  $N$  must be of order 1, 2 or 4. We

now define the *order* of  $N$ , written  $\text{order}.N$ , to be the maximal of the orders of the groups of isometries acting freely on  $N$ . This concept is useful for:

**6.2. Commutativity Theorem.** *Let  $M'$  be a compact connected simply connected RIEMANNIAN symmetric manifold of nonzero characteristic. Then these are equivalent:*

1. *A group of isometries acting freely on  $M'$  is necessarily abelian.*
2. *Any group of isometries acting freely on  $M'$  is a direct product of some number  $m \geq 0$  of groups  $\mathbf{Z}_2$ , or is cyclic of order 4.*
3. *If one of the irreducible factors of  $M'$  has order 4, then all the others have order 1. If  $M'$  has two isometric irreducible factors of order 2, then all the others have order 1.*

**Complement to the Commutativity Theorem.** *Let  $N$  be an irreducible compact connected RIEMANNIAN symmetric manifold of nonzero characteristic.*

1. *These are equivalent:*
  - (a)  *$N$  has order 2.*
  - (b)  *$\mathbf{Z}_2$  acts freely by isometries on  $N$ , but  $\mathbf{Z}_4$  does not.*
  - (c)  *$\mathbf{Z}_2$  acts freely by isometries on  $N$ , but  $\mathbf{Z}_2 \times \mathbf{Z}_2$  does not.*
  - (d)  *$N$  is isometric to  $\mathbf{G}_{q,n}(\mathbf{R})$  where  $n \neq 2q$  and  $q(n - q)$  is even, or to  $\mathbf{G}_{q,n}(\mathbf{C})$  where either  $2q = n$  or  $q(n - q)$  is odd, or to  $\mathbf{G}_{q,2q}(\mathbf{H})$ , or to  $\mathbf{SO}(2n)/\mathbf{U}(n)$  where  $n > 2$ , or to  $\mathbf{Sp}(n)/\mathbf{U}(n)$  where  $n \geq 1$ , or to  $\mathbf{E}_7/A_7$ , or to  $\mathbf{E}_7/\mathbf{E}_6 \cdot T^1$ .*
2. *These are equivalent:*
  - (a)  *$N$  has order 4.*
  - (b)  *$\mathbf{Z}_4$  acts freely by isometries on  $N$ .*
  - (c)  *$\mathbf{Z}_2 \times \mathbf{Z}_2$  acts freely by isometries on  $N$ .*
  - (d)  *$N$  is isometric to  $\mathbf{G}_{2n,4n}(\mathbf{R})$  where  $n > 1$ .*

Here  $\mathbf{Z}_m$  denotes the cyclic group of order  $m$ .

The Complement follows trivially from the results of § 5. The remainder of § 6 is devoted to the proof of the Commutativity Theorem. As (2) obviously implies (1) there, we need only prove that (3) implies (2) and that (1) implies (3).

**6.3.** The proof that (3) implies (2) is based on Theorem 3.2 and on

**Lemma.** *Let  $\Delta$  be a nontrivial group of isometries acting freely on  $N \times N$  where  $N$  is a complete connected simply connected irreducible RIEMANNIAN symmetric manifold of nonzero characteristic and order 2. Then  $\Delta$  is isomorphic to  $\mathbf{Z}_2$ ,  $\mathbf{Z}_2 \times \mathbf{Z}_2$  or  $\mathbf{Z}_4$ .*

*Proof.*  $\Delta \cap \{\mathbf{I}(N) \times \mathbf{I}(N)\}$  has order  $\leq 4$  and has index  $\leq 2$  in  $\Delta$ , by Theorem 3.2; it suffices to prove that  $\Delta$  is not a nonabelian group of order 8.

Suppose  $\Delta$  nonabelian of order 8. Then  $\Delta$  is generated by an element  $\gamma$  of order 4 and an element  $\delta$  of order 2 or 4, where  $\delta\gamma\delta^{-1} = \gamma^{-1}$ . By Theorem 3.2,

we may assume  $\gamma$  to be given by  $(x, y) \rightarrow (\tau y, x)$  where  $\tau$  is a fixed point free isometry of order 2 on  $N$ . If  $\delta^2 = 1$ , then  $\delta(x, y) = (\delta_1 x, \delta_2 y)$  where  $\delta_i$  is an isometry of square 1 on  $N$ . Then  $\delta\gamma\delta = \gamma^{-1}$  implies  $\delta_1 = \tau\delta_2$ ; it follows that  $\gamma\delta(x, y) = (\delta_1 y, \delta_1 x)$ ; thus  $(x, \delta_1 x)$  is a fixed point for  $\gamma\delta$ . This proves  $\delta^2 \neq 1$ .

Now  $\delta$  must have order 4, and is thus given by  $(x, y) \rightarrow (\delta_1 y, \delta_2 x)$  where  $\delta_i$  are isometries of  $N$ . Thus  $\sigma = \delta\gamma$  is given by  $(x, y) \rightarrow (\sigma_1 x, \sigma_2 y)$  where  $\sigma_i$  are isometries of  $N$ . By Theorem 3.2 we have  $\sigma^2 = 1$ . But  $\sigma^2 \neq 1$  because  $\Delta$  is the quaternion group. The Lemma follows. Q.E.D.

We will prove that (3) implies (2) in Theorem 6.2. Assume (3) and let  $\Gamma$  be a group of isometries acting freely on  $M'$ . If  $\gamma \in \Gamma$ , then  $\gamma^4 = 1$  by Theorem 3.2. If  $\Gamma$  has no element of order 4, it must be a product of groups  $\mathbf{Z}_2$ , and we are done. Now suppose that  $\Gamma$  has an element of order 4. By our assumption (3) and by Theorem 3.2, there is a RIEMANNIAN product decomposition  $M' = S \times X$  where  $X$  is a product of irreducible manifolds of order 1 and either  $S$  is irreducible with order.  $S = 4$  or  $S = S_1 \times S_2$ ,  $S_1$  isometric to  $S_2$ , with order.  $S_i = 2$ . Let  $\Delta$  be the restriction of  $\Gamma$  to  $S$ . The restriction  $\Gamma \rightarrow \Delta$  is an isomorphism; thus it suffices to prove  $\Delta$  isomorphic to  $\mathbf{Z}_4$ .

Observe that  $\Delta$  acts freely on  $S$ . If  $S$  is irreducible of order 4, then  $\Delta \cong \mathbf{Z}_4$  by the Complement, by Theorem 5.3, and because it contains an element of order 4. If  $S$  is reducible, then  $\Delta \cong \mathbf{Z}_4$  by the Lemma above.

**6.4.** To prove that (1) implies (3) it suffices to exhibit a noncommutative group of isometries acting freely on a direct factor of  $M'$ , in case the conditions of (3) do not hold. For this noncommutative group will then act freely by isometries on  $M'$ . Thus we need only take compact connected simply connected irreducible RIEMANNIAN symmetric manifolds  $N$  and  $L$ , order.  $L > 1$ , and prove:

*If order.  $N = 4$ , then there is a noncommutative group of isometries acting freely on  $N \times L$ . If order.  $N = 2$ , then there is a noncommutative group of isometries acting freely on  $N \times N \times L$ .*

We will construct examples of such groups which are dihedral groups of order 8.

Suppose that  $N$  has order 4. Then  $N = \mathbf{G}_{2n,4n}(\mathbf{R})$ ,  $n \geq 2$ , and (Theorem 5.3)  $\beta g_{2v-1} = \nu$  generates a cyclic group of order 4 of isometries acting freely on  $N$ . Let  $\gamma = \nu \times 1 \in \mathbf{I}(N \times L)$ . Choose a fixed point free isometry  $\tau$  of order 2 on  $L$  and define  $\delta = g_{2v-1} \times \tau$ . Then  $\gamma$  has order 4,  $\delta$  has order 2, and  $\delta\gamma\delta = \gamma^{-1}$  because  $g_{2v-1}\beta g_{2v-1} = \omega\beta$ . Now

$$\Gamma = \{1, \gamma, \gamma^2, \gamma^3; \delta, \delta\gamma, \delta\gamma^2, \delta\gamma^3\}$$

is the (dihedral) group generated by  $\gamma$  and  $\delta$ . The powers of  $\gamma$  act freely on the

$N$ -component, and the last four elements move every  $L$ -coordinate. Thus  $\Gamma$  is a noncommutative group of isometries acting freely on  $N \times L$ .

Suppose that  $N$  has order 2. Let  $\nu$  and  $\tau$  be involutive fixed point free isometries of  $N$  and  $L$ , respectively. We define elements  $\gamma$  and  $\delta$  of  $\mathbf{I}(N \times N \times L)$  by  $\gamma(x, y, z) = (\nu y, x, z)$  and  $\delta(x, y, z) = (\nu x, y, \tau z)$ .  $\gamma$  has order 4 and its powers act freely.  $\delta$  has order 2 and any  $\delta\gamma^a$  moves the  $L$ -coordinate.  $\delta\gamma\delta = \gamma^{-1}$  is easily checked. Thus the group  $\Gamma$  generated by  $\gamma$  and  $\delta$  is a noncommutative group acting freely by isometries on  $N \times N \times L$ .

Theorem 6.2 is now proven.

*Remark.* The other noncommutative group of order 8, the quaternion group, can act freely by isometries on  $N \times N \times N \times N$  where  $N$  is as above with order.  $N > 1$ .

**6.5. Corollary.** *Let  $M \rightarrow N$  be the universal RIEMANNIAN covering of a complete connected locally symmetric RIEMANNIAN manifold  $N$  with every sectional curvature  $\geq 0$  and characteristic  $\chi(N) \neq 0$ . Suppose, if one of the compact irreducible factors of  $M$  has order 4, that all the others have order 1; suppose, if  $M$  has a pair of isometric compact irreducible factors of order 2, that all the others have order 1. Then the fundamental group  $\pi_1(N)$  is a finite direct product of groups  $\mathbf{Z}_2$ , or is cyclic of order 4.*

This follows immediately from Theorems 3.2 and 6.2.

We can give a good description of the manifold  $N$  of the Corollary. One has

$$M = M_0 \times M_1 \times \dots \times M_t$$

where  $M_0$  is a EUCLIDEAN space  $\mathbf{R}^m$  and each  $M_i$  ( $i > 0$ ) is compact and irreducible with  $\chi(M_i) > 0$ . If  $\pi_1(N) \cong \mathbf{Z}_4$ , there are two sorts of possibilities: some  $M_i$  has order 4 or two isometric  $M_i$  have order 2. We permute the  $M_i$  and obtain  $M = M_0 \times S \times X$  where  $S$  is irreducible of order 4 or the product of two isometric irreducible manifolds  $S_i$  of order 2.  $N = M/\Gamma$  where  $\Gamma$  is generated by an element  $\gamma = \gamma_0 \times \gamma_S \times \gamma_X$ ,  $\gamma_0^4 = 1$ ,  $\gamma_X^4 = 1$ , and  $\gamma_S$  is given by: If  $S$  is irreducible,  $S = \mathbf{G}_{2n, 4n}(\mathbf{R})$  with  $n \geq 2$ , then  $\gamma_S$  is conjugate to an isometry  $\beta g_{2n-1}$  of  $S$ . In the other case,  $\gamma_S$  is conjugate to an isometry  $(s_1, s_2) \rightarrow (\tau s_2, s_1)$  of  $S = S_1 \times S_2$  where  $\tau$  is a fixed point free involutive isometry of  $S_1 = S_2$ .

Suppose  $\pi_1(N) \neq \mathbf{Z}_4$ ; then  $\pi_1(N)$  is a product of some number  $k \geq 0$  of groups  $\mathbf{Z}_2$ , and  $N = M/\Gamma$  for a group  $\Gamma$  isomorphic to  $\pi_1(N)$  and given as follows.  $\Gamma$  has generators  $\{\gamma_1, \dots, \gamma_k\}$ . Suppose first that some  $M_i$  (say  $S$ ) is of order 4, or that two isometric  $M_i$  (say  $S_1$  and  $S_2$ ; let  $S = S_1 \times S_2$ ) are of order two. Then  $M = M_0 \times S \times X$ , where  $X$  is a product of irreducible manifolds of order 1,  $k \leq 2$ , and each  $\gamma_i = \gamma_{i,0} \times \gamma_{i,S} \times \gamma_{i,X}$ ;  $\gamma_i \rightarrow \gamma_{i,S}$

is an isomorphism of  $\Gamma$  onto a group  $\Sigma$  of isometries acting freely on  $S$  and each  $\gamma_{i,s}$  preserves each  $S$  if  $S_j$  is a product, so the possibilities for  $\Sigma$  are given in § 5; the  $\gamma_{i,0}$  commute and have square 1, as do the  $\gamma_{i,x}$ . We now consider the other possibility—the case where no  $M_i$  has order 4 and no two  $M_i$  of order 2 are isometric. Re-ordering the  $M_i$ , we may assume that  $M_1, \dots, M_k$  each has order 2 and is preserved by each  $\gamma_j$ , and that  $\gamma_j$  induces a fixed point free involutive isometry of  $M_j$ . Then  $M = M_0 \times M_1 \times \dots \times M_k \times X$ ,  $\gamma_j = \gamma_{j,0} \times \dots \times \gamma_{j,k} \times \gamma_{j,x}$ ,  $\gamma_j \rightarrow \gamma_{j,s}$  ( $s = 0, 1, \dots, k, X$ ) is a homomorphism of  $\Gamma$ , and  $\gamma_{j,j}$  has no fixed point.

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