A Discrete Renewal Theorem with Infinite Mean.

A Discrete Renewal Theorem with Infinite Mean

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1. Statement of results

1.1. This paper is devoted to a problem which can reasonably be considered as belonging to pure analysis, but which was suggested by, and has applications to, the theory of probability. Suppose that $\{f_n\}$, $n = 1, 2, \ldots$ is a sequence of real numbers such that

$$\left.\begin{array}{l} f_n \geq 0, \qquad \sum_{n=1}^{\infty} f_n = 1 \\ \text{g.c.d. } \{n : f_n > 0\} = 1 . \end{array}\right\} \tag{1.1.1}$$

Define another sequence, say $\{u_n\}$, by setting

$$\left.\begin{array}{l} u_0 = 1 \\ u_n = \sum_{k=1}^{n} f_k \, u_{n-k}, \quad n \geq 1 . \end{array}\right\} \tag{1.1.2}$$

It is easy to see (recursively) that $0 \leq u_n \leq 1$. It is known [5] that

$$\lim_{n\to\infty} u_n = \frac{1}{\sum_{k=1}^{\infty} k \, f_k} \tag{1.1.3}$$

where the right hand side is interpreted as zero if the sum in the demonimator diverges.

Our problem is to study the manner in which $u_n \to 0$, under certain additional hypotheses which we shall impose on the sequence $\{f_n\}$. The main results are summarized in the following

Theorem 1.1. *Suppose that, in addition to* 1.1.1, *the sequence* $\{f_n\}$ *satisfies*

$$\sum_{k=n+1}^{\infty} f_k = L(n) \, n^{-\alpha} \tag{1.1.4}$$

where $0 < \alpha < 1$ *and* $L(n)$ *is a slowly varying function*[2]). *Then*

$$\liminf_{n\to\infty} n^{1-\alpha} L(n) \, u_n = \frac{\sin \pi\alpha}{\pi} . \tag{1.1.5}$$

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[1]) In carrying out this work the first mentioned author was supported by Contract Nonr-220(31) between the Office of Naval Research and the California Institute of Technology; the second author (part time) by a National Science Foundation grant to Dartmouth College.

[2]) A function $L(x)$ is said to be "slowly varying" or "of slow growth" if it is positive, measurable and for every $\lambda > 0$, $L(\lambda x)/L(x) \to 1$ as $x \to \infty$.

If $\frac{1}{2} < \alpha < 1$, *this assertion can be sharpened to*

$$\lim_{n \to \infty} n^{1-\alpha} L(n) u_n = \frac{\sin \pi\alpha}{\pi}, \tag{1.1.6}$$

while for $0 < \alpha < \frac{1}{2}$ *this limit does not, in general, exist. However, for* $0 < \alpha \leq \frac{1}{2}$, 1.1.6 *does hold provided that the limit is taken excluding a set of integers having density* 0.

1.2. The probabilistic interpretation of sequences $\{f_n\}$ and $\{u_n\}$ satisfying 1.1.2 in terms of "recurrent events" is explained in [6]. Here we shall put the matter in the setting of "renewal theory". Suppose that $\{X_i\}$, $i = 1, 2, \ldots$ is a sequence of independent positive integer valued random variables each satisfying

$$\Pr\{X_i = k\} = f_k .$$

Let $\{S_n\}$ be the partial sums of the X_i, with $S_0 = 0$. Then it is easy to see that the quantities u_n defined in 1.1.2 have the interpretation

$$u_n = \Pr\{\exists k : S_k = n\} = \sum_{k=0}^{n} \Pr\{S_k = n\} . \tag{1.2.1}$$

The latter equality holds since the events $\{S_k = n\}$ are disjoint as k varies.

The significance of the assumption 1.1.4 can now be seen in that it implies that the random variables $\{X_i\}$ belong to the "domain of attraction of a stable law" (see for instance [9]). This fact has many consequences in the theory of renewals and MARKOV chains; in addition to [6] see [14] and the references cited there.

We shall first approach the study of $\{u_n\}$ by applying certain probability limit theorems together with the representation 1.2.1. This will yield the result in 1.1.5 with "$\geq$" in place of "$=$". Some TAUBERIAN theorems then provide the estimate

$$\sum_{k=1}^{n} u_k \sim \frac{\sin \pi\alpha}{\pi\alpha} \frac{n^\alpha}{L(n)} . \tag{1.2.2}$$

A comparsion of these two results yields the equality in 1.1.5 and also the existence of the limit 1.1.6 except for a set of density 0; we can not obtain 1.1.6 itself in this way, however.

In the last section a separate attack is made to obtain in a self-contained fashion 1.1.6 and 1.1.5. Although in this treatment probability does not figure explicitly, the methods are related to those used in the proofs of limit theorems for sums of independent random variables. Also in this section is the explanation why 1.1.6 is in general false for $\alpha < \frac{1}{2}$.

The problem of the asymptotic behavior of $\{u_n\}$ when $\sum_{k=1}^{\infty} k f_k = \infty$ has already attracted some attention. The paper that is most closely related to the present one is that of DE BRUIJN-ERDÖS [2]. These authors obtain among other things a result from which 1.1.6 can easily be deduced. However, they do so only under much more restrictive conditions than ours. Namely, in addition to 1.1.1 and 1.1.4 they assume the condition

$$f_{n-1} f_{n+1} > f_n^2 \quad \text{for all} \quad n > 1 .$$

Other related works are those of GARSIA, OREY and RODEMICH [8], GARSIA [7] and OREY [15]. These papers study conditions under which

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1 ; \tag{1.2.3}$$

1.1.1 is assumed but the other hypotheses are not similar to 1.1.4. Our results have the obvious.

Corollary. Under the hypothesis of Theorem 1.1, *for* $\frac{1}{2} < \alpha < 1$ *we have* 1.2.3, *while for* $0 < \alpha \leq \frac{1}{2}$, 1.2.3 *holds when* n *is restricted to vary outside a set of integers of density zero.*

It would be interesting to discover if the latter conclusion is true under conditions weaker than 1.1.4.

2. Probabilistic approach

2.1. One method of attack is based on the random walk interpretation outlined in 1.2, together with probability limit theorems associated with sums of independent random variables which are "attracted" to a "stable law" [9]. For k large, we can apply these theorems to obtain good estimates for $\Pr\{S_k = n\}$; by summing these, a lower bound for u_n is obtained. An upper bound seems difficult to obtain this way, since the limit theorem does not yield useful information about some of the terms (with small k) making up u_n.

The local limit theorem which we shall use [9, p. 236] asserts that

$$\lim_{k\to\infty} [B_k \Pr\{S_k = n\} - g_\alpha(n/B_k)] = 0 , \tag{2.1.1}$$

where the limit is uniform in n. The assumptions here are that 1.1.1 and 1.1.4 hold and that the sequence $\{B_k\}$ is chosen so that

$$\lim_{k\to\infty} \Pr\{S_k \leq x B_k\} = G_\alpha(x) ; \tag{2.1.2}$$

$G_\alpha(x)$ is a stable law and has the continuous derivative $g_\alpha(x)$ which appears in 2.1.1. From [9][1]) we learn that $\{B_k\}$ may be chosen in such a way that

[1]) It appears that in certain statements in [9], concerning the attraction of sums of random variables to stable laws, the constants defining the limit laws are slightly in error. These things are given accurately in a convenient form for our purposes in [6].

$$\lim_{k\to\infty} k \sum_{l>B_k} f_l = 1\,, \tag{2.1.3}$$

and that the corresponding $g_\alpha(x)$ is characterized by

$$\int_0^\infty e^{izx} g_\alpha(x)dx = \gamma_\alpha(z) = \exp\{-|z|^\alpha\left(\cos\frac{\pi\alpha}{2} - i\sin\frac{\pi\alpha}{2}\frac{z}{|z|}\right)\Gamma(1-\alpha)\}\,. \tag{2.1.4}$$

Perhaps we should also mention some properties of functions of slow growth which will be relevant in our subsequent arguments. First of all any function of slow growth $L(x)$ can be given a representation (see [12]) of the form

$$L(x) = c(x)\exp\left[\int_0^x \varepsilon(t)\frac{dt}{t}\right]$$

where $c(x)$ is convergent to a number different from zero and $\varepsilon(t)\to 0$ as $t\to\infty$. We also have that the limit $L(\lambda x)/L(x)\to 1$ is uniform whenever λ stays away from zero and infinity. Furthermore the function $A(x) = \exp\int_0^x \varepsilon(t)dt/t$ has the property that, for any $\varepsilon > 0$, $x^\varepsilon A(x)$ and $x^{-\varepsilon}A(x)$ are eventually monotonic, increasing and decreasing respectively.

2.2. Using these results we shall first prove

Lemma 2.2.1. *Under the conditions of Theorem* 1.1,

$$u_n \geq \frac{n^{\alpha-1}}{L(n)}\int_0^\infty g_\alpha(x^{-1/\alpha})\,x^{-1/\alpha}dx[1+o(1)]\,. \tag{2.2.1}$$

Proof. From 1.2.1 and 2.1.1 we clearly have

$$u_n \geq \sum_{k=a_n}^{b_n} B_k^{-1} g_\alpha\left(\frac{n}{B_k}\right) + \sum_{k=a_n}^{b_n} B_k^{-1} o(1)\,, \tag{2.2.2}$$

where the two limits a_n and b_n can be anywhere between 0 and n. We should choose them in such a way that the first term in 2.2.2 approaches the first in 2.2.1, and the second term in 2.2.2 is negligible. This can be achieved by setting

$$a_n = \frac{An^\alpha}{L(n)}\,, \qquad b_n = \frac{Bn^\alpha}{L(n)}\,. \tag{2.2.3}$$

The conditions in 2.1.3 and 1.1.4 then imply that

$$\sum_{k=an}^{bn} B_k^{-1} = O\left(\frac{n^{\alpha-1}}{L(n)}\right),$$

and from the uniformity of o(1) we obtain

$$\sum_{k=a_n}^{bn} B_k^{-1} o(1) = o\left[\frac{n^{\alpha-1}}{L(n)}\right]. \tag{2.2.4}$$

Using 2.1.3 and basic properties of slowly varying functions we deduce that

$$\frac{n}{B_k} \sim \left[\frac{B_k^\alpha}{L(B_k)} \frac{L(n)}{n^\alpha}\right]^{-1/\alpha} \sim \left[\frac{kL(n)}{n^\alpha}\right]^{-1/\alpha}$$

uniformly for $a_n \leq k \leq b_n$, so that setting $x_k = \dfrac{kL(n)}{n^\alpha}$ the first term in 2.2.2 after multiplication by $L(n)\,n^{1-\alpha}$ can be transformed into the RIEMANN sum

$$\sum_{A \leq x_k \leq B} x_k^{-1/\alpha}\, g_\alpha(x_k^{-1/\alpha})\, [x_{k+1} - x_k] .$$

Substituting this result in 2.2.2 and taking account of 2.2.4 and the arbitrariness in the choice of A and B, we can easily obtain 2.2.1.

The assertion in 1.1.5 with $\geq$ in place of $=$ is a consequence of lemma 2.2.1 and the identity

$$\int_0^\infty g_\alpha(x^{-1/\alpha})\, x^{-1/\alpha}\, dx = \frac{\sin \pi\alpha}{\pi} . \tag{2.2.5}$$

2.3. Some of the further conclusions of the theorem can be established by means of the following

Lemma 2.3.1. *Under the hypotheses* 1.1.1 *and* 1.1.4

$$\sum_{k=1}^{n} u_k \sim \frac{\sin \pi\alpha}{\pi\alpha} \frac{n^\alpha}{L(n)} . \tag{2.3.1}$$

Proof. We shall use generating functions. We set

$$U(t) = \sum_{n=0}^{\infty} u_n t^n , \quad R(t) = \sum_{n=0}^{\infty} r_n t^n = \sum_{n=0}^{\infty} \Big(\sum_{k=n+1}^{\infty} f_k \Big) t^n . \tag{2.3.2}$$

From 1.1.4, we find by an ABELIAN theorem that

$$R(t) \sim \Gamma(1 - \alpha)\,(1 - t)^{\alpha - 1}\, L\left(\frac{1}{1 - t}\right) . \tag{2.3.3}$$

The definition 1.1.2 can be expressed in the form

$$U(t) = \frac{1}{(1-t)\,R(t)} , \tag{2.3.4}$$

so that from 2.3.3 we obtain

$$U(t) \sim \frac{(1 - t)^{-\alpha}}{\Gamma(1 - \alpha)\, L\left(\frac{1}{1-t}\right)} . \tag{2.3.5}$$

KARAMATA'S TAUBERIAN theorem applied to 2.3.5 then yields the conclusion 2.3.1[1]).

[1]) We should mention that 2.3.1 can also be obtained, with an equivalent amount of labor, from a result of DE BRUIJN and ERDÖS [2] (cf. Theorem 2, p. 161).

By comparing 2.3.1 with the lower bound 1.1.5 (with "$\geq$"), it is not difficult to see that, for all $\alpha \varepsilon (0, 1)$, the limit 1.1.6 holds except for a set of integers of density 0. Indeed, if this were not so, the right side of 2.3.1 would have to be increased. This justifies the equality in 1.1.5 as well as establishing the last assertion of Theorem 1.1.

3. Analytic approach

3.1. The methods of this section are based upon a representation of the u_n's as Fourier coefficients of an integrable function. To facilitate our exposition we shall use the functions $U(t)$, $R(t)$ in 2.3.2 and two more functions

$$F(t) = \sum_{n=1}^{\infty} f_n t^n, \qquad \Phi(\theta) = \sum_{n=1}^{\infty} f_n e^{in\theta}. \tag{3.1.1}$$

The definition 1.1.2 then yields

$$U(t) = \frac{1}{1 - F(t)}, \qquad U(e^{i\theta}) = \frac{1}{1 - \Phi(\theta)}. \tag{3.1.2}$$

We observe that the condition 1.1.1 implies that $\Phi(\theta) = 1$ if and only if $e^{i\theta} = 1$. The function $1 - F(t)$ is thus bounded away from zero when t is bounded away from one. We deduce that $U(t)$ is defined and continuous for $t \neq 1$ and $|t| \leq 1$.

The next two lemmas give our basic tools.

Lemma 3.1.1. *Under assumption* 1.1.1, *the function* $\operatorname{Re} U(e^{i\theta})$ *is in* $L_1(-\pi, \pi)$ *and*

$$u_n = \frac{1}{\sum_{n=1}^{\infty} n f_n} + \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \operatorname{Re} U(e^{i\theta}) d\theta. \tag{3.1.3}$$

This result is not new; it has recently appeared in the probabilistic literature in work of D. G. Kendall [13], but the basic idea goes back to work of Herglotz [11] on functions analytic and with positive real part in the unit circle. In our cases the formula can be written in the form

$$u_n = \frac{1}{\pi} \operatorname{Re} \int_{0}^{\pi} e^{-in\theta} \frac{d\theta}{1 - \Phi(\theta)}. \tag{3.1.4}$$

3.2. Lemma 3.2.1. *Let $G(x)$ be the distribution function of a positive random variable; suppose that*

$$1 - G(x) = L(x)/x^{\alpha} \tag{3.2.1}$$

where $0 < \alpha < 1$ *and* $L(x)$ *is a function of slow growth. Then the characteristic function*

$$\Phi(\theta) = \int_0^\infty e^{ix\theta}\, dG(x) \tag{3.2.2}$$

has near $\theta = 0^+$ *the asymptotic behavior:*

$$1 - \Phi(\theta) \sim L(1/\theta)\, \theta^\alpha \frac{1}{i} \int_0^\infty e^{i\sigma} \frac{d\sigma}{\sigma^\alpha}\,. \tag{3.2.3}$$

Proof. A first result of this nature goes back to HARDY [10], and these things have been discussed in the probabilistic literature (cf. [6]). For convenience we shall give a proof, similar to that in [6] but slightly completed.

We first write $\Phi(\theta)$ in the form

$$1 - \Phi(\theta) = \int_0^\infty (1 - e^{ix\theta}) dG(x)\,. \tag{3.2.4}$$

Then for some fixed $M > 0$, we consider separately the two integrals

$$I_1 = \int_0^{M/\theta} \frac{(1 - e^{ix\theta})}{1 - G(1/\theta)}\, dG(x)\,, \qquad I_2 = \int_{M/\theta}^\infty \frac{(1 - e^{ix\theta})}{1 - G(1/\theta)}\, dG(x)\,.$$

The second integral presents no difficulty. In fact, by 3.2.1 we have

$$|I_2| \leq 2\frac{1 - G(M/\theta)}{1 - G(1/\theta)} \sim \frac{2}{M^\alpha} \quad (\text{as } \theta \to 0^+). \tag{3.2.5}$$

The first integral requires more care. An integration by parts yields

$$-I_1 = \int_0^{M/\theta} \frac{(1 - e^{ix\theta})}{1 - G(1/\theta)}\, d[1 - G(x)] = O\left[\frac{1 - G(M/\theta)}{1 - G(1/\theta)}\right] + i\theta \int_0^{M/\theta} e^{ix\theta} \frac{[1 - G(x)]}{1 - G(1/\theta)}\, dx\,,$$

and substituting $x\theta = \sigma$ we get

$$\lim_{\theta \to 0+} I_1 = 0[1/M^\alpha] + \frac{1}{i} \lim_{\theta \to 0+} \int_0^M e^{i\sigma} \frac{[1 - G(\sigma/\theta)]}{1 - G(1/\theta)}\, d\sigma\,. \tag{3.2.6}$$

By 3.2.1 we have that for each fixed $\sigma > 0$, $\dfrac{1 - G(\sigma/\theta)}{1 - G(1/\theta)} \to \dfrac{1}{\sigma^\alpha}$.

Although the limit is not uniform, the passage to the limit under the integral sign can be justified by use of some properties of slowly varying functions. Since similar steps will be necessary in later proofs, we shall carry out the justification once in detail. The treatment of this point is not new with us [1], but seems not to be too widely known.

Since $L(x)$ is slowly varying, for a given $\varepsilon > 0$ such that $\alpha + \varepsilon < 1$ we can find a $\lambda(\varepsilon)$ such that for all $\sigma \leq 1$ and $\dfrac{\sigma}{\theta} \geq \lambda(\varepsilon)$ we have

$$\left(\frac{\sigma}{\theta}\right)^{\varepsilon} L\left(\frac{\sigma}{\theta}\right) \leq \frac{1+\varepsilon}{1-\varepsilon}\left(\frac{1}{\theta}\right)^{\varepsilon} L\left(\frac{1}{\theta}\right)$$

or better,

$$\frac{1-G(\sigma/\theta)}{1-G(1/\theta)} \leq \frac{1+\varepsilon}{1-\varepsilon}\frac{1}{\sigma^{\alpha-\varepsilon}} \quad \text{for all } \sigma : \theta\lambda(\varepsilon) \leq \sigma \leq 1 . \tag{3.2.7}$$

Also we observe that

$$\left| \int_0^{\theta\lambda(\varepsilon)} e^{i\sigma} \frac{1-G(\sigma/\theta)}{1-G(1/\theta)} d\sigma \right| \leq \frac{\theta\lambda(\varepsilon)}{1-G(1/\theta)} = o(1) . \tag{3.2.8}$$

The fact that the integrand in 3.2.6 does converge uniformly to its limit for $1 \leq \sigma \leq M$, together with the inequalities 3.2.7 and 3.2.8, permits the passage to the limit under the integral sign in 3.2.6. Thereafter by the arbitrariness of M we deduce 3.2.3.

3.3. The following lemmas furnish the key estimates for establishing formula 1.1.6.

Lemma 3.3.1. *If $G(x)$ is the distribution function of the previous lemma, then as $\lambda \to \infty$*

$$\int_0^{\lambda} x\, dG(x) \sim \frac{\alpha}{1-\alpha} \lambda^{1-\alpha} L(\lambda) . \tag{3.3.1}$$

Proof. We have

$$\int_0^{\lambda} x dG(x) = -\int_0^{\lambda} x d[1-G(x)] = -\lambda[1-G(\lambda)] + \int_0^{\lambda} (1-G(x)\, dx .$$

Thus, making the substitution $x = \lambda\sigma$ and using 3.2.1 we get

$$\int_0^{\lambda} x\, dG(x) = -\lambda^{1-\alpha} L(\lambda) + \lambda^{1-\alpha} L(\lambda) \int_0^1 \frac{L(\lambda\sigma)}{L(\lambda)} \frac{d\sigma}{\sigma^{\alpha}} \sim -\lambda^{1-\alpha} L(\lambda) \left[1 - \int_0^1 \frac{d\sigma}{\sigma^{\alpha}}\right]$$

This result implies 3.3.1.

Lemma 3.3.2. *If $\Phi(\theta)$ is defined as in 3.2.2 and 3.2.1 holds, then there exists a constant M such that for all $\theta_1 \neq \theta_2$*

$$|\Phi(\theta_1) - \Phi(\theta_2)| \leq ML\left(\frac{1}{|\theta_1-\theta_2|}\right) |\theta_1 - \theta_2|^{\alpha} . \tag{3.3.3}$$

Proof. We start by writing the difference $\Phi(\theta_1) - \Phi(\theta_2)$ in the form

$$\Phi(\theta_1) - \Phi(\theta_2) = \int_0^N [e^{ix\theta_1} - e^{ix\theta_2}]\, dG(x) + \int_N^{\infty} e^{ix\theta_1} dG(x) - \int_N^{\infty} e^{ix\theta_2} dG(x) ,$$

where $N|\theta_1 - \theta_2| = 1$. Then estimate as follows

$$|\Phi(\theta_1) - \Phi(\theta_2)| \leq 2\int_0^N \left|\sin\left(|\theta_1-\theta_2|\frac{x}{2}\right)\right| dG(x) + 2(1-G(N))$$

$$\leq |\theta_1 - \theta_2| \int_0^N x\, dG(x) + 2\frac{L(N)}{N^\alpha}.$$

Using lemma 3.3.1 we deduce that for all sufficiently large N we must have

$$|\Phi(\theta_1) - \Phi(\theta_2)| \leq |\theta_1 - \theta_2| \frac{1}{1-\alpha} L(N)\, N^{1-\alpha} + 2\frac{L(N)}{N^\alpha}.$$

In other words, for all sufficiently small $|\theta_1 - \theta_2|$ we obtain

$$|\Phi(\theta_1) - \Phi(\theta_2)| \leq \left[2 + \frac{1}{1-\alpha}\right] |\theta_1 - \theta_2|^\alpha L\left(\frac{1}{|\theta_1 - \theta_2|}\right).$$

The conclusion of the lemma follows then from the boundedness of $\dot{\Phi}(\theta)$.

3.4. We are now in a position to establish 1.1.6. For the purposes of this section we set $G(x) = \sum_{k \leq x} f_k$ and then observe that, if we assume 1.1.1 and 1.1.4, lemmas 3.1.1, 3.2.1, 3.3.1 and 3.3.2 become applicable. In particular, the representation 3.1.3 holds; we shall use it in the form 3.1.4. To study the behavior of u_n we shall consider separately the two integrals

$$\alpha_n(a) = \int_0^{a/n} e^{-in\theta} \frac{d\theta}{1 - \Phi(\theta)}, \tag{3.4.1}$$

$$\beta_n(a) = \int_{a/n}^{\pi} e^{-in\theta} \frac{d\theta}{1 - \Phi(\theta)}. \tag{3.4.2}$$

Our aim is to first let $n \to \infty$ and then let $a \to \infty$.

The first integral always behaves in the desired fashion, according to the following

Lemma 3.4.1. *Under the hypotheses* 1.1.1 *and* 1.1.4 *we have*

$$\lim_{n\to\infty} n^{1-\alpha} L(n)\, \alpha_n(a) = i \frac{\int_0^a e^{-i\sigma} \sigma^{-\alpha}\, d\sigma}{\int_0^\infty e^{i\sigma}\, \sigma^{-\alpha}\, d\sigma}. \tag{3.4.3}$$

Proof. By the change of variables $n\theta = \sigma$ we can write $n^{1-\alpha} L(n)\, \alpha_n(a)$ in the form

$$n^{1-\alpha} L(n)\, \alpha_n(a) = \int_0^a e^{-i\sigma} \frac{L(n)\, d\sigma}{n^\alpha[1 - \Phi(\sigma/n)]}. \tag{3.4.4}$$

Note that by lemma 3.2.1 we have

$$\frac{n^\alpha}{L(n)}[1-\Phi(\sigma/n)] \sim \sigma^\alpha \frac{1}{i}\int_0^\infty e^{it}\frac{dt}{t^\alpha}.$$

Therefore passing to the limit under the integral sign in 3.4.4 we get 3.4.3.

We can proceed further towards 1.1.6. In fact, from 3.4.3 and the formula

$$\int_0^\infty e^{i\sigma}\frac{d\sigma}{\sigma^\alpha} = i\Gamma(1-\alpha)\left[\cos\frac{\pi\alpha}{2} - i\sin\frac{\pi\alpha}{2}\right]$$

we obtain

$$\lim_{a\to\infty}\left[\lim_{n\to\infty} n^{1-\alpha}L(n)\,\alpha_n(a)\right] = \sin\pi\alpha - i\cos\pi\alpha\,. \tag{3.4.5}$$

Combining this result with 3.1.4 and 3.4.1 we deduce that 1.1.6 will necessarily hold whenever we can conclude that

$$\lim_{a\to\infty}\left[\limsup_{n\to\infty} n^{1-\alpha}L(n)\,|\beta_n(a)|\right] = 0\,. \tag{3.4.6}$$

It turns out that we can always draw this conclusion only when $\frac{1}{2}<\alpha<1$. For $0<\alpha<\frac{1}{2}$, we can still show that

$$\beta_n(a) = O\left[\frac{L(n)}{n^\alpha}\right]. \tag{3.4.7}$$

This is very near to a best possible estimate. In fact, any improvement beyond changing $O[\quad]$ into $o[\quad]$ is bound to fail for the following reason. From 1.1.4 we can deduce that

$$f_n = o[r_n] = o\left(\frac{L(n)}{n^\alpha}\right), \tag{3.4.8}$$

and it can be easily shown by examples that this order condition cannot be further improved. On the other hand, by 1.1.2 and lemma 3.4.1 we have that

$$f_n \le u_n = O\left(\frac{n^{\alpha-1}}{L(n)}\right) + |\beta_n(a)|\,. \tag{3.4.9}$$

These considerations show why 1.1.6 fails, in general, for $\alpha<\frac{1}{2}$. When $\alpha=\frac{1}{2}$ the estimates are more complicated since then the behavior of $L(n)$ plays a role. We do not know whether our results in this case can be improved in any significant way.

3.5. We proceed to estimate $\beta_n(a)$. To this end we write it in the form

$$\beta_n(a) = \tfrac{1}{2}\int_{(a+\pi)/n}^{\pi} e^{-in\theta} \frac{[\Phi(\theta) - \Phi(\theta - \pi/n)]}{[1 - \Phi(\theta)][1 - \Phi(\theta - \pi/n)]} d\theta \tag{3.5.1}$$

$$+ \tfrac{1}{2}\int_{a/n}^{(a+\pi)/n} e^{-in\theta} \frac{d\theta}{1 - \Phi(\theta)} - \tfrac{1}{2}\int_{\pi}^{\pi+\pi/n} e^{-in\theta} \frac{d\theta}{1 - \Phi(\theta - \pi/n)}$$

$$= \beta_n^{(1)}(a) + \beta_n^{(2)}(a) + \beta_n^{(3)}(a).$$

This identity may be obtained by first making the substitution $\theta \to \theta - \pi/n$ in 3.4.2 and then averaging out the resulting integral with its original expression. We shall consider each of the terms in 3.5.1 separately.

The third term is easiest to estimate. When θ is bounded away from zero, $\Phi(\theta)$ is bounded away from one; we therefore have

$$L(n)\, n^{1-\alpha}\, \beta_n^{(3)}(a) = O\,[L(n) n^{1-\alpha} \int_{\pi}^{\pi+\pi/n} d\theta] = O\left[\frac{L(n)}{n^{\alpha}}\right]. \tag{3.5.2}$$

For the second term we make the substitution $n\theta = \sigma$ and use lemma 3.2.1 to obtain

$$\limsup_{n\to\infty} L(n)\, n^{1-\alpha}\, |\beta_n^{(2)}(a)| = O\left[\lim_{n\to\infty} \int_a^{a+\pi} \frac{L(n)}{L\left(\frac{n}{\sigma}\right)} \frac{d\sigma}{\sigma^{\alpha}}\right] = O\left[\int_a^{a+\pi} \frac{d\sigma}{\sigma^{\alpha}}\right]. \tag{3.5.3}$$

Finally, using lemmas 3.3.2 and 3.2.1 we get the estimate

$$\beta_n^{(1)}(a) = O\left[\frac{L(n)}{n^{\alpha}} \int_{\frac{a+\pi}{n}}^{\pi} \frac{d\theta}{\left[L\left(\frac{1}{\theta}\right)\right]^2 \theta^{2\alpha}}\right]. \tag{3.5.4}$$

We shall now distinguish the cases $0 < \alpha < \frac{1}{2}$, $\frac{1}{2} < \alpha < 1$ and $\alpha = \frac{1}{2}$. In the first case the integral in 3.5.4 is convergent as $n \to \infty$. This yields 3.4.7. In the second case we multiply 3.5.4 by $n^{1-\alpha}L(n)$ and make the substitution $\sigma = n\theta$ to obtain

$$\limsup_{n\to\infty} n^{1-\alpha}\, L(n)\, |\beta_n^{(1)}(a)| = O\left[\lim_{n\to\infty} \int_{a+\pi}^{n\pi} \left[\frac{L(n)}{L\left(\frac{n}{\sigma}\right)}\right]^2 \frac{d\sigma}{\sigma^{2\alpha}}\right] = O\left[\int_{a+\pi}^{\infty} \frac{d\sigma}{\sigma^{2\alpha}}\right].$$

This inequality together with 3.5.1, 3.5.2 and 3.5.3 implies 3.4.6. Therefore 1.1.6 is established in this case.

Concerning $\alpha = \frac{1}{2}$ we want to mention only that when the integral in 3.5.4 converges we obtain again 3.4.7, and when $L(x)$ approaches a nonvanishing limit, we obtain

$$\beta_n^{(1)}(a) = O\left[\frac{\log n}{\sqrt{n}}\right].$$

Remark. Before closing we would like to sketch another way of estimating $\beta_n^{(1)}(a)$. HÖLDER's inequality, lemma 3.2.1 and the substitution $\sigma = n\theta$ yield

$$\beta_n^{(1)}(a) = O\left[\left\{\int_{-\pi}^{\pi}\left|\Phi(\theta) - \Phi\left(\theta - \frac{\pi}{n}\right)\right|^p d\theta\right\}^{1/p} n^{2\alpha-1/q}\left\{\int_{a+\pi}^{n\pi}\frac{d\sigma}{\left[L\left(\frac{n}{\sigma}\right)\right]^{2q}\sigma^{2\alpha q}}\right\}^{1/q}\right].$$

Assuming that q was chosen so that $2\alpha q > 1$ we obtain

$$L(n)\, n^{1-\alpha}\, \beta_n^{(1)}(a) = O\left[\left\{\int_{-\pi}^{\pi}\left|\Phi(\theta) - \Phi\left(\theta - \frac{\pi}{n}\right)\right|^p d\theta\right\}^{1/p} \frac{n^{\alpha+1/p}}{L(n)}\left\{\int_{a+\pi}^{\infty}\frac{d\sigma}{\sigma^{2\alpha q}}\right\}^{1/q}\right].$$

This implies that an estimate such as

$$\int_{-\pi}^{\pi}\left|\Phi(\theta) - \Phi\left(\theta - \frac{\pi}{n}\right)\right|^p d\theta = O\left[\frac{[L(n)]^p}{n^{1+p\alpha}}\right] \tag{3.5.5}$$

is sufficient to guarantee 3.4.6, and hence the limit 1.1.6.

It is worth noting that 3.5.5 can actually be established in several interesting cases. For instance, for $\frac{1}{4} < \alpha < \frac{1}{2}$, if we assume that $f_n = 0[L(n)/n^{1+\alpha}]$, then 3.5.5 holds for $p = 2$.

3.6. We shall terminate with a self contained proof of the lower bound for u_n; i.e. 1.1.5 with "$\geq$". First note that if a_n is any sequence of integers and we set

$$U^{(n)}(t) = \sum_{k=0}^{\infty} u_k^{(n)} t^k = \frac{[F(t)]^{a_n}}{1 - F(t)}$$

then we shall necessarily have

$$u_n \geq u_n^{(n)}. \tag{3.6.1}$$

On the other hand, under our assumptions we can establish the formula

$$u_n^{(n)} = \frac{1}{\pi}\,\mathrm{Re}\int_0^{\pi} e^{-in\theta}\frac{[\Phi(\theta)]^{a_n}}{1 - \Phi(\theta)}\, d\theta\,. \tag{3.6.2}$$

It is thus easy to see that, in view of 3.6.1, the lower estimate in 1.1.5 is a consequence of the following

Theorem 3.6.1. *If the conditions* 1.1.1 *and* 1.1.4 *hold and the sequence* $\{a_n\}$ *is chosen so that for some* $a > 0$

$$a_n \sim \frac{a}{L(n)}\, n^{\alpha} \tag{3.6.3}$$

then we have

$$\lim_{n\to\infty} n^{1-\alpha} L(n)\int_0^{\pi} e^{-in\theta}\frac{[\Phi(\theta)]^{a_n}}{1 - \Phi(\theta)}\, d\theta = \frac{1}{c}\int_0^{\infty} e^{-i\sigma} e^{-ac\sigma^{\alpha}}\frac{d\sigma}{\sigma^{\alpha}}, \tag{3.6.4}$$

where

$$c = \frac{1}{i}\int_0^\infty e^{i\sigma}\frac{d\sigma}{\sigma^\alpha} = \Gamma(1-\alpha)\left[\cos\frac{\pi\alpha}{2} - i\sin\frac{\pi\alpha}{2}\right].$$

Proof. The contribution to the limit in 3.6.4 can only come from an interval $[0,\varepsilon]$, and this no matter how small is ε. This is because in any interval $[\varepsilon,\pi]$, $\Phi(\theta)$ is necessarily bounded away from one. We shall therefore study the quantity

$$\gamma_n = n^{1-\alpha} L(n)\int_0^\varepsilon e^{-in\theta}\frac{[\Phi(\theta)]^{a_n}}{1-\Phi(\theta)}\,d\theta\,, \tag{3.6.5}$$

for some suitable choice of ε.

We make the substitution $\sigma = n\theta$ in 3.6.5 and write the result in the form

$$\gamma_n = \int_0^{n\varepsilon} e^{-i\sigma} k_n(\sigma)\,d\sigma \tag{3.6.6}$$

with

$$k_n(\sigma) = \frac{L(n)}{n^\alpha}\frac{[\Phi(\sigma/n)]^{a_n}}{1-\Phi(\sigma/n)}\,.$$

Using 3.6.3 and lemma 3.2.1 we obtain that for each $\sigma > 0$ we have

$$\lim_{n\to\infty} k_n(\sigma) = \frac{1}{c}\lim_{n\to\infty}\left[\frac{L(n)}{L\left(\frac{n}{\sigma}\right)\sigma^\alpha}\left(1 - c\,L(n)\left(\frac{\sigma}{n}\right)^\alpha\right)^{\frac{an}{L(n)}}\right] = \frac{1}{c}\,\frac{e^{-ac\sigma^a}}{\sigma^\alpha}\,.$$

Thus all we have to do to obtain 3.6.4 is to show that we can carry out the passage to the limit under the integral sign in 3.6.6.

To this end, in view of lemma 3.2.1, we observe that because $\operatorname{Re} c > 0$ it is possible to choose $\delta > 0$ and ε so small that, for all $0 \le \theta \le \varepsilon$,

$$|\Phi(\theta)| \le |\exp[-e^{i\delta}c\,L(1/\theta)\,\theta^\alpha]| = e^{-bL(1/\theta)\theta^a}$$

where we assume that $b = \operatorname{Re}[e^{-i\delta}c] > 0$. Thus we obtain

$$\left|\Phi\left(\frac{\sigma}{n}\right)\right|^{a_n} \le e^{-a_n bL(n/\sigma)(\sigma/n)^a} \qquad \text{for all } 0 \le \frac{\sigma}{n} \le \varepsilon\,.$$

On the other hand, by lemma 3.2.1, there exists a constant M such that

$$\frac{L(n)}{n^\alpha\left|1-\Phi\left(\frac{\sigma}{n}\right)\right|} \le \frac{L(n)}{L\left(\frac{n}{\sigma}\right)}\frac{M}{\sigma^\alpha} \qquad \text{for all } 0 \le \frac{\sigma}{n} \le \varepsilon\,.$$

Combining these last two inequalities and 3.6.3 yields

$$k_n(\sigma) = O\left[\frac{e^{-ba\frac{L(n/\sigma)}{L(n)}\sigma^\alpha}}{\frac{L(n/\sigma)}{L(n)}\sigma^\alpha}\right]. \tag{3.6.8}$$

Choose $\gamma > 0$ such that $\alpha + \gamma < 1$. Note then that, when n is sufficiently large,

$$n^\gamma L(n) \leq \frac{1+\gamma}{1-\gamma} \left(\frac{n}{\sigma}\right)^\gamma L\left(\frac{n}{\sigma}\right).$$

Thus 3.6.8 gives

$$k_n(\sigma) = O\left[\frac{1+\gamma}{1-\gamma} \frac{1}{\sigma^{\alpha+\gamma}}\right] \qquad \text{for all } 0 \leq \sigma \leq 1 . \qquad (3.6.9)$$

When $\sigma > 1$, $n/\sigma < n$ so that for a sufficiently small ε

$$\frac{L(n)}{L(n/\sigma)} \leq \frac{1+\gamma}{1-\gamma} \sigma^\gamma \qquad \text{for all } 1 \leq \sigma \leq n\varepsilon .$$

This with 3.6.8 gives

$$k_n(\sigma) = O\left[\frac{1+\gamma}{1-\gamma} \frac{e^{-ba\frac{1+\gamma}{1-\gamma}\sigma^{\alpha-\gamma}}}{\sigma^{\alpha-\gamma}}\right] \qquad \text{for all } 1 \leq \sigma \leq n\varepsilon . \qquad (3.6.10)$$

The inequalities 3.6.9 and 3.6.1.0 permit the use of Lebesgue's dominated convergence theorem in 3.6.6; this completes our proof.

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(Received August 4, 1962)