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The equivalence of two definitions of quasiconformal mappings¹⁾

by LIPMAN BERS

We give here a new proof to the known fact (cf. MORI [7], BERS [3], YÛJÔBÔ [12], PFLUGER [11]) that the so-called analytic and geometric definitions of quasiconformality are equivalent. The proof uses a minimum of real variable techniques; no mention is made of absolute continuity in the sense of TONELLI. We rely instead on the theory of BELTRAMI's equations as exposed in AHLFORS-BERS [2] and on a theorem of BEURLING-AHLFORS [5].

Let $z \rightarrow w(z) = u(x, y) + iv(x, y)$ be an orientation preserving homeomorphism of a plane domain D onto another. If the partial derivatives of w , in the sense of distribution theory, are locally square integrable functions, we denote by $K_a(D, w)$ the smallest constant $K \geq 1$ such that the inequality

$$\left| \frac{\partial w}{\partial x} \right|^2 + \left| \frac{\partial w}{\partial y} \right|^2 \leq \left(K + \frac{1}{K} \right) \frac{\partial(u, v)}{\partial(x, y)} \quad (1)$$

holds a.e. in D . If there is no such number, or if w does not have locally square integrable generalized derivatives, we set $K_a(D, w) = \infty$. If

$$K_a(D, w) < \infty,$$

w is said to be K -quasiconformal according to the *analytic definition* [4, 6, 9].

A *topological rectangle* R is a conformal image of a closed rectangle

$$0 \leq \xi \leq m, \quad 0 \leq \eta \leq 1,$$

the images of the vertices being distinguished. We write $m = \text{mod } R$. For $R \subset D$, $w(R)$ is also a topological rectangle, in view of RIEMANN's mapping theorem. We set

$$K_g(D, w) = \sup \left(\frac{\text{mod } w(R)}{\text{mod } R} \right), \quad R \subset D. \quad (2)$$

It is immediate that $K_g(D, w) = K_g(w(D), w^{-1})$. Mappings with

$$K_g(D, w) < \infty$$

are called K -quasiconformal according to the *geometric definition* [1, 8, 10].

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Theorem. $K_a(D, w) = K_g(D, w)$.

The proof requires several lemmas. The crux of the argument is contained in Lemma 8 below.

If $K_a(D, w) \leq K < \infty$, then $w(z)$ satisfies a BELTRAMI equation

$$\frac{\partial w}{\partial \bar{z}} = \mu(z) \frac{\partial w}{\partial z} \tag{3}$$

with a measurable coefficient $\mu(z)$ satisfying

$$|\mu(z)| \leq \frac{K - 1}{K + 1}. \tag{4}$$

Indeed, inequality (1) may be written as

$$\left| \frac{\partial w}{\partial \bar{z}} \right| \leq \frac{K - 1}{K + 1} \left| \frac{\partial w}{\partial z} \right|.$$

Thus the theory exposed in [2] is applicable. In particular

$$K_a(D, \varphi \circ w \circ \psi) = K_a(\psi(D), w)$$

if φ and ψ are conformal.

Now let D and $w(D)$ be JORDAN domains, z_0 a point in D and φ and ψ conformal mappings of D and $w(D)$ onto the unit disc with

$$\varphi(z_0) = \psi(w(z_0)) = 0.$$

Then $W = \psi \circ w \circ \varphi^{-1}$ is a self-mapping of the unit disc with $W(0) = 0$. If $K(D, w) \leq K < \infty$, W is a solution of a BELTRAMI equation with a coefficient satisfying (4). Using [2] we obtain

Lemma 1. *If $K_a(D, w) < \infty$, D and $w(D)$ are JORDAN domains and $z_0 \in D$, then w has a uniform modulus of continuity depending only on $K_a(D, w)$, D , $w(D)$, z_0 and $w(z_0)$.*

Let R be a topological rectangle, $\mu(z)$, $z \in R$, a measurable function satisfying (4) for some $K \geq 1$, w and w_1 two homeomorphisms of D satisfying (3). Then $w \circ w_1^{-1}$ is a conformal mapping so that $\text{mod } w(R) = \text{mod } w_1(R)$. Hence we may define: $\text{mod } (R, \mu) = \text{mod } w(R)$.

Lemma 2. *Let R be a topological rectangle and $\{\mu_j(z)\}$ a sequence of measurable functions in R such that $|\mu_j(z)| \leq k < 1$ and $\mu_j(z) \rightarrow \mu(z)$ a.e. Then $\text{mod } (R, \mu_j) \rightarrow \text{mod } (R, \mu)$.*

Proof. We may assume that R is the unit disc made into a topological rectangle by choosing four "vertices" $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ on the boundary. Let W^{μ_j} be the homeomorphism of $|z| \leq 1$ onto itself with $W^{\mu_j}(0) = 0$, $W^{\mu_j}(1) = 1$

and $\partial W^{\mu_j}/\partial \bar{z} = \mu_j(z) \partial W^{\mu_j}/\partial z$ and let W^μ be defined similarly. By [2], p. 399, $W^{\mu_j} \rightarrow W^\mu$ uniformly in the closed unit disc. Since $\text{mod}(R, \mu_j) = \text{mod } W^{\mu_j}(R)$ is a continuous function of the cross-ratio of the points $W^{\mu_j}(\zeta_i)$, $i = 1, 2, 3, 4$, and similarly for $\text{mod}(R, \mu)$, the conclusion follows.

Lemma 3. *Let R be a topological rectangle, $\mu(z)$, $z \in R$, a measurable function satisfying (4). Then $\text{mod}(R, \mu) \leq K \text{mod } R$.*

Proof. If $\mu(z)$ is smooth, every homeomorphic solution of (3) is smooth and has a positive jacobian (cf. [2], p. 391). In this case the desired inequality follows by GRÖTZSCH's classical argument [7]. The general case is reduced to this special one by Lemma 2, since it is easy to find a sequence of smooth μ_j satisfying (4) and converging a. e. to μ .

Lemma 4. $K_g(D, w) \leq K_a(D, w)$.

This is an immediate corollary of Lemma 3.

Lemma 5. *If $K_a(D, w) < \infty$, then $K_g(D, w) = K_a(D, w)$.*

Proof. Set $K_a(D, w) = K$ and assume that $1 < K < \infty$. (Otherwise there is nothing to prove.) In view of Lemma 4 it suffices to show that for every δ , $0 < \delta < K - 1$, there exists a sequence of squares $Q_j \subset D$ with

$$\lim \text{mod } w(Q_j) \geq K - \delta. \quad (5)$$

Set $\mu(z) = (\partial w/\partial \bar{z})/(\partial w/\partial z)$ for $\partial w/\partial z \neq 0$, $\mu(z) = 0$ for $\partial w/\partial z = 0$; then w satisfies (4) and $\text{ess. sup } |\mu(z)| = (K - 1)/(K + 1)$. Let Δ denote the annulus

$$\frac{K - \delta - 1}{K + \delta + 1} \leq |\mu| \leq \frac{K - 1}{K + 1}$$

in the μ -plane; then $\mu^{-1}(\Delta) \subset D$ has positive measure. Let $\epsilon_j \downarrow 0$ be a given sequence. We can find a sequence of measurable sets Δ_j such that

$$\Delta_{j+1} \subset \Delta_j \subset \Delta, \quad \text{diam } \Delta_j \leq \epsilon_j, \quad \text{mes } \mu^{-1}(\Delta_j) > 0.$$

Indeed, if Δ is subdivided into finitely many measurable sets of diameter not exceeding ϵ_1 , at least one of them, say Δ_1 must be such that $\text{mes } \mu^{-1}(\Delta) > 0$. If Δ_1 is subdivided into finitely many measurable sets of diameter not exceeding ϵ_2 , at least one of them, say Δ_2 , is such that $\text{mes } \mu^{-1}(\Delta_2) > 0$, etc. Let $\mu_0 = |\mu_0| e^{i\alpha}$ be the intersection of the closures of the Δ_j . For each j let $z_j \in D$ be a point at which the set $\mu^{-1}(\Delta_j)$ has metric density one; such points exist by LEBESGUE's theorem. Each z_j is the center of a square Q_j with one side parallel to the ray $z = r e^{i\alpha/2}$, $0 < r < \infty$, and such that

$$Q_j \subset D, \quad \text{mes } Q_j = m_j^2 < \epsilon_j, \quad \text{mes } [Q_j \cap \mu^{-1}(\Delta_j)] \geq (1 - \epsilon_j) m_j^2.$$

Hence

$$\text{mes} \{z \mid z \in Q_j, \mid \mu(z) - \mu_0 \mid > \epsilon_j\} \leq \epsilon_j m_j^2.$$

Let Q be the square obtained from Q_j by the mapping $z \rightarrow (z - z_j)/m_j$. For $z \in Q$ set $\mu_j(z) = \mu(z_j + m_j z)$. Then $\mu_j(z) \rightarrow \mu_0$ in measure. Selecting if need be a subsequence we may assume that $\mu_j(z) \rightarrow \mu_0$ a. e. in Q . By Lemma 2

$$\text{mod} (Q, \mu_j) \rightarrow \text{mod} (Q, \mu_0) = \frac{1 + \mid \mu_0 \mid}{1 - \mid \mu_0 \mid} \geq K - \delta.$$

Noting that $\text{mod} w(Q_j) = \text{mod} (Q, \mu_j)$ we obtain (5).

Lemma 6. *If $w_j \rightarrow w$ uniformly in D and $K_a(D, w_j) \leq K < \infty$, then $K_a(D, w) \leq K$.*

Proof. Set $w = u + iv$, $w_j = u_j + iv_j$. Let D_0 be a relatively compact subdomain of D . It suffices to show that $K_a(D_0, w) \leq K$. By [2] Theorem 5, and the hypothesis

$$\int \int_{D_0} \frac{\partial(u_j, v_j)}{\partial(x, y)} dx dy = \text{mes} w_j(D_0) = O(1).$$

Since (1) holds for each w_j ,

$$\int \int_{D_0} \left(\left| \frac{\partial w_j}{\partial x} \right|^2 + \left| \frac{\partial w_j}{\partial y} \right|^2 \right) dx dy = O(1).$$

This shows that the partial derivatives of w_j are square-integrable functions in D_0 and that we may assume, selecting if need be a subsequence, that

$$\frac{\partial w_j}{\partial x} \rightarrow \frac{\partial w}{\partial x}, \quad \frac{\partial w_j}{\partial y} \rightarrow \frac{\partial w}{\partial y} \quad \text{weakly in } L_2(D_0). \tag{6}$$

Next, let ω be a smooth function with compact support in D_0 . Then

$$\begin{aligned} \int \int_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)} dx dy &= \int \int_{D_0} v_j \frac{\partial(\omega, u_j)}{\partial(x, y)} dx dy, \\ \int \int_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)} dx dy &= \int \int_{D_0} v \frac{\partial(\omega, u)}{\partial(x, y)} dx dy. \end{aligned} \tag{7}$$

If w and w_j are smooth, this follows by integration by parts. In the general case one approximates w (or w_j) together with its first derivatives, in the mean, by smooth functions. If ω is also non-negative, then

$$\int \int_{D_0} \omega \left(\left| \frac{\partial w_j}{\partial x} \right|^2 + \left| \frac{\partial w_j}{\partial y} \right|^2 \right) dx dy \leq \left(K + \frac{1}{K} \right) \int \int_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)} dx dy. \tag{8}$$

But by (6)

$$\int \int_{D_0} \omega \left(\left| \frac{\partial w}{\partial x} \right|^2 + \left| \frac{\partial w}{\partial y} \right|^2 \right) dx dy \leq \liminf \int \int_{D_0} \omega \left(\left| \frac{\partial w_j}{\partial x} \right|^2 + \left| \frac{\partial w_j}{\partial y} \right|^2 \right) dx dy$$

and by (6) and (7)

$$\lim \int \int_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)} dx dy = \int \int_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)} dx dy ,$$

so that by (8)

$$\int \int_{D_0} \omega \left(\left| \frac{\partial w}{\partial x} \right|^2 + \left| \frac{\partial w}{\partial y} \right|^2 \right) dx dy \leq \left(K + \frac{1}{K} \right) \int \int_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)} dx dy .$$

A simple limiting argument shows that this holds also if ω is the characteristic function of a rectangle in D_0 . This implies that (1) holds a.e.

We note now a corollary of the BEURLING-AHLFORS theorem [5].

Lemma 7. *For every $K \geq 1$ there exists a number K^* with the following property. Let $K_g(D, w) \leq K < \infty$ and let D_0 be a relatively compact JORDAN subdomain of D . Then there exists a homeomorphism Ω of the closure of D_0 onto that of $w(D_0)$ with $K_a(D, \Omega) \leq K^*$ and $w(z) = \Omega(z)$ on the boundary \dot{D}_0 of D_0 .*

Proof. Choose a point \hat{z} on \dot{D}_0 and set $\hat{Z} = w(\hat{z})$. Let $z \rightarrow \varphi(z)$ and $Z \rightarrow \psi(Z)$ be conformal homeomorphisms, of D_0 and $w(D_0)$, respectively, onto the half-plane $U = \{ \zeta \mid \text{Im } \zeta > 0 \}$ with $\varphi(\hat{z}) = \psi(\hat{Z}) = \infty$. Set

$$\gamma(\xi) = \psi \circ w \circ \varphi^{-1}(\xi), \quad -\infty < \xi < +\infty .$$

For a real ξ and an $h > 0$ make D_0 into a topological rectangle $R_{\xi, h}$ by choosing as "vertices" the points $\varphi^{-1}(\xi - h)$, $\varphi^{-1}(\xi)$, $\varphi^{-1}(\xi + h)$ and \hat{z} . Then $\text{mod } R_{\xi, h} = 1$. The "vertices" of $w(R_{\xi, h})$ are the points

$$\psi^{-1}(\gamma(\xi - h)), \psi^{-1}(\gamma(\xi)), \psi^{-1}(\gamma(\xi + h))$$

and \hat{Z} , and $\text{mod } w(R_{\xi, h})$ is a continuous function of the ratio

$$(\gamma(\xi + h) - \gamma(\xi)) / (\gamma(\xi) - \gamma(\xi - h)) .$$

Since $K^{-1} \leq \text{mod } w(R_{\xi, h}) \leq K$, there exists a $\varrho > 0$ depending only on K such that

$$0 < \frac{1}{\varrho} \leq \frac{\gamma(\xi + h) - \gamma(\xi)}{\gamma(\xi) - \gamma(\xi - h)} \leq \varrho . \tag{9}$$

According to [5], condition (9) implies that

$$\zeta = \xi + i\eta \rightarrow F(\zeta) = \frac{1}{2} \int_0^1 [(1 + i)\gamma(\xi + \tau\eta) + (1 - i)\gamma(\xi - \tau\eta)] d\tau$$

is a homeomorphism of the closed upper half-plane onto itself with $F(\xi) = g(\xi)$ and $K_a(U, F) = K^* < \infty$, where K^* depends only on K . Set $\Omega = \psi^{-1} \circ F \circ \varphi$; this mapping has the required properties.

Lemma 8. *If $K_g(D, w) < \infty$, then $K_a(D, w) < \infty$.*

Proof. Set $K_g(D, w) = K$. We show that for every square $Q \subset D$,

$$K_a(Q, w) \leq K^*,$$

the number in Lemma 7. For every integer $j > 0$ subdivide Q into 4^j congruent squares. Lemma 7 implies that there exists a homeomorphism w_j of Q such that $K_a(q, w_j) \leq K^*$ for each of the 4^j small squares q and $w_j = w$ on the boundary of each small square. Hence w_j is a homeomorphism of Q onto $w(Q)$ and $K_a(Q, w_j) \leq K^*$. By Lemma 1 the w_j are equicontinuous and, by construction, $w_j \rightarrow w$ on a dense set. Hence $w_j \rightarrow w$ uniformly and, by Lemma 6, $K_a(Q, w) \leq K^*$.

Combining Lemmas 5 and 8 we obtain the theorem.

Now set

$$K_1(D, w) = \inf \text{mod } w(R) \quad \text{for all } R \subset D \text{ and } \text{mod } R = 1$$

where R is a topological rectangle.

The argument used in proving Lemma 5 shows that $K_1(D, w) = K_a(D, w)$ whenever $K_a(D, w) < \infty$. The argument used in proving Lemmas 7 and 8 shows that $K_a(D, w)$ is finite whenever $K_1(D, w)$ is. Thus

$$K_1(D, w) = K_a(D, w).$$

The geometric definition can be given a local form by setting (cf. PFLUGER [9]).

$$K^*(D, w) = \sup_{z_0 \in D} \lim_{r \rightarrow 0} K_g(S_r(z_0), w)$$

$S_r(z_0)$ being the disc $|z - z_0| < r$. We have that

$$K^*(D, w) = K_g(D, w);$$

the proof is immediate via the equivalence theorem.

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