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A Proof of Thom's Theorem¹)

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§ 0. Introduction

The paper is designed to give a simple proof of a theorem of Thom (Théorème II. 10 of [11]), which states that the cohomology of the stable Thom object MO is a free module over the STEENROD algebra A over Z_2 .

The proof is divided into three parts: we first recall that the stable cohomology is a coalgebra M over Z_2 , and show that the graded dual M^* is a polynomial algebra; we then prove that M^* is an algebra over A^* (the graded dual of A); lastly we show that M^* is isomorphic to a free comodule over A^* . As a corollary of the proof of the main theorem, we give a short proof of the structure theorem for the unoriented cobordism ring \mathfrak{N}_* .

It seems possible to prove the theorems of Wall [12] on MSO in a similar way.

The author wishes to thank D.B.A.EPSTEIN for many chats about HOPF algebras.

§ 1. Cohomology of the Thom Spectrum

Let 0(n) be the *n*-dimensional real orthogonal group, $B_{0(n)}$ the classifying space for 0(n), γ_n the classifying *n*-plane bundle over $B_{0(n)}$. Let $\eta_n : E \to B_{0(n)}$ be the *n*-disk bundle associated with γ_n , $\dot{\eta}_n : \partial E \to B_{0(n)}$ the (n-1)-sphere bundle associated with η_n . Let MO(n) be the space obtained from E by collapsing ∂E to a point. MO(n) is called the Thom space of 0(n) ([11], [7], [3]).

The inclusion $0(n) \times 1 \subset 0(n+1)$ induces a map

$$MO(n) \times S^1 \rightarrow MO(n+1)$$
 (1.1)

which yields isomorphisms of cohomology and homotopy in dimensions

$$n + k, k < n$$
.

Thus a spectrum MO is obtained:

$$M0 = (point, MO(1), MO(2), ..., MO(k), MO(k+1), ...)$$
. (1.2)

The cohomology groups of MO are defined as follows (we will only consider coefficients \mathbb{Z}_2):

¹⁾ The paper was written at The Institute for Advanced Study while the author held a National Science Foundation post-doctoral fellowship.

$$H^k(MO; Z_2) = H^{n+k}(MO(n); Z_2) \quad k < n.$$
 (1.3)

We will write M for $\sum_{k} H^{k}(MO; \mathbb{Z}_{2})$. The Steenrod algebra operates on M. The A-module structure of M is given by Thom's theorem:

Theorem 1. (Thom). The A-module M is a free A-module, with free generators $u(\omega)$ in one-to-one correspondence with partitions ω of integers into integers, none of which have the form $2^t - 1$ for t > 0.

The theorem was first proved in [11]. A new proof will be given in § 3.

The additive structure of M is easily determined. Let $s: B_{0(n)} \to E$ be the zero cross section of η_n , above. We still denote by s the map induced by s into $MO(n) = E/\partial E$. It is well known [7] that s^* is a monomorphism, and that Image $s^* = w_n H^*(B_{0(n)}; Z_2)$, where w_n is the top Stiefel-Whitney class.

Since $H^*(B_{0(n)}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_n]$, we have the result that

$$M \cong Z_2[w_1, \ldots w_k, \ldots], \qquad (1.4)$$

as graded vector spaces, where grade $(w_k) = k$.

It has been noted [9] that, although M does not have a natural algebra structure, it does have a natural coalgebra [8] structure. Consider the usual inclusion

$$0(m) \times 0(n) \subset 0(m+n); \tag{1.5}$$

it induces a map

$$\varrho_{m,n}: MO(m) \otimes MO(n) \to MO(m+n)$$
. (1.6)

The maps $\varrho_{m,n}$ induce

$$\varrho^*: M \to M \otimes M , \qquad (1.7)$$

which make M into a coalgebra over Z_2 (the symbol \otimes of course stands for \otimes_{Z_1}), and the coproduct ϱ^* is consistent with the operation of A on M, that is, the following diagram is commutative:

where $\pi: A \otimes M \to M$ is the action of A on M, $\psi: A \to A \otimes A$ is the coproduct [6] in A, and T is the twist map which interchanges factors.

We can describe the map ϱ^* very easily, because the following diagram is commutative:

$$MO(m) \otimes MO(n) \xrightarrow{\varrho} MO(m+n)$$

$$\uparrow \approx \qquad \qquad \uparrow s$$

$$B_{0(m)\times 0(n)} \xrightarrow{\sigma} B_{0(m+n)},$$

$$(1.9)$$

where σ is the Whitney direct sum map, induced from (1.5).

Under the isomorphism (1.4) ϱ^* corresponds to σ^* , but σ^* is well known (as the Whitney direct sum theorem [5]):

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j. \tag{1.10}$$

§ 2. Comodules over A^*

Let A^* be the graded dual of the STEENROD algebra A over Z_2 . Let φ be the product and ψ the coproduct of A; we will denote by φ^* the coproduct and ψ^* the product of A^* . If we let $\varepsilon: A \to Z_2$ be the augmentation of A and $\eta: Z_2 \to A$ the unit of A, then the dual maps ε^* and η^* are the unit and augmentation of A^* . According to [6], A^* is the algebra of polynomials $Z_2[\xi_1, \ldots, \xi_n, \ldots]$, grade $\xi_n = 2^n - 1$, with the coproduct given by

$$\varphi^*(\xi_n) = \sum_{i=0}^n \xi_{n-i}^{2^i} \otimes \xi_i.$$
 (2.1)

The notion of a comodule L over A^* is just the obvious dualization of the notion of a module over A:

Definition. A \mathbb{Z}_2 -module L is called a comodule over A^* if there exists a map

$$\mu: L \to A^* \otimes L \,, \tag{2.2}$$

called the coaction of A^* , such that the following two diagrams are commutative:

$$L \xrightarrow{\mu} A^* \otimes L$$

$$\downarrow \mu \downarrow \qquad \downarrow 1 \otimes \mu$$

$$A^* \otimes L \xrightarrow{\varphi^* \otimes 1} A^* \otimes A^* \otimes L,$$

$$(2.3)$$

$$\begin{array}{ccc}
L & \xrightarrow{\mu} A^* \otimes L \\
1 & & \downarrow \eta^* \otimes 1 \\
L & \xrightarrow{\cong} Z_2 \otimes L.
\end{array} (2.4)$$

We immediately cite examples of A^* -comodules.

- 1. A^* itself is a comodule over A^* under φ^* as coaction.
- 2. If N is a graded module over A (suppose that N is finite dimensional in each grading) with action

$$\lambda: A \otimes N \to N , \qquad (2.5)$$

then the graded dual N^* is a comodule over A^* with coaction the dual of λ :

$$\lambda^*: N^* \to A^* \otimes N^* . \tag{2.6}$$

3. If V is a vector space over Z_2 , we can construct a free comodule $F = A^* \otimes V$ by letting

$$\mu: F \to A^* \otimes F \tag{2.7}$$

be just $\varphi^* \otimes 1$.

Free comodules have the expected properties: we just quote two, which we will use in the proof of Theorem 1.

Proposition 1. Let V be a \mathbb{Z}_2 -module and $F=A^*\otimes V$ a free A^* -comodule on V. Suppose we are given a comodule L over A^* and a \mathbb{Z}_2 -map

$$f: L \to V$$
 . (2.8)

Then there exists a unique A^* -comodule map

$$g: L \to F \tag{2.9}$$

which makes the following diagram commutative:

$$L \xrightarrow{g} F$$

$$\mu \downarrow \qquad \downarrow 1$$

$$A^* \otimes L \xrightarrow{1 \otimes f} A^* \otimes V.$$

$$(2.10)$$

The map g is said to be induced by f.

Proof. Define $g = (1 \otimes f) \mu$. The following commutative diagram proves that g is a map of A^* -comodules:

$$L \xrightarrow{\mu} A^* \otimes L \xrightarrow{1 \otimes f} A^* \otimes V$$

$$\downarrow \mu \downarrow \qquad \qquad \downarrow \varphi^* \otimes 1 \qquad \qquad \downarrow \varphi^* \otimes 1$$

$$A^* \otimes L \xrightarrow{1 \otimes \mu} A^* \otimes A^* \otimes L \xrightarrow{1 \otimes 1 \otimes f} A^* \otimes A^* \otimes V.$$

$$(2.11)$$

Definition. We say that the Z_2 -vector space L is an algebra over A^* if 1) it is an A^* -comodule with coaction μ (2.2), and 2) it is a Z_2 -algebra with multiplication

$$h: L \otimes L \to L \tag{2.12}$$

such that the following diagram is commutative:

$$\begin{array}{c}
L \otimes L & \xrightarrow{h} & L \\
\mu \otimes \mu \downarrow & \downarrow \\
A^* \otimes L \otimes A^* \otimes L & \downarrow \mu \\
1 \otimes T \otimes 1 \downarrow & \downarrow \\
A^* \otimes A^* \otimes L \otimes L & \xrightarrow{\psi^* \otimes h} & A^* \otimes L
\end{array} (2.13)$$

Proposition 2. Let V be a \mathbb{Z}_2 -algebra, $F=A^*\otimes V$ the free A^* -comodule on V. Then

- i) F is an A^* -algebra under $(\psi^* \otimes h')$ $(1 \otimes T \otimes 1)$, where $h': V \otimes V \to V$ is the product in V,
 - ii) If L is an A^* -algebra, and

$$f: L \to V \tag{2.14}$$

is a map of \mathbb{Z}_2 -algebras, then the comodule map induced by f

$$g: L \to F \tag{2.15}$$

is a map of A^* -algebras.

Proof. Part i) is an immediate consequence of the fact that A^* is a Hopf algebra under ψ^* , φ^* . The reader is invited to draw the appropriate commutative diagram.

We prove that g is a map of algebras by referring to the commutative diagram (2.16):

§ 3. Proof of Thom's Theorem

Let n be a fixed positive integer,

$$R^{(n)} = Z_2[w_1, \dots, w_n] \tag{3.1}$$

a graded polynomial algebra on n indeterminates $w_i, i = 1, \ldots, n$, with grade $(w_k) = k$. We make $R^{(n)}$ into a Hopf algebra by setting

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j. \tag{3.2}$$

Let

$$S^{(n)} = Z_2[y_1, \dots, y_n], \tag{3.3}$$

where grade $(y_i) = 1, i = 1, \ldots, n$.

Suppose ω is a partition of a non-negative integer k:

$$\omega = (i_1, \ldots, i_q), \, \omega \in \Pi(k) \,. \tag{3.4}$$

If all of i_1, \ldots, i_q are positive, we write

$$||\omega|| = q, \qquad (3.5)$$

if k = 0, we set $||\omega|| = 0$.

If $||\omega|| \leq n$, we will denote by $s(\omega)$ the smallest symmetric polynomial in $S^{(n)}$ containing the monomial $y_1^{i_1} \ldots y_q^{i_q}$ (see [7], for example).

Let us make $S^{(n)}$ into a Hopf algebra by setting

$$\sigma^*(y_i) = y_i \otimes 1 + 1 \otimes y_i ; \qquad (3.6)$$

then

$$\sigma^*(s(\omega)) = \sum_{(\omega_1, \omega_2) = \omega} s(\omega_1) \otimes s(\omega_2)$$
 (3.7)

(compare [5]). We may thus consider $R^{(n)}$ as a Hopf subalgebra of $S^{(n)}$, by identifying w_i with $s((1,\ldots,1)), (1,\ldots,1) \in \Pi(i)$. Under this identification, a \mathbb{Z}_2 -basis of $R^{(n)}$ is furnished by the set of elements

$$\{s(\omega) \mid \omega \in \Pi(k), k \geq 0, ||\omega|| \leq n\}. \tag{3.8}$$

If we consider the normal inclusions $R^{(n)} \subset R^{(n+1)}$, $S^{(n)} \subset S^{(n+1)}$, we see that we can define Hopf algebra retractions $f^{(n+1)}: R^{(n+1)} \to R^{(n)}$, $g^{(n+1)}: S^{(n+1)} \to S^{(n)}$ which make the following diagram commutative:

The maps are defined by:

$$f^{(n+1)}(w_{j}) = w_{j} \text{ if } j \leq n$$

$$= 0 \text{ if } j = n + 1$$

$$g^{(n+1)}(y_{j}) = y_{j} \text{ if } j \leq n$$

$$= 0 \text{ if } j = n + 1.$$
(3.10)

We remark that $f^{(n+1)}$ is an isomorphism in gradings < n+1. If we now consider the Hopf algebra

$$R = Z_2[w_1, \dots, w_k, \dots], \qquad (3.11)$$

where we set

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j , \qquad (3.12)$$

we can define Hopf algebra epimorphisms

$$h^{(n)}: R \to R^{(n)}$$
 $h^{(n)}(w_j) = w_j \quad j \le n$,
 $h^{(n)}(w_j) = 0 \quad j > n$.
$$(3.13)$$

Given $\omega \in \Pi(k)$, we define

$$\tilde{s}(\omega) = h^{(n)-1}(s(\omega)), \quad n > k. \tag{3.14}$$

The definition makes sense, for $h^{(n)}$ is an isomorphism in gradings < n + 1, and $s(\omega)$ is independent of the choice of n > k, according to (3.9).

From (3.8) we see that the set of elements

$$\{\widetilde{s}(\omega) \mid \omega \in \Pi(k), \ k \ge 0\}$$
(3.15)

forms a Z_2 -basis of R.

Let R^* be the graded dual of R. Let $\tilde{s}(\omega)^*$ be the dual basis to (3.15). The elements $\tilde{s}(\omega)^*$ are characterized by:

$$\langle \widetilde{s}(\omega)^*, \widetilde{s}(\omega') \rangle = \begin{cases} 1 & \omega' = \omega, \\ 0 & \omega' \neq \omega. \end{cases}$$
 (3.16)

Let

$$x_k = \tilde{s}((k))^*. \tag{3.17}$$

Proposition 3. As an algebra,

$$R^* = Z_2[x_1, \ldots, x_k, \ldots]. {(3.18)}$$

Proof. Let $T=Z_2[\widetilde{x}_1,\ldots,\widetilde{x}_k,\ldots]$, grade $(\widetilde{x}_k)=k$. Since R has commutative, associative coproduct, R^* is a commutative, associative algebra, therefore the assignment $f(\widetilde{x}_k)=x_k$ defines an algebra map

$$f: T \to R^* \ . \tag{3.19}$$

We claim that f is an epimorphism. To prove this, it is sufficient to show that for each $\omega \in \Pi(k)$, $k \geq 0$ the element $\tilde{s}(\omega)^*$ is in the image of f. This follows from the

Lemma. If $\omega = 1^{\lambda_1} \dots q^{\lambda_q} \dots k^{\lambda_k}$ (where λ_q is the number of times q occurs in ω), then

$$\widetilde{s}(\omega)^* = x_1^{\lambda_1} \dots x_q^{\lambda_q} \dots x_k^{\lambda_k}$$
.

Proof of Lemma. The result follows from the equation

$$\langle x_1^{\lambda_1} \dots x_k^{\lambda_k}, \widetilde{s}(\omega') \rangle = \langle \underbrace{x_1 \otimes \dots \otimes x_1}_{\lambda_1} \otimes \dots \otimes \underbrace{x_k \otimes \dots \otimes x_k}_{\lambda_k}, \sigma^{(m)} \widetilde{s}(\omega') \rangle, \quad (3.20)$$

where $m = \sum_{i} \lambda_{i}$, and $\sigma^{(m)}$ denotes the coproduct σ^{*} iterated m-1 times.

The proof of Proposition 3 is now immediate: since f preserves grading, and T with R have the same dimension in each grading, we know that since f is an epimorphism, it is also a monomorphism.

Corollary: As an algebra,

$$M^* = Z_2[x_1, \dots, x_k, \dots],$$
 (3.21)
 $x_k = \tilde{s}(k)^*, \text{ grade } (x_k) = k.$

where

Proof. Proposition 3 and (1.4), (1.10).

For the next proposition, we hark back to the isomorphism

$$s^*: M_t = w_n H^t(B_{0(n)}; Z_2)$$

of A-modules for t < n (1.3). For what follows, we always suppose that n was picked large. The elements $\tilde{s}(\omega)$ (3.14) satisfy

$$s^*(\tilde{s}(\omega)) = w_n s(\omega). \tag{3.22}$$

Proposition 4. Let $k=2^t-1$, $\vartheta \in A$, $\omega \in \Pi(q)$, grade $\vartheta=k-q$. Then

$$\langle x_k, \vartheta \ \widetilde{s} (\omega) \rangle = 0 \text{ if } \omega \neq (q), q = 2^s - 1, \langle x_k, \vartheta \ \widetilde{s} ((q)) \rangle = \langle \xi_{t-s}^{2^s}, \vartheta \rangle \text{ if } q = 2^s - 1.$$
 (3.23)

Proof. Consider the A-map $h: A \to M$ defined by $h(1) = \tilde{s}(0)$. This is the well-known Cartan-Serre representation of A([4], [10]), for

$$s^*h(\vartheta) = s^*(\vartheta \, \widetilde{s} \, (0))) = \vartheta \, s^* \, \widetilde{s} \, (0)) = \vartheta \, w_n \, . \tag{3.24}$$

If we identify w_n with $s(1^n) = y_1 \dots y_n$, we get ([2], p. 43)

$$\vartheta w_n = \vartheta(y_1 \dots y_n) = \sum_{(i_1, \dots, i_n)} \langle \xi_{i_1} \dots \xi_{i_n}, \vartheta \rangle y_1^{2^{i_1}} \dots y_n^{2^{i_n}}.$$
 (3.25)

To find $\vartheta \tilde{s}(\omega)$, where $\omega = 1^{\lambda_1} \dots k^{\lambda_k}$, it is sufficient to take

$$\vartheta(y_1^{\lambda_1+1}\ldots y_k^{\lambda_k+1}y_{k+1}\ldots y_n)$$

and symmetrize the result. In particular, if $\omega = (2^s - 1)$, we see that

$$\vartheta(y_1^{2^8} y_2 \dots y_n) = \Sigma \langle \xi_{i_1}^{2^8} \xi_{i_2} \dots \xi_{i_n}, \vartheta \rangle y_1^{2^{i_1+8}} y_2^{2^{i_2}} \dots y_n^{2^{i_n}}, \qquad (3.26)$$

which proves part of Proposition 4. Let us call a partition

$$\omega \in \Pi(k), \ \omega = 1^{\lambda_1} \dots k^{\lambda_k}$$

honest, if for at least one λ_j we have $0 < \lambda_j < k$. It is then an immediate consequence of (3.25) that if ω is an honest partition, $\vartheta \in A$ and $\vartheta \widetilde{s}(\omega) = \sum c_{\omega'} \widetilde{s}(\omega')$, $c_{\omega'} \in \mathbb{Z}_2$, then $c_{\omega'} \neq 0$ implies ω' is an honest partition. For partitions $\omega = (q)$, $q \neq 2^s - 1$, we prove again using (3.25) that $\vartheta \widetilde{s}(\omega)$ is in the subspace spanned by elements $\widetilde{s}(\omega')$, where ω' is an honest partition.

Proposition 5. Let $\mu^*: M^* \to A^* \otimes M^*$ be the coaction of A^* on M^* . Then

$$\mu^*(x_{2t-1}) = \sum_{s=0}^t \xi_{t-s}^{2s} \otimes x_{2s-1}, \qquad (3.27)$$

where we set $x_0 = 1$.

Proof. Let $\mu^*(x_k) = \sum \alpha_{\omega} \otimes \widetilde{s}(\omega)^*$. The term $\alpha_{\omega} \otimes \widetilde{s}(\omega)^*$ occurs in $\mu^*(x_k)$ with a non-zero coefficient if and only if for $\vartheta \in A$, grade $\vartheta = \operatorname{grade} \alpha_{\omega}$ we have

$$\langle x_k, \vartheta \ \widetilde{s}(\omega) \rangle = \langle \alpha_\omega, \vartheta \rangle .$$
 (3.28)

Proposition 4 completes the proof.

Corollary. Let $q: A^* \to M^*$ be a map of \mathbb{Z}_2 -algebras, defined by

$$q(\xi_k) = x_{2k-1}.$$

Then q is a monomorphism of A^* -algebras.

Proof. (2.1) and (3.27).

Let $H^* = \mathbb{Z}_2[u_2, \ldots, u_k, \ldots], \ k \neq 2^t - 1$, any t > 0, grade $(u_k) = k$. Let

$$f: M^* \to H^* \tag{3.29}$$

be an epimorphism of algebras, defined by

$$f(x_k) = u_k \text{ if } k \neq 2^t - 1 \text{ for any } t > 0,$$

= 0 if $k = 2^t - 1, t > 0.$ (3.30)

Consider the free A^* -comodule $F = A^* \otimes H^*$. According to Proposition 2, F is an A^* -algebra. Furthermore, Proposition 1 shows that there exists a comodule map g induced by f; Proposition 2 asserts that g is a map of algebras.

Let $H^{*(m)}$ be the subalgebra of H^* generated by $1, f(x_1), \ldots, f(x_m)$.

Lemma.

$$g(x_{2t-1}) = \xi_t \otimes 1 , \qquad (3.31)$$

$$g(x_k) \equiv 1 \otimes u_k \mod \overline{A}^* \otimes H^{*(k-1)}$$
 (3.32)

Proof. Formula (3.31) follows from (3.27). The assertion (3.26) follows from the remark that $\mu^*(x_k) \equiv 1 \otimes x_k \mod \overline{A}^* \otimes M^{*(k-1)}$, where $M^{*(k-1)}$ is the subalgebra generated by $1, x_1, \ldots, x_{k-1}$.

Proposition 6. The map

$$g: M^* \to A^* \otimes H^* \tag{3.33}$$

induced by f (3.30) yields an isomorphism of algebras over A^* .

Proof. Since M^* and $A^* \otimes H^*$ are graded, have the same (finite) dimension in each grading as Z_2 -modules, and g is grading preserving, it is sufficient to prove that g is an epimorphism. Let us prove this by showing that the image of g contains $A^* \otimes H^{*(m)}$. This is true for m = 1, for $H^{*(1)} = \{1\}$, and $\xi_t \otimes 1 \in \text{Image } g$, according to (3.31). Suppose $\text{Im}(g) \supset A^* \otimes H^{*(m-1)}$. If $m = 2^t - 1$ for some t > 0, then $H^{*(m)} = H^{*(m-1)}$, and we are done; suppose, therefore, that $m \neq 2^t - 1$ for any t > 0. According to (3.32) and the induction hypothesis, there is an element $z_m \in \overline{A}^* \otimes M^{*(m-1)}$ such that

$$g(x_m + z_m) = 1 \otimes u_m.$$

Since g is a map of algebras, this proves that $A^* \otimes M^{*(m)} \subset \operatorname{Im} g$. Induction completes the proof.

Proof of Theorem 1.

Consider the dual map to g:

$$g^*: A \otimes H \to M . \tag{3.34}$$

Since g^* is an isomorphism of A^* -algebras, g is an isomorphism of A-coalgebras. A Z_2 -basis of H is given by the dual basis of the basis of H^* consisting of monomials in the u_k , $k \neq 2^t - 1$, t > 0.

This completes the proof of Thom's Theorem. We cannot, however, restrain ourselves from taking the argument one step further. Let \mathfrak{N}_* be the unoriented cobordism ring [11]. According to a fundamental theorem of Thom (Théorème IV. 8 [11]), there is a naturally defined isomorphism

$$T: \Pi_{n+k}(MO(n)) \to \mathfrak{N}_{*k} \quad k < n . \tag{3.35}$$

Furthermore, the product in \mathfrak{N}^* corresponds under this isomorphism to the map induced by (1.6) [9].

We can use the Adams spectral sequence [1] as in [7] to compute the homotopy of MO. It is sufficient to look at the Adams spectral sequence for p=2. The E_2 -term is given by

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(M, Z_2). \tag{3.36}$$

Since M is a coalgebra over A with coproduct ϱ^* , $\operatorname{Ext}_A^{s_it}(M, Z_2)$ is an algebra; furthermore, the multiplication in the E_{∞} terms corresponds to the multiplication in homotopy induced by $\varrho \otimes$. However, since M is $A \otimes H$ as an A-coalgebra, we have

$$\operatorname{Ext}_{A}^{*,*}(M, Z_{2}) = \operatorname{Ext}_{A}^{0,*}(M, Z_{2}) \cong H^{*}$$
 (3.37)

as an algebra. Thus $E_2^{s,t} = 0$ unless s = 0, hence the Adams spectral sequence collapses in the nicest way imaginable—and we have the following theorem, also first proved by Thom:

Theorem 2. The ring \mathfrak{N}_* is a polynomial ring over \mathbb{Z}_2 in generators u_k , where $k=2,\ldots,\ k\neq 2^t-1$ for any t>0.

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