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A Proof of Thom's Theorem¹⁾

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§ 0. Introduction

The paper is designed to give a simple proof of a theorem of THOM (Théorème II. 10 of [11]), which states that the cohomology of the stable THOM object \mathbf{MO} is a free module over the STEENROD algebra A over Z_2 .

The proof is divided into three parts: we first recall that the stable cohomology is a coalgebra M over Z_2 , and show that the graded dual M^* is a polynomial algebra; we then prove that M^* is an algebra over A^* (the graded dual of A); lastly we show that M^* is isomorphic to a free comodule over A^* . As a corollary of the proof of the main theorem, we give a short proof of the structure theorem for the unoriented cobordism ring \mathfrak{N}_* .

It seems possible to prove the theorems of WALL [12] on \mathbf{MSO} in a similar way.

The author wishes to thank D.B.A. EPSTEIN for many chats about HOPF algebras.

§ 1. Cohomology of the Thom Spectrum

Let $O(n)$ be the n -dimensional real orthogonal group, $B_{O(n)}$ the classifying space for $O(n)$, γ_n the classifying n -plane bundle over $B_{O(n)}$. Let $\eta_n: E \rightarrow B_{O(n)}$ be the n -disk bundle associated with γ_n , $\eta_n: \partial E \rightarrow B_{O(n)}$ the $(n-1)$ -sphere bundle associated with η_n . Let $MO(n)$ be the space obtained from E by collapsing ∂E to a point. $MO(n)$ is called the THOM space of $O(n)$ ([11], [7], [3]).

The inclusion $O(n) \times 1 \subset O(n+1)$ induces a map

$$MO(n) \otimes S^1 \rightarrow MO(n+1) \quad (1.1)$$

which yields isomorphisms of cohomology and homotopy in dimensions

$$n+k, \quad k < n.$$

Thus a spectrum \mathbf{MO} is obtained:

$$\mathbf{MO} = (\text{point}, MO(1), MO(2), \dots, MO(k), MO(k+1), \dots). \quad (1.2)$$

The cohomology groups of MO are defined as follows (we will only consider coefficients Z_2):

¹⁾ The paper was written at The Institute for Advanced Study while the author held a National Science Foundation post-doctoral fellowship.

$$H^k(MO; Z_2) = H^{n+k}(MO(n); Z_2) \quad k < n. \tag{1.3}$$

We will write M for $\sum_k H^k(MO; Z_2)$. The STEENROD algebra operates on M . The A -module structure of M is given by THOM's theorem:

Theorem 1. (THOM). The A -module M is a free A -module, with free generators $u(\omega)$ in one-to-one correspondence with partitions ω of integers into integers, none of which have the form $2^t - 1$ for $t > 0$.

The theorem was first proved in [11]. A new proof will be given in § 3.

The additive structure of M is easily determined. Let $s: B_{0(n)} \rightarrow E$ be the zero cross section of η_n , above. We still denote by s the map induced by s into $MO(n) = E/\partial E$. It is well known [7] that s^* is a monomorphism, and that $\text{Image } s^* = w_n H^*(B_{0(n)}; Z_2)$, where w_n is the top STIEFEL-WHITNEY class.

Since $H^*(B_{0(n)}; Z_2) = Z_2[w_1, \dots, w_n]$, we have the result that

$$M \cong Z_2[w_1, \dots, w_k, \dots], \tag{1.4}$$

as graded vector spaces, where grade $(w_k) = k$.

It has been noted [9] that, although M does not have a natural algebra structure, it does have a natural coalgebra [8] structure. Consider the usual inclusion

$$O(m) \times O(n) \subset O(m+n); \tag{1.5}$$

it induces a map

$$\varrho_{m,n}: MO(m) \otimes MO(n) \rightarrow MO(m+n). \tag{1.6}$$

The maps $\varrho_{m,n}$ induce

$$\varrho^*: M \rightarrow M \otimes M, \tag{1.7}$$

which make M into a coalgebra over Z_2 (the symbol \otimes of course stands for \otimes_{Z_2}), and the coproduct ϱ^* is consistent with the operation of A on M , that is, the following diagram is commutative:

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\psi \otimes \varrho^*} & A \otimes A \otimes M \otimes M \xrightarrow{1 \otimes T \otimes 1} & A \otimes M \otimes A \otimes M \\ \pi \downarrow & & & \downarrow \pi \otimes \pi \\ M & \xrightarrow{\varrho^*} & & M \otimes M, \end{array} \tag{1.8}$$

where $\pi: A \otimes M \rightarrow M$ is the action of A on M , $\psi: A \rightarrow A \otimes A$ is the co-product [6] in A , and T is the twist map which interchanges factors.

We can describe the map ϱ^* very easily, because the following diagram is commutative:

$$\begin{array}{ccc}
 MO(m) \otimes MO(n) & \xrightarrow{\varrho} & MO(m+n) \\
 \uparrow \approx & & \uparrow s \\
 M(0(m) \times 0(n)) & & \\
 \uparrow s & \xrightarrow{\sigma} & B_{0(m+n)} \\
 B_{0(m) \times 0(n)} & &
 \end{array}
 \tag{1.9}$$

where σ is the WHITNEY direct sum map, induced from (1.5).

Under the isomorphism (1.4) ϱ^* corresponds to σ^* , but σ^* is well known (as the WHITNEY direct sum theorem [5]):

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j.
 \tag{1.10}$$

§ 2. Comodules over A^*

Let A^* be the graded dual of the STEENROD algebra A over Z_2 . Let φ be the product and ψ the coproduct of A ; we will denote by φ^* the coproduct and ψ^* the product of A^* . If we let $\varepsilon: A \rightarrow Z_2$ be the augmentation of A and $\eta: Z_2 \rightarrow A$ the unit of A , then the dual maps ε^* and η^* are the unit and augmentation of A^* . According to [6], A^* is the algebra of polynomials $Z_2[\xi_1, \dots, \xi_n, \dots]$, grade $\xi_n = 2^n - 1$, with the coproduct given by

$$\varphi^*(\xi_n) = \sum_{i=0}^n \xi_{n-i}^{2^i} \otimes \xi_i.
 \tag{2.1}$$

The notion of a comodule L over A^* is just the obvious dualization of the notion of a module over A :

Definition. A Z_2 -module L is called a comodule over A^* if there exists a map

$$\mu: L \rightarrow A^* \otimes L,
 \tag{2.2}$$

called the coaction of A^* , such that the following two diagrams are commutative:

$$\begin{array}{ccc}
 L & \xrightarrow{\mu} & A^* \otimes L \\
 \mu \downarrow & & \downarrow 1 \otimes \mu \\
 A^* \otimes L & \xrightarrow{\varphi^* \otimes 1} & A^* \otimes A^* \otimes L,
 \end{array}
 \tag{2.3}$$

$$\begin{array}{ccc}
 L & \xrightarrow{\mu} & A^* \otimes L \\
 1 \downarrow & & \downarrow \eta^* \otimes 1 \\
 L & \xrightarrow{\cong} & Z_2 \otimes L.
 \end{array}
 \tag{2.4}$$

We immediately cite examples of A^* -comodules.

1. A^* itself is a comodule over A^* under φ^* as coaction.
2. If N is a graded module over A (suppose that N is finite dimensional in each grading) with action

$$\lambda: A \otimes N \rightarrow N, \quad (2.5)$$

then the graded dual N^* is a comodule over A^* with coaction the dual of λ :

$$\lambda^*: N^* \rightarrow A^* \otimes N^*. \quad (2.6)$$

3. If V is a vector space over Z_2 , we can construct a free comodule $F = A^* \otimes V$ by letting

$$\mu: F \rightarrow A^* \otimes F \quad (2.7)$$

be just $\varphi^* \otimes 1$.

Free comodules have the expected properties: we just quote two, which we will use in the proof of Theorem 1.

Proposition 1. Let V be a Z_2 -module and $F = A^* \otimes V$ a free A^* -comodule on V . Suppose we are given a comodule L over A^* and a Z_2 -map

$$f: L \rightarrow V. \quad (2.8)$$

Then there exists a unique A^* -comodule map

$$g: L \rightarrow F \quad (2.9)$$

which makes the following diagram commutative:

$$\begin{array}{ccc} L & \xrightarrow{g} & F \\ \mu \downarrow & & \downarrow 1 \\ A^* \otimes L & \xrightarrow{1 \otimes f} & A^* \otimes V. \end{array} \quad (2.10)$$

The map g is said to be induced by f .

Proof. Define $g = (1 \otimes f) \mu$. The following commutative diagram proves that g is a map of A^* -comodules:

$$\begin{array}{ccccc} L & \xrightarrow{\mu} & A^* \otimes L & \xrightarrow{1 \otimes f} & A^* \otimes V \\ \mu \downarrow & & \downarrow \varphi^* \otimes 1 & & \downarrow \varphi^* \otimes 1 \\ A^* \otimes L & \xrightarrow{1 \otimes \mu} & A^* \otimes A^* \otimes L & \xrightarrow{1 \otimes 1 \otimes f} & A^* \otimes A^* \otimes V. \end{array} \quad (2.11)$$

Definition. We say that the Z_2 -vector space L is an algebra over A^* if 1) it is an A^* -comodule with coaction μ (2.2), and 2) it is a Z_2 -algebra with multiplication

$$h: L \otimes L \rightarrow L \tag{2.12}$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
 L \otimes L & \xrightarrow{h} & L \\
 \downarrow \mu \otimes \mu & & \downarrow \mu \\
 A^* \otimes L \otimes A^* \otimes L & & A^* \otimes L \\
 \downarrow 1 \otimes T \otimes 1 & & \downarrow \psi^* \otimes h \\
 A^* \otimes A^* \otimes L \otimes L & \xrightarrow{\psi^* \otimes h} & A^* \otimes L.
 \end{array} \tag{2.13}$$

Proposition 2. Let V be a Z_2 -algebra, $F = A^* \otimes V$ the free A^* -comodule on V . Then

- i) F is an A^* -algebra under $(\psi^* \otimes h')(1 \otimes T \otimes 1)$, where $h': V \otimes V \rightarrow V$ is the product in V ,
- ii) If L is an A^* -algebra, and

$$f: L \rightarrow V \tag{2.14}$$

is a map of Z_2 -algebras, then the comodule map induced by f

$$g: L \rightarrow F \tag{2.15}$$

is a map of A^* -algebras.

Proof. Part i) is an immediate consequence of the fact that A^* is a HOPF algebra under ψ^*, φ^* . The reader is invited to draw the appropriate commutative diagram.

We prove that g is a map of algebras by referring to the commutative diagram (2.16):

$$\begin{array}{ccccc}
 L \otimes L & \xrightarrow{h} & & & L \\
 \downarrow \mu \otimes \mu & & & & \downarrow \mu \\
 A^* \otimes L \otimes A^* \otimes L & \xrightarrow{1 \otimes T \otimes 1} & A^* \otimes A^* \otimes L \otimes L & \xrightarrow{\psi^* \otimes h} & A^* \otimes L \\
 \downarrow 1 \otimes f \otimes 1 \otimes f & & \downarrow 1 \otimes 1 \otimes f \otimes f & & \downarrow 1 \otimes f \\
 A^* \otimes V \otimes A^* \otimes V & \xrightarrow{1 \otimes T \otimes 1} & A^* \otimes A^* \otimes V \otimes V & \xrightarrow{\psi^* \otimes h'} & A^* \otimes V.
 \end{array} \tag{2.16}$$

§ 3. Proof of Thom's Theorem

Let n be a fixed positive integer,

$$R^{(n)} = Z_2[w_1, \dots, w_n] \tag{3.1}$$

a graded polynomial algebra on n indeterminates $w_i, i = 1, \dots, n$, with grade $(w_k) = k$. We make $R^{(n)}$ into a HOPF algebra by setting

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j. \quad (3.2)$$

Let

$$S^{(n)} = \mathbb{Z}_2[y_1, \dots, y_n], \quad (3.3)$$

where $\text{grade}(y_i) = 1, i = 1, \dots, n$.

Suppose ω is a partition of a non-negative integer k :

$$\omega = (i_1, \dots, i_q), \omega \in \Pi(k). \quad (3.4)$$

If all of i_1, \dots, i_q are positive, we write

$$\|\omega\| = q, \quad (3.5)$$

if $k = 0$, we set $\|\omega\| = 0$.

If $\|\omega\| \leq n$, we will denote by $s(\omega)$ the smallest symmetric polynomial in $S^{(n)}$ containing the monomial $y_1^{i_1} \dots y_q^{i_q}$ (see [7], for example).

Let us make $S^{(n)}$ into a HOPF algebra by setting

$$\sigma^*(y_i) = y_i \otimes 1 + 1 \otimes y_i; \quad (3.6)$$

then

$$\sigma^*(s(\omega)) = \sum_{(\omega_1, \omega_2) = \omega} s(\omega_1) \otimes s(\omega_2) \quad (3.7)$$

(compare [5]). We may thus consider $R^{(n)}$ as a HOPF subalgebra of $S^{(n)}$, by identifying w_i with $s((1, \dots, 1)), (1, \dots, 1) \in \Pi(i)$. Under this identification, a \mathbb{Z}_2 -basis of $R^{(n)}$ is furnished by the set of elements

$$\{s(\omega) \mid \omega \in \Pi(k), k \geq 0, \|\omega\| \leq n\}. \quad (3.8)$$

If we consider the normal inclusions $R^{(n)} \subset R^{(n+1)}, S^{(n)} \subset S^{(n+1)}$, we see that we can define HOPF algebra retractions $f^{(n+1)}: R^{(n+1)} \rightarrow R^{(n)}, g^{(n+1)}: S^{(n+1)} \rightarrow S^{(n)}$ which make the following diagram commutative:

$$\begin{array}{ccc} R^{(n+1)} & \xrightarrow{f^{(n+1)}} & R^{(n)} \\ \cap \downarrow & & \downarrow \cap \\ S^{(n+1)} & \xrightarrow{g^{(n+1)}} & S^{(n)}. \end{array} \quad (3.9)$$

The maps are defined by:

$$\begin{aligned} f^{(n+1)}(w_j) &= w_j & \text{if } j \leq n \\ &= 0 & \text{if } j = n + 1 \\ g^{(n+1)}(y_j) &= y_j & \text{if } j \leq n \\ &= 0 & \text{if } j = n + 1. \end{aligned} \quad (3.10)$$

We remark that $f^{(n+1)}$ is an isomorphism in gradings $< n + 1$. If we now consider the HOPF algebra

$$R = \mathbb{Z}_2[w_1, \dots, w_k, \dots], \tag{3.11}$$

where we set

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j, \tag{3.12}$$

we can define HOPF algebra epimorphisms

$$\begin{aligned} h^{(n)} : R &\rightarrow R^{(n)} \\ h^{(n)}(w_j) &= w_j \quad j \leq n, \\ h^{(n)}(w_j) &= 0 \quad j > n. \end{aligned} \tag{3.13}$$

Given $\omega \in \Pi(k)$, we define

$$\tilde{s}(\omega) = h^{(n)-1}(s(\omega)), \quad n > k. \tag{3.14}$$

The definition makes sense, for $h^{(n)}$ is an isomorphism in gradings $< n + 1$, and $s(\omega)$ is independent of the choice of $n > k$, according to (3.9).

From (3.8) we see that the set of elements

$$\{\tilde{s}(\omega) \mid \omega \in \Pi(k), k \geq 0\} \tag{3.15}$$

forms a \mathbb{Z}_2 -basis of R .

Let R^* be the graded dual of R . Let $\tilde{s}(\omega)^*$ be the dual basis to (3.15). The elements $\tilde{s}(\omega)^*$ are characterized by:

$$\langle \tilde{s}(\omega)^*, \tilde{s}(\omega') \rangle = \begin{cases} 1 & \omega' = \omega, \\ 0 & \omega' \neq \omega. \end{cases} \tag{3.16}$$

Let

$$x_k = \tilde{s}((k))^*. \tag{3.17}$$

Proposition 3. As an algebra,

$$R^* = \mathbb{Z}_2[x_1, \dots, x_k, \dots]. \tag{3.18}$$

Proof. Let $T = \mathbb{Z}_2[\tilde{x}_1, \dots, \tilde{x}_k, \dots]$, grade $(\tilde{x}_k) = k$. Since R has commutative, associative coproduct, R^* is a commutative, associative algebra, therefore the assignment $f(\tilde{x}_k) = x_k$ defines an algebra map

$$f : T \rightarrow R^*. \tag{3.19}$$

We claim that f is an epimorphism. To prove this, it is sufficient to show that for each $\omega \in \Pi(k)$, $k \geq 0$ the element $\tilde{s}(\omega)^*$ is in the image of f . This follows from the

Lemma. If $\omega = 1^{\lambda_1} \dots q^{\lambda_q} \dots k^{\lambda_k}$ (where λ_q is the number of times q occurs in ω), then

$$\tilde{s}(\omega)^* = x_1^{\lambda_1} \dots x_q^{\lambda_q} \dots x_k^{\lambda_k}.$$

Proof of Lemma. The result follows from the equation

$$\langle x_1^{\lambda_1} \dots x_k^{\lambda_k}, \tilde{s}(\omega') \rangle = \langle \underbrace{x_1 \otimes \dots \otimes x_1}_{\lambda_1} \otimes \dots \otimes \underbrace{x_k \otimes \dots \otimes x_k}_{\lambda_k}, \sigma^{(m)} \tilde{s}(\omega') \rangle, \quad (3.20)$$

where $m = \sum_i \lambda_i$, and $\sigma^{(m)}$ denotes the coproduct σ^* iterated $m - 1$ times.

The proof of Proposition 3 is now immediate: since f preserves grading, and T with R have the same dimension in each grading, we know that since f is an epimorphism, it is also a monomorphism.

Corollary: As an algebra,

$$M^* = Z_2[x_1, \dots, x_k, \dots], \quad (3.21)$$

where $x_k = \tilde{s}((k))^*$, grade $(x_k) = k$.

Proof. Proposition 3 and (1.4), (1.10).

For the next proposition, we hark back to the isomorphism

$$s^* : M_t = w_n H^t(B_{0(n)}; Z_2)$$

of A -modules for $t < n$ (1.3). For what follows, we always suppose that n was picked large. The elements $\tilde{s}(\omega)$ (3.14) satisfy

$$s^*(\tilde{s}(\omega)) = w_n s(\omega). \quad (3.22)$$

Proposition 4. Let $k = 2^t - 1$, $\vartheta \in A$, $\omega \in \Pi(q)$, grade $\vartheta = k - q$. Then

$$\begin{aligned} \langle x_k, \vartheta \tilde{s}(\omega) \rangle &= 0 \text{ if } \omega \neq (q), q = 2^s - 1, \\ \langle x_k, \vartheta \tilde{s}((q)) \rangle &= \langle \xi_{t-s}^{2^s}, \vartheta \rangle \text{ if } q = 2^s - 1. \end{aligned} \quad (3.23)$$

Proof. Consider the A -map $h : A \rightarrow M$ defined by $h(1) = \tilde{s}((0))$. This is the well-known CARTAN-SERRE representation of A ([4], [10]), for

$$s^*h(\vartheta) = s^*(\vartheta \tilde{s}((0))) = \vartheta s^* \tilde{s}((0)) = \vartheta w_n. \quad (3.24)$$

If we identify w_n with $s(1^n) = y_1 \dots y_n$, we get ([2], p. 43)

$$\vartheta w_n = \vartheta(y_1 \dots y_n) = \sum_{(i_1, \dots, i_n)} \langle \xi_{i_1} \dots \xi_{i_n}, \vartheta \rangle y_1^{2^{i_1}} \dots y_n^{2^{i_n}}. \quad (3.25)$$

To find $\vartheta \tilde{s}(\omega)$, where $\omega = 1^{\lambda_1} \dots k^{\lambda_k}$, it is sufficient to take

$$\vartheta(y_1^{\lambda_1+1} \dots y_k^{\lambda_k+1} y_{k+1} \dots y_n)$$

and symmetrize the result. In particular, if $\omega = (2^s - 1)$, we see that

$$\vartheta(y_1^{2^s} y_2 \dots y_n) = \sum \langle \xi_{i_1}^{2^s} \xi_{i_2} \dots \xi_{i_n}, \vartheta \rangle y_1^{2^{i_1+2^s}} y_2^{2^{i_2}} \dots y_n^{2^{i_n}}, \quad (3.26)$$

which proves part of Proposition 4. Let us call a partition

$$\omega \in \Pi(k), \quad \omega = 1^{\lambda_1} \dots k^{\lambda_k}$$

honest, if for at least one λ_j we have $0 < \lambda_j < k$. It is then an immediate consequence of (3.25) that if ω is an honest partition, $\vartheta \in A$ and $\vartheta \tilde{s}(\omega) = \sum c_{\omega'} \tilde{s}(\omega')$, $c_{\omega'} \in \mathbb{Z}_2$, then $c_{\omega'} \neq 0$ implies ω' is an honest partition. For partitions $\omega = (q)$, $q \neq 2^s - 1$, we prove again using (3.25) that $\vartheta \tilde{s}(\omega)$ is in the subspace spanned by elements $\tilde{s}(\omega')$, where ω' is an honest partition.

Proposition 5. Let $\mu^* : M^* \rightarrow A^* \otimes M^*$ be the coaction of A^* on M^* . Then

$$\mu^*(x_{2^t-1}) = \sum_{s=0}^t \xi_{2^s} \otimes x_{2^{t-s}}, \tag{3.27}$$

where we set $x_0 = 1$.

Proof. Let $\mu^*(x_k) = \sum \alpha_{\omega} \otimes \tilde{s}(\omega)^*$. The term $\alpha_{\omega} \otimes \tilde{s}(\omega)^*$ occurs in $\mu^*(x_k)$ with a non-zero coefficient if and only if for $\vartheta \in A$, $\text{grade } \vartheta = \text{grade } \alpha_{\omega}$ we have

$$\langle x_k, \vartheta \tilde{s}(\omega) \rangle = \langle \alpha_{\omega}, \vartheta \rangle. \tag{3.28}$$

Proposition 4 completes the proof.

Corollary. Let $q : A^* \rightarrow M^*$ be a map of \mathbb{Z}_2 -algebras, defined by

$$q(\xi_k) = x_{2^k-1}.$$

Then q is a monomorphism of A^* -algebras.

Proof. (2.1) and (3.27).

Let $H^* = \mathbb{Z}_2[u_2, \dots, u_k, \dots]$, $k \neq 2^t - 1$, any $t > 0$, $\text{grade}(u_k) = k$. Let

$$f : M^* \rightarrow H^* \tag{3.29}$$

be an epimorphism of algebras, defined by

$$\begin{aligned} f(x_k) &= u_k \text{ if } k \neq 2^t - 1 \text{ for any } t > 0, \\ &= 0 \text{ if } k = 2^t - 1, t > 0. \end{aligned} \tag{3.30}$$

Consider the free A^* -comodule $F = A^* \otimes H^*$. According to Proposition 2, F is an A^* -algebra. Furthermore, Proposition 1 shows that there exists a comodule map g induced by f ; Proposition 2 asserts that g is a map of algebras.

Let $H^{*(m)}$ be the subalgebra of H^* generated by $1, f(x_1), \dots, f(x_m)$.

Lemma.

$$g(x_{2^t-1}) = \xi_t \otimes 1, \tag{3.31}$$

$$\begin{aligned} g(x_k) &\equiv 1 \otimes u_k \text{ mod } \bar{A}^* \otimes H^{*(k-1)} \\ &\text{if } k \neq 2^t - 1, t > 0. \end{aligned} \tag{3.32}$$

Proof. Formula (3.31) follows from (3.27). The assertion (3.26) follows from the remark that $\mu^*(x_k) \equiv 1 \otimes x_k \pmod{\bar{A}^* \otimes M^{*(k-1)}}$, where $M^{*(k-1)}$ is the subalgebra generated by $1, x_1, \dots, x_{k-1}$.

Proposition 6. The map

$$g: M^* \rightarrow A^* \otimes H^* \quad (3.33)$$

induced by f (3.30) yields an isomorphism of algebras over A^* .

Proof. Since M^* and $A^* \otimes H^*$ are graded, have the same (finite) dimension in each grading as Z_2 -modules, and g is grading preserving, it is sufficient to prove that g is an epimorphism. Let us prove this by showing that the image of g contains $A^* \otimes H^{*(m)}$. This is true for $m = 1$, for $H^{*(1)} = \{1\}$, and $\xi_t \otimes 1 \in \text{Image } g$, according to (3.31). Suppose $\text{Im}(g) \supset A^* \otimes H^{*(m-1)}$. If $m = 2^t - 1$ for some $t > 0$, then $H^{*(m)} = H^{*(m-1)}$, and we are done; suppose, therefore, that $m \neq 2^t - 1$ for any $t > 0$. According to (3.32) and the induction hypothesis, there is an element $z_m \in \bar{A}^* \otimes M^{*(m-1)}$ such that

$$g(x_m + z_m) = 1 \otimes u_m.$$

Since g is a map of algebras, this proves that $A^* \otimes M^{*(m)} \subset \text{Im } g$. Induction completes the proof.

Proof of Theorem 1.

Consider the dual map to g :

$$g^*: A \otimes H \rightarrow M. \quad (3.34)$$

Since g^* is an isomorphism of A^* -algebras, g is an isomorphism of A -co-algebras. A Z_2 -basis of H is given by the dual basis of the basis of H^* consisting of monomials in the u_k , $k \neq 2^t - 1$, $t > 0$.

This completes the proof of THOM's Theorem. We cannot, however, restrain ourselves from taking the argument one step further. Let \mathfrak{N}_* be the unoriented cobordism ring [11]. According to a fundamental theorem of THOM (Théorème IV. 8 [11]), there is a naturally defined isomorphism

$$T: \Pi_{n+k}(MO(n)) \rightarrow \mathfrak{N}_{*k} \quad k < n. \quad (3.35)$$

Furthermore, the product in \mathfrak{N}^* corresponds under this isomorphism to the map induced by (1.6) [9].

We can use the ADAMS spectral sequence [1] as in [7] to compute the homotopy of \mathbf{MO} . It is sufficient to look at the ADAMS spectral sequence for $p = 2$. The E_2 -term is given by

$$E_2^{s,t} = \text{Ext}_A^{s,t}(M, Z_2). \quad (3.36)$$

Since M is a coalgebra over A with coproduct ϱ^* , $\text{Ext}_A^{s,t}(M, Z_2)$ is an algebra; furthermore, the multiplication in the E_∞ terms corresponds to the multiplication in homotopy induced by $\varrho \times$. However, since M is $A \otimes H$ as an A -coalgebra, we have

$$\text{Ext}_A^{s,t}(M, Z_2) = \text{Ext}_A^{0,t}(M, Z_2) \cong H^* \quad (3.37)$$

as an algebra. Thus $E_2^{s,t} = 0$ unless $s = 0$, hence the ADAMS spectral sequence collapses in the nicest way imaginable—and we have the following theorem, also first proved by THOM:

Theorem 2. The ring \mathfrak{N}_* is a polynomial ring over Z_2 in generators u_k , where $k = 2, \dots, k \neq 2^t - 1$ for any $t > 0$.

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