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**Autor:** Gehring, F.W.  
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# A remark on the moduli of rings

F. W. GEHRING

**1. Modulus of a ring.** A finite doubly-connected plane domain is called a ring. Given any ring  $R$  we let  $C_0$  and  $C_1$  denote, respectively, the bounded and unbounded components of the complement of  $R$  in the extended plane. We further let  $B_0 = \partial C_0$  and  $B_1 = \partial C_1$ , where  $\partial E$  denotes the boundary of the set  $E$ .  $B_0$  and  $B_1$  are then the components of  $\partial R$ .

Each ring  $R$  in the  $z$ -plane can be mapped conformally by  $w(z)$  onto an annulus  $0 \leq a < |w| < b \leq \infty$  so that  $B_0$  corresponds to  $|w| = a$  and  $B_1$  to  $|w| = b$ <sup>1)</sup>. The conformal invariant

$$\text{mod } R = \log \frac{b}{a} \quad (1)$$

is called the modulus of  $R$ . When  $B_0$  and  $B_1$  are both non-degenerate,  $0 < a < b < \infty$  and the function

$$u(z) = \frac{\log \left| \frac{w(z)}{a} \right|}{\log \frac{b}{a}} \quad (2)$$

is harmonic in  $R$  with boundary values 0 on  $B_0$  and 1 on  $B_1$ . It is also easy to verify that

$$\frac{2\pi}{\text{mod } R} = \iint_R |\nabla u|^2 d\sigma.$$

If we now appeal to the DIRICHLET principle, we obtain

$$\frac{2\pi}{\text{mod } R} = \inf_v \iint_R |\nabla v|^2 d\sigma, \quad (3)$$

where  $v$  is any function which is continuously differentiable in  $R$  and has boundary values 0 on  $B_0$  and 1 on  $B_1$ . When  $B_0$  or  $B_1$  reduces to a point,  $\text{mod } R = \infty$  and the infimum on the right hand side of (3) is 0. Hence (3) yields an alternative definition for the modulus of a ring which does not depend upon conformal mapping.

It is sometimes convenient to work with a slightly larger class of competing functions  $v$ . For example if  $v$  satisfies a LIPSCHITZ condition on each compact

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<sup>1)</sup> See p. 203 of [4].

subset of  $R$ , then  $v$  is differentiable almost everywhere in  $R$ . If, in addition,  $v$  has boundary values 0 on  $B_0$  and 1 on  $B_1$ , we can apply GREEN's theorem to show that

$$\frac{2\pi}{\text{mod } R} \leq \iint_R |\nabla v|^2 d\sigma^2. \quad (4)$$

Hence the infimum in (3) can be taken over the class of all such functions  $v$ .

**2. Continuity of the modulus.** A sequence of sets  $\{E_n\}$  is said to *converge uniformly* to a set  $E$  if, for each  $\varepsilon > 0$ , there exists an  $N$  such that  $n > N$  implies each point of  $E_n$  lies within distance  $\varepsilon$  of  $E$  and each point of  $E$  lies within distance  $\varepsilon$  of  $E_n$ .

In the preceding paper [2] VÄISÄLÄ and I appealed to the following continuity property of  $\text{mod } R$ .

**Theorem.** *Let  $\{R_n\}$  be a sequence of rings and let  $R$  be a bounded ring. If each of the components of  $\partial R_n$  converges uniformly to the corresponding component of  $\partial R$ , then*

$$\text{mod } R = \lim_{n \rightarrow \infty} \text{mod } R_n^3. \quad (5)$$

It is easy to establish this result by a direct argument in the special cases we required, that is when  $R$  is bounded by concentric rectangles or by concentric ellipses. I present here an elementary proof for the general case where  $R$  is an arbitrary bounded ring.

**3.** The proof depends upon an equicontinuity property for the harmonic functions  $u$  defined in (2). (See also p. 386 in [5].)

**Lemma 1.** *Let  $0 < a < b$ , let  $R$  be a bounded ring and let the diameter of  $B_0$  exceed  $b$ . Let  $u$  be the harmonic function defined in (2) and extend  $u$  so that  $u$  is 0 on  $C_0$  and 1 on  $C_1$ <sup>4</sup>). Then*

$$|u(z_1) - u(z_2)| \leq c \quad (6)$$

whenever  $|z_1 - z_2| \leq a$ , where

$$c = 2\pi \left( \text{mod } R \cdot \log \frac{b}{a} \right)^{-\frac{1}{2}}. \quad (7)$$

**Proof.** Fix  $z_1$  and  $z_2$  so that  $|z_1 - z_2| \leq a$ . Since  $0 \leq u(z_1), u(z_2) \leq 1$ , we may clearly assume that  $c < 1$  for otherwise (6) follows trivially.

<sup>2</sup>) This follows, for example, from the proof of Theorem 4.3 in [3].

<sup>3</sup>) The restriction that  $R$  be bounded can be omitted by redefining the notion of uniform convergence in terms of the metric on the RIEMANN sphere.

<sup>4</sup>) The fact that  $R$  is bounded implies that  $B_1$  is also non-degenerate.

Now let  $\gamma = \gamma(r)$  denote the circle with center  $\frac{1}{2}(z_1 + z_2)$  and radius  $r$ ,  $r > 0$ . By the SCHWARZ inequality

$$\left( \int_{\gamma \cap R} |\nabla u| ds \right)^2 \leq 2\pi r \int_{\gamma \cap R} |\nabla u|^2 ds$$

and, with FUBINI's theorem, we obtain

$$\int_{\frac{1}{2}a}^{\frac{1}{2}b} \left( \int_{\gamma \cap R} |\nabla u| ds \right)^2 \frac{dr}{r} \leq 2\pi \iint_R |\nabla u|^2 d\sigma = \frac{(2\pi)^2}{\text{mod } R}.$$

Hence there exists an  $r$ ,  $\frac{1}{2}a \leq r \leq \frac{1}{2}b$ , for which

$$\int_{\gamma \cap R} |\nabla u| ds \leq c.$$

Suppose that the corresponding circle  $\gamma$  lies in  $R$ . Since the diameter of  $\gamma$  is less than that of  $B_0$ ,  $R$  contains  $\Delta$ , the disk bounded by  $\gamma$ . Thus  $u$  satisfies the maximum principle in  $\Delta$  and

$$|u(z_1) - u(z_2)| \leq \underset{\gamma}{\text{osc } u} \leq \int_{\gamma} |\nabla u| ds \leq c,$$

as desired.

Next suppose that  $\gamma$  meets both  $R$  and  $C_0$ . Then  $\gamma \cap R$  is the union of open arcs  $\alpha$  and, since

$$\underset{\alpha}{\text{osc } u} \leq \int_{\gamma \cap R} |\nabla u| ds \leq c < 1,$$

it follows that  $\gamma$  does not meet  $C_1$ . Thus each  $\alpha$  has its endpoints in  $B_0$  and hence

$$\sup_{\gamma} u \leq c.$$

We see  $\partial(\Delta \cap R) \subset \gamma \cup B_0$  and, since  $u$  satisfies the maximum principle in  $\Delta \cap R$  and vanishes on  $\Delta \cap C_0$  and  $B_0$ ,

$$|u(z_1) - u(z_2)| \leq \sup_{\Delta \cap R} u = \sup_{\gamma} u \leq c.$$

A slight modification of the above argument handles the case where  $\gamma$  meets  $R$  and  $C_1$ .

Finally suppose that  $\gamma$  is contained in a component of the complement of  $R$ . The fact that the diameter of  $\gamma$  is less than that of  $B_0$  implies  $\Delta$  lies in the same component. Hence  $u(z_1) = u(z_2)$  and the proof is complete.

4. We now use Lemma 1 to establish the following result.

**Lemma 2.** *Let  $0 < a < b$  and let  $R$  and  $R'$  be bounded rings with boundary components  $B_0, B_1$  and  $B'_0, B'_1$ , respectively. Let all points of  $B'_0$  and  $B'_1$  lie within distance  $a$  of  $B_0$  and  $B_1$ , respectively, and let the diameter of  $B_0$  exceed  $b$ . Then*

$$\text{mod } R' \geq (1 - 4c) \text{ mod } R, \quad (8)$$

where  $c$  is as in (7).

**Proof.** We may assume that  $c < \frac{1}{4}$  for otherwise (8) follows trivially. Now let  $u$  be the harmonic function defined in (2) and extend  $u$  so that  $u$  is 0 on  $C_0$  and 1 on  $C_1$ . Then define  $v$  as follows:

$$v = \begin{cases} 0 & \text{if } u < c, \\ \frac{u - c}{1 - 2c} & \text{if } c \leq u \leq 1 - c, \\ 1 & \text{if } 1 - c < u. \end{cases}$$

The set where  $c \leq u \leq 1 - c$  is a compact subset of  $R$ . Hence  $u$  satisfies a LIPSCHITZ condition at each point of this set. From this it follows that  $v$  satisfies a LIPSCHITZ condition at all points of the ring  $R'$ . Next let  $z'$  be a point of  $B'_0$ . By hypothesis  $z'$  lies within distance  $a$  of some point  $z$  of  $B_0$  and Lemma 1 yields  $u(z') = u(z') - u(z) \leq c$ . Thus  $v$  is 0 on  $B'_0$ . Arguing similarly we see that  $v$  is 1 on  $B'_1$ . From (4) it follows that

$$\frac{2\pi}{\text{mod } R'} \leq \iint_{R'} |\nabla v|^2 d\sigma \leq (1 - 2c)^{-2} \iint_R |\nabla u|^2 d\sigma = (1 - 2c)^{-2} \frac{2\pi}{\text{mod } R},$$

and, since  $1 - 4c \leq (1 - 2c)^2$ , we obtain (8) as desired.

**5. Proof of the theorem.** Suppose first that  $B_0$  is non-degenerate and let  $B_{0,n}$  and  $B_{1,n}$  denote the boundary components of  $R_n$ . Then  $\text{mod } R < \infty$  and we can find a sequence  $\{a_n\}$ ,  $a_n > 0$ , and a number  $b > 0$  with the following properties. All points of  $B_{0,n}$  and  $B_{1,n}$  lie within  $a_n$  of  $B_0$  and  $B_1$  respectively, the diameter of  $B_0$  exceeds  $b$ , and  $a_n \rightarrow 0$ . Lemma 2 then yields  $\text{mod } R_n \geq (1 - 4c_n) \text{ mod } R$  for large  $n$ , where

$$c_n = 2\pi \left( \text{mod } R \cdot \log \frac{b}{a_n} \right)^{-\frac{1}{2}},$$

and, since  $c_n \rightarrow 0$ , we obtain

$$\liminf_{n \rightarrow \infty} \text{mod } R_n \geq \text{mod } R. \quad (9)$$

The uniform convergence next implies the existence of a second sequence  $\{a'_n\}$ ,  $a'_n > 0$ , such that all points of  $B_0$  and  $B_1$  lie within  $a'_n$  of  $B_{0,n}$  and  $B_{1,n}$  respectively and  $a'_n \rightarrow 0$ . Since the diameter of  $B_{0,n}$  eventually exceeds  $b$ ,  $\text{mod } R \geq (1 - 4c'_n) \text{ mod } R_n$  for large  $n$ , where

$$c'_n = 2\pi \left( \text{mod } R_n \cdot \log \frac{b}{a'_n} \right)^{-\frac{1}{2}}.$$

Inequality (9) implies that  $c'_n \rightarrow 0$ . Hence

$$\lim_{n \rightarrow \infty} \sup \operatorname{mod} R_n \leq \operatorname{mod} R$$

and we obtain (5) for the case where  $B_0$  is non-degenerate.

Now suppose that  $B_0$  reduces to a point  $P$ , let  $r_n$  be the radius of the smallest closed disk with center at  $P$  and containing  $B_{0,n}$ , and let  $d_n$  be the distance from  $P$  to  $B_{1,n}$ . Then  $\operatorname{mod} R = \infty$  and

$$\operatorname{mod} R_n \geq \log \frac{d_n}{r_n} \rightarrow \infty.$$

Hence we again obtain (5) and this completes the proof of the theorem.

**6. Remarks.** This result can also be proved directly using theorems on conformal mapping. On the other hand the above method can be used to establish the same continuity property for the moduli of rings in space [1].

Finally examining the above argument we can split up the hypotheses and conclusions for the theorem as follows.

*If, for each  $\varepsilon > 0$ , there exists an  $N$  such that  $n > N$  implies the points of each component of  $\partial R_n$  lie within distance  $\varepsilon$  of the corresponding component of  $\partial R$ , then*

$$\operatorname{mod} R \leq \lim_{n \rightarrow \infty} \inf \operatorname{mod} R_n.$$

*If, for each  $\varepsilon > 0$ , there exists an  $N$  such that  $n > N$  implies the points of each component of  $\partial R$  lie within distance  $\varepsilon$  of the corresponding component of  $\partial R_n$ , then*

$$\operatorname{mod} R \geq \lim_{n \rightarrow \infty} \sup \operatorname{mod} R_n.$$

It is clear that neither of the above inequalities can be replaced by equality.

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