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VINCENT's Conjecture on CLIFFORD Translations of the Sphere

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I. Introduction and statements of theorems

G. VINCENT has suggested the possibility that every finite group of CLIFFORD translations of a sphere is either cyclic or binary polyhedral [2, § 10.5]. In a recent *Comptes rendus* note [3] I stated that this is the case; the purpose of this note is to supply a proof.

S^n is the unit sphere in EUCLIDEAN space R^{n+1} , and carries the induced RIEMANNIAN structure; hence the group of isometries of S^n is the orthogonal group $O(n+1)$. Recall that an isometry f of S^n is a CLIFFORD translation if the distance between a point $x \in S^n$ and its image $f(x)$ is independent of x . This just means that either $f = \pm I$ (I = identity) or $n+1 = 2m$ and there is a unimodular complex number λ such that f has m eigenvalues equal to λ and m eigenvalues equal to the complex conjugate $\bar{\lambda}$ of λ .

We recall the binary polyhedral groups. The *polyhedral groups* are the dihedral groups D_m , the tetrahedral group T , the octahedral group O and the icosahedral group I —the respective groups of symmetries of the regular m -gon, the regular tetrahedron, the regular octahedron and the regular icosahedron. Each polyhedral group can, in a natural fashion, be considered as a subgroup of the special orthogonal group $SO(3)$. Let $\pi: Spin(3) \rightarrow SO(3)$ be the universal covering. The *binary polyhedral groups*²) are the *binary dihedral groups* $D_m^* = \pi^{-1}(D_m)$, the *binary tetrahedral group* $T^* = \pi^{-1}(T)$, the *binary octahedral group* $O^* = \pi^{-1}(O)$, and the *binary icosahedral group* $I^* = \pi^{-1}(I)$.

We can now state

Theorem 1 (conjectured by VINCENT). *If Γ is a finite group of CLIFFORD translations of a sphere, then Γ is either a cyclic group or a binary polyhedral group.*

In fact, one can add

Theorem 2. *Let Γ be a finite group of CLIFFORD translations of a sphere $S^n \subset R^{n+1}$. If Γ is cyclic of order 1 or 2, then $\Gamma = \{I\}$ or $\{\pm I\}$. If Γ is cyclic of order $q > 2$, then $n+1$ is even (say $n+1 = 2s$) and Γ is the*

¹) This work was done while the author held a National Science Foundation fellowship.

²) This definition was brought to my attention by J. TITS.

image of a representation ϱ of the abstract cyclic group Z_q where A is a generator of Z_q and ϱ is $SO(2s)$ -equivalent to the representation

$$A^t \rightarrow \begin{pmatrix} R(t/s) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & R(t/s) \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}.$$

If Γ is binary polyhedral and noncyclic, then 4 divides $n+1$ (say $n+1=4s$) and Γ is the image of a representation ϱ of an abstract binary polyhedral group $\mathcal{P}^* \cong \Gamma$ where ϱ is $SO(4s)$ -equivalent to a sum of s copies of the $SO(4)$ -representation

$$\mathcal{P}^* \subset Spin(3) = SU(2) \subset SO(4).$$

Finally, the images of these representations are finite groups of CLIFFORD translations of S^n .

Using Theorem 2 we will prove

Theorem 3. Let Γ be a finite group of CLIFFORD translations of a sphere $S^n \subset R^{n+1}$. Then the centralizer of Γ in $O(n+1)$ is transitive on S^n .

Theorem 4. Let Γ be a finite subgroup of $O(n+1)$. Then these are equivalent:

- (1) Γ is a group of CLIFFORD translations of S^n .
- (2) Γ is the image, by one of the representations described in Theorem 2, of a cyclic or binary polyhedral group.
- (3) The centralizer of Γ in $O(n+1)$ is transitive on S^n .
- (4) The quotient S^n/Γ is a RIEMANNIAN homogeneous manifold.

II. Proof of VINCENT's conjecture

We must give an abstract characterization of finite groups of CLIFFORD translations of a sphere.

Definition. Let $\varphi: \Gamma \rightarrow U(q)$ be a faithful unitary representation of an abstract finite group Γ such that, for every $\gamma \in \Gamma$, either $\varphi(\gamma) = \pm I$ or q is even (say $q = 2s$) and there is a unimodular complex number λ such that $\varphi(\gamma)$ is $U(q)$ -conjugate to

$$\begin{pmatrix} \lambda & & & \\ & \bar{\lambda} & & \\ & & \ddots & \\ & & & \lambda & \bar{\lambda} \end{pmatrix}.$$

Then φ is a CLIFFORD representation of Γ . Let Δ be an abstract finite group which has a CLIFFORD representation. Then Δ is a CLIFFORD group.

Note that a CLIFFORD representation φ of Γ gives a representation $\Gamma \xrightarrow{\varphi} U(q) \subset SO(2q)$ of Γ as CLIFFORD translations of S^{2q-1} , and a finite group Δ of CLIFFORD translations of S^n admits a CLIFFORD representation $\Delta \subset O(n+1) \subset U(n+1)$.

Lemma 1. *Let Γ be a noncyclic CLIFFORD group. Then*

- (1) *Every abelian subgroup of Γ is cyclic.*
- (2) *Given primes p and q , every subgroup of Γ of order pq is cyclic.*
- (3) *Γ has a unique element of order 2. It generates the center of Γ .*
- (4) *If α and α^t are conjugate elements of Γ , then $\alpha = \alpha^t$ or $\alpha^{-1} = \alpha^t$.*

Proof. Statements (1), (2) and the uniqueness of elements of order 2 in Γ are well known to follow from the fact that Γ has a free action on a sphere; see [2], [4] or [5], for example. As Γ has even order [2, § 10.5], (3) follows when we show that a central element $\neq 1$ of Γ has order 2.

Let φ be a CLIFFORD representation of Γ . Looking at characters, we see that the irreducible components of φ are equal and are CLIFFORD representations, so we may assume φ irreducible. If $\gamma \neq 1$ is central in Γ , SCHUR's lemma shows that $\varphi(\gamma)$ is scalar,

$$\varphi(\gamma) = \begin{pmatrix} \lambda & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots & \lambda \end{pmatrix}.$$

Hence $\lambda = \bar{\lambda}$ so $|\lambda| = 1$ implies $(\gamma \neq 1) \quad \varphi(\gamma) = -I$, so that $\gamma^2 = 1$ and (3) is proved. In (4), we may assume α not central in Γ , so

$$\varphi(\alpha) = \begin{pmatrix} \lambda & & & \\ & \bar{\lambda} & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} \quad \text{and} \quad \varphi(\alpha^t) = \varphi(\alpha)^t = \begin{pmatrix} \lambda^t & & & \\ & \bar{\lambda}^t & & \\ & & \ddots & \\ & & & \lambda^t \end{pmatrix}$$

have the same eigenvalues. Thus either $\lambda = \lambda^t$ and $\alpha = \alpha^t$, or $\bar{\lambda} = \lambda^t$ and $\alpha^{-1} = \alpha^t$. Q.E.D.

Lemma 2. *Let Γ_1 be a normal subgroup of a CLIFFORD group Γ , assume Γ_1 cyclic or binary dihedral $\mathcal{D}_m^*(m \neq 2)$, and suppose Γ generated by Γ_1 and some element $\tau \in \Gamma$. Then Γ is cyclic or binary dihedral.*

Proof. First suppose Γ_1 cyclic of order $m: \alpha^m = 1$. $\tau\alpha\tau^{-1} = \alpha$ or α^{-1} by Lemma 1. If $\tau\alpha\tau^{-1} = \alpha$, Γ is abelian and thus cyclic by Lemma 1. Now

assume $\tau\alpha\tau^{-1} = \alpha^{-1} \neq \alpha$. τ is not central in Γ so $\tau^2 \neq 1$, but τ^2 is central in Γ and Γ is not cyclic, so τ has order 4. Thus Γ is binary dihedral \mathcal{D}_m^* if m is odd, \mathcal{D}_s^* if $m = 2s$.

Now suppose Γ_1 binary dihedral \mathcal{D}_m^* with $m \neq 2$: $\alpha^m = 1 = \beta^4$, $\beta\alpha\beta^{-1} = \alpha^{-1}$ for m odd; $\alpha^{2m} = 1$, $\beta^2 = \alpha^m$, $\beta\alpha\beta^{-1} = \alpha^{-1}$ for m even. As $m \neq 2$, the cyclic group $\{\alpha\}$ is a characteristic subgroup of Γ_1 , hence a normal subgroup of Γ . Thus $\tau\alpha\tau^{-1}$ is either α or α^{-1} . β^2 is central in Γ because it has order 2, so the subgroup Γ' generated by β^2 , α , and either τ or $\tau\beta$ is abelian and thus cyclic. Γ is generated by Γ' and β . $\tau\beta\tau^{-1}$ has order 4, hence is of the form $\beta\alpha^u$ or $\beta^3\alpha^u$; thus $\beta^{-1}\tau\beta$ is of the form $\alpha^u\tau$ or $\alpha^u\tau\beta^2$ and $\beta^{-1}(\tau\beta)\beta$ is of the form $\alpha^u(\tau\beta)$ or $\alpha^u(\tau\beta)\beta^2$. Thus Γ_1 is normal in Γ and we are done by the first paragraph of the proof. Q.E.D.

The next lemma depends on a procedure of H. ZASSENHAUS [5, proof of Satz 7] which depends on his result [5, Satz 6]: *Let G be a finite solvable group of order not divisible by 2^{s+1} , and which contains an element of order 2^{s-1} ($s > 1$). Then G has a normal subgroup G_1 , with cyclic 2-Sylow subgroup, such that G/G_1 is the cyclic group Z_2 of order 2, the alternating group A_4 on 4 letters, or the symmetric group S_4 on 4 letters.* The lemma also uses a result of G. VINCENT [2, Théorème X] which implies that a CLIFFORD group with all SYLOW subgroups cyclic is either cyclic or binary dihedral \mathcal{D}_m^* (m odd).

Lemma 3. *A solvable CLIFFORD group is cyclic, binary dihedral, binary tetrahedral or binary octahedral.*

Proof. Let Γ be a solvable CLIFFORD group. We recall [2,5] that the odd SYLOW subgroups of Γ are cyclic and the 2-SYLOW subgroups are either cyclic or generalized quaternionic (binary dihedral \mathcal{D}_m^* where $m > 1$ is a power of 2) because every abelian subgroup of Γ is cyclic. If the 2-SYLOW subgroups of Γ are cyclic, we are done by the above-mentioned result of VINCENT. Otherwise, Γ has order $2^s n$ with n odd and $s > 2$, and an element of order 2^{s-1} . Using the above-mentioned result of ZASSENHAUS, we take a normal subgroup Γ_1 of Γ with all SYLOW subgroups cyclic and $\Gamma/\Gamma_1 = Z_2, A_4$ or S_4 . Note that Γ_1 is either cyclic or \mathcal{D}_m^* (m odd) by the result of VINCENT.

Case 1: $\Gamma/\Gamma_1 = Z_2$. By Lemma 2, Γ is cyclic or binary dihedral.

Case 2: $\Gamma/\Gamma_1 = A_4$. As the 2-SYLOW subgroups of Γ are generalized quaternionic and those of Γ/Γ_1 are $Z_2 \times Z_2$, Γ_1 must have some even order $2t$. Γ/Γ_1 is given in generators and relations by $\hat{\mu}^2 = \hat{\nu}^2 = \hat{\omega}^3 = 1$, $\hat{\mu}\hat{\nu} = \hat{\nu}\hat{\mu}$, $\hat{\omega}\hat{\mu}\hat{\omega}^{-1} = \hat{\nu}$ and $\hat{\omega}\hat{\nu}\hat{\omega}^{-1} = \hat{\mu}$. We choose representatives μ, ν, ω in Γ for $\hat{\mu}, \hat{\nu}, \hat{\omega}$ in Γ/Γ_1 .

First suppose that Γ_1 is cyclic: $\alpha^{2t} = 1$. Lemma 1 shows that one of $\nu\mu$, ν and μ commutes with α , so we can assume $\mu\alpha = \alpha\mu$. Then μ and α generate a cyclic group of order $4t$, which is normal in the group Γ' generated by μ, α and ν . Lemma 2 shows that Γ' is either cyclic order $8t$ or binary dihedral \mathcal{D}_{2t}^* of order $8t$. Note that Γ' is normal in Γ . If $t \neq 1$, Lemma 2 shows that Γ is binary dihedral. If $t = 1$, $\Gamma' = \mathcal{D}_2^*$ has automorphism group \mathcal{S}_4 , so an automorphism of Γ' of order $3k$ has order 3, and thus ω^3 is central in Γ . Replacing ω by $\alpha\omega$ if necessary, we see that Γ is the binary tetrahedral group $\mathcal{T}^* : \mu^4 = 1, \mu^2 = \nu^2 = \alpha, \omega^3 = 1, \mu\nu\mu^{-1} = \nu^{-1}, \omega\mu\omega^{-1} = \nu$ and $\omega\nu\omega^{-1} = \nu\mu$.

Now suppose that $\Gamma_1 = \mathcal{D}_m^*$ (m odd): $\alpha^m = \beta^4 = 1, \beta\alpha\beta^{-1} = \alpha^{-1}$. The cyclic group generated by α is characteristic in Γ_1 , hence normal in Γ . As before we can assume $\mu\alpha = \alpha\mu$, so μ and α generate a cyclic group, evidently normal in the group Γ' generated by μ, ν and α . By Lemma 2, Γ' is either cyclic or binary dihedral. As the order of Γ' is not 8 and Γ' is normal in the group Γ'' generated by Γ' and β , Γ'' is binary dihedral by Lemma 2. Γ'' is normal in Γ because it is generated by Γ_1, μ and ν ; a final application of Lemma 2 shows that Γ is binary dihedral.

Case 3: $\Gamma/\Gamma_1 = \mathcal{S}_4$. We have a natural homomorphism $\psi : \Gamma \rightarrow \mathcal{S}_4$ of Γ onto \mathcal{S}_4 with kernel Γ_1 , and we set $\Gamma' = \psi^{-1}(\mathcal{A}_4)$. Γ' is a normal subgroup of index 2 in Γ . By Case 2, Γ' is either binary dihedral \mathcal{D}_q^* ($q \neq 2$) or binary tetrahedral \mathcal{T}^* . If $\Gamma' = \mathcal{D}_q^*$ ($q \neq 2$), Lemma 2 shows $\Gamma = \mathcal{D}_{2q}^*$. If $\Gamma' = \mathcal{T}^*$, then Γ_1 is cyclic order 2, is the center of Γ' and is the center of Γ . It is now easy to see that Γ is the binary octahedral group \mathcal{O}^* . Q.E.D.

It now remains only to show that a non-solvable CLIFFORD group is the binary icosahedral group \mathcal{I}^* . Our proof depends on the isomorphism of \mathcal{I}^* with the group $SL(2, 5)$ of unimodular 2×2 matrices over the field \mathbb{Z}_5 of 5 elements, as well as a result of M. SUZUKI which implies [1, Theorem E] that a non-solvable group with every abelian subgroup cyclic has a normal subgroup isomorphic to some $SL(2, p)$ with $p > 3$ prime.

Lemma 4. *If p is a prime and $SL(2, p)$ is a CLIFFORD group, then $p = 3$ or $p = 5$.*

Proof. Let ω be a generator of the multiplicative group of non-zero elements of the field \mathbb{Z}_p of p elements, and set

$$\nu = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ in } SL(2, p).$$

$\nu\alpha\nu^{-1} = \alpha^{(\omega^3)}$ so $\omega^2 \equiv \pm 1 \pmod{p}$ by Lemma 1. Hence $\omega^4 \equiv 1 \pmod{p}$

so, as ω has order $p - 1$ in the multiplicative group, $p - 1$ divides 4. Thus p is 2, 3, or 5. $p \neq 2$ because $SL(2,2)$ has several elements of order 2. Q.E.D.

Lemma 5. *Let Γ be a CLIFFORD group, and suppose that Γ has a normal subgroup Γ_1 isomorphic to $SL(2,5)$. Then $\Gamma = \Gamma_1$.*

Proof. Given $\gamma \in \Gamma$, let $ad(\gamma)$ denote the automorphism $\alpha \mapsto \gamma\alpha\gamma^{-1}$ of Γ_1 . Let $\gamma \in \Gamma$ and assume that $ad(\gamma)$ is an inner automorphism of Γ_1 . There is a $\gamma' \in \Gamma_1$ with $ad(\gamma\gamma') = 1$, so $\gamma\gamma'$ is central in the noncyclic CLIFFORD group generated by $\gamma\gamma'$ and Γ_1 . Thus $\gamma\gamma' \in \Gamma_1$, for either $\gamma\gamma' = 1$, or $\gamma\gamma'$ is the unique element of Γ of order 2, and that is contained in Γ_1 . Thus $\gamma' \in \Gamma_1$ implies $\gamma \in \Gamma_1$. It follows that Γ/Γ_1 is isomorphic to a group of outer automorphisms of $SL(2,5)$. The group of outer automorphisms of $SL(2,5)$ has order 2, so Γ_1 has index 1 or 2 in Γ .

Now assume $\Gamma \neq \Gamma_1$, and let $\sigma \in \Gamma$ such that $ad(\sigma)$ is the outer automorphism of $SL(2,5) = \Gamma_1$ which is conjugation by $\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$. σ cannot have order 2 but $\sigma^2 = -I \in SL(2,5)$, being central in Γ . In $SL(2,5)$ we have

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As $ad(\sigma)\alpha = \beta^3$ and $\gamma\alpha\gamma^{-1} = \beta^{-1}$, β is conjugate in Γ to $\beta^{-3} = \beta^2$. As Γ is CLIFFORD, it follows that $\beta = I$ or β has order 3. This is a contradiction. Q.E.D.

Lemma 6. *Let Γ be a non-solvable CLIFFORD group. Then Γ is a binary icosahedral group \mathcal{I}^* .*

Proof. Lemmas 4 and 5 and the result mentioned of SUZUKI [1, Theorem E] show $\Gamma \cong SL(2,5)$. But $SL(2,5) \cong \mathcal{I}^*$. Q.E.D.

Theorem 1 is an immediate consequence of Lemmas 3 and 6.

III. Representations of CLIFFORD groups

Given an abstract CLIFFORD group Γ , we will find the faithful orthogonal representations $\varphi: \Gamma \rightarrow O(n+1)$ such that $\varphi(\Gamma)$ is a group of CLIFFORD translations of S^n . This will provide proofs of Theorems 2 and 3.

Lemma 7. *Let γ generate a cyclic group Γ of finite order q and let $\psi: \Gamma \rightarrow O(n+1)$ be a faithful orthogonal representation such that $\psi(\Gamma)$ is a group of CLIFFORD translations of S^n . If $q \leq 2$, $\psi(\Gamma) = \{I\}$ or $\{\pm I\}$. If*

$q > 2$, then $n + 1 = 2s$ and ψ is $O(n + 1)$ -equivalent to a sum of s copies of one of the representations given by

$$\sigma_t(\gamma) = R(t/q) = \begin{pmatrix} \cos(2\pi t/q) \sin(2\pi t/q) \\ -\sin(2\pi t/q) \cos(2\pi t/q) \end{pmatrix}, \quad t \text{ prime to } q.$$

Conversely, $\{I\}$, $\{\pm I\}$ and $O(2s)$ -conjugates of images of sums of s copies of a σ_t are groups of CLIFFORD translations.

Proof. The statement for $q \leq 2$ is clear; assume $q > 2$. As $\psi(\gamma)$ is a CLIFFORD translation of order q , it has $(n + 1 = 2s)$ s eigenvalues $\exp(2\pi it/q)$ and s eigenvalues $\exp(-2\pi it/q)$, where t is prime to q .

Thus $\psi(\gamma)$ is $O(n + 1)$ -conjugate to $\begin{pmatrix} R(t/q) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & R(t/q) \end{pmatrix}$, so ψ is $O(n + 1)$ -equivalent to $\sigma_t \oplus \dots \oplus \sigma_t$. The rest is clear. Q.E.D.

Lemma 8. *An irreducible CLIFFORD representation φ of a non-cyclic group Γ has degree 2.*

Proof. Γ is binary polyhedral. Suppose first that $\Gamma = \mathcal{D}_m^*$. $m > 1$ as \mathcal{D}_1^* is cyclic. \mathcal{D}_m^* has $m + 3$ conjugacy classes of elements, hence $m + 3$ inequivalent irreducible unitary representations, say of degrees d_i . The commutator subgroup has index 4 so we may assume $d_1 = d_2 = d_3 = d_4 = 1$, and the other $d_i > 1$. $\sum d_i^2 = 4m$ as \mathcal{D}_m^* has order $4m$, so each d_i is 1 or 2. φ has even degree as Γ is non-cyclic, so the degree of φ is 2.

Now suppose $\Gamma = \mathcal{T}^*$ binary tetrahedral group. As above, we see that the degrees of the irreducible representations are 1, 2 and 3. As φ has even degree, it has degree 2.

Suppose that $\Gamma = \mathcal{O}^*$. \mathcal{O}^* has a subgroup \mathcal{T}^* of index 2 such that φ is irreducible if and only if its restriction to \mathcal{T}^* is irreducible. Hence φ has degree 2.

Finally, suppose that $\Gamma = \mathcal{J}^*$. \mathcal{J}^* has 9 conjugacy classes, order 120, and presentation: $\alpha^{10} = 1$, $\alpha^5 = \gamma^3$, $\gamma\alpha\gamma^{-1} = \alpha^{-1}\gamma$. As φ has even degree $q = 2r$, $\varphi(\alpha)$ has r eigenvalues $\exp(2\pi i v/10)$ and r eigenvalues $\exp(-2\pi i v/10)$, for some integer v prime to 10. Thus the character χ_φ of φ is determined on 6 conjugacy classes by r and v : $\chi_\varphi(1) = 2r$, $\chi_\varphi(\alpha) = 2r \cos(\pi v/5)$, $\chi_\varphi(\alpha^2) = 2r \cos(2\pi v/5)$, $\chi_\varphi(\alpha^3) = 2r \cos(3\pi v/5)$, $\chi_\varphi(\alpha^4) = 2r \cos(4\pi v/5)$ and $\chi_\varphi(\alpha^5) = -2r$.

Let b be an eigenvalue of $\varphi(\gamma)$. As $\varphi(\gamma)^3 = \varphi(\alpha)^5 = -I$, b is a cube root of -1 . $\varphi(\gamma) \neq I$ so $b = \exp(2\pi i/6)$ or $b = \exp(-2\pi i/6)$. Thus $\chi_\varphi(\gamma) = r(b + \bar{b}) = 2r \cdot \cos(\pi/3) = r$ and $\chi_\varphi(\gamma^2) = r(b^2 + \bar{b}^2) = 2r \cdot \cos(2\pi/3)$

$= -r$. Finally χ_φ is zero on the conjugacy class consisting of elements of order 4, so χ_φ is determined on all 9 conjugacy classes—hence is completely determined—by r and v . We notice that χ_φ is precisely r times the character of one of the representations $\mathcal{I}^* \subset \text{Spin}(3) = \text{SU}(2) \subset \text{U}(2)$, so the irreducibility of φ implies $r = 1$. Q.E.D.

We remark that we have just seen: *If $\varphi: \mathcal{I}^* \rightarrow \text{U}(q)$ is an irreducible CLIFFORD representation, then $q = 2$ and φ is equivalent to one of the representations $\mathcal{I}^* \subset \text{Spin}(3) = \text{SU}(2) \subset \text{U}(2)$.* In fact we have

Lemma 9. *Let $\varphi: \Gamma \rightarrow \text{U}(q)$ be an irreducible CLIFFORD representation of a noncyclic group. Then $q = 2$, Γ is binary polyhedral, and φ is equivalent to one of the representations $\Gamma \subset \text{Spin}(3) = \text{SU}(2) \subset \text{U}(2)$.*

Proof. We need only check the equivalence class of φ for $\Gamma = \mathcal{D}_m^*(m > 1)$, \mathcal{I}^* and \mathcal{O}^* . As with \mathcal{I}^* , we calculate the character χ_φ and see that it is the same as the character of one of the representations $\Gamma \subset \text{Spin}(3) = \text{SU}(2) \subset \text{U}(2)$. Q.E.D.

Proof of Theorem 3. Given a finite group Γ of CLIFFORD translations of $S^n \subset \mathbb{R}^{n+1}$, we will show the centralizer \mathbf{G} of Γ in $O(n+1)$ to be transitive on S^n . This is obvious if Γ is cyclic of order 1 or 2, so we first suppose Γ cyclic of order q ($q > 2$). Let $2s = n+1$, as $n+1$ is even; let $\Gamma' \subset \text{U}(s)$ be the cyclic group generated by $\exp(2\pi i 1/q)$. Γ' is central in $\text{U}(s)$ so its centralizer in $\text{U}(s)$ is transitive on the unit sphere in complex euclidean space \mathbb{C}^s . By Lemma 7 we can assume that Γ' goes onto Γ , and its centralizer $\text{U}(s)$ into \mathbf{G} , under the inclusion $\text{U}(s) \subset O(n+1)$ induced by an isometry of \mathbb{C}^s onto \mathbb{R}^{n+1} which sends the unit sphere of \mathbb{C}^s onto S^n . Hence \mathbf{G} is transitive on S^n .

Now suppose Γ noncyclic. Γ is isomorphic to a binary polyhedral group \mathcal{P}^* . Let \mathbf{K} be the algebra of quaternions and let \mathbf{K}' be the multiplicative group of unit quaternions. Under the inclusion and identification $\mathcal{P}^* \subset \text{Spin}(3) = \mathbf{K}'$, we'll view \mathcal{P}^* as a subgroup of \mathbf{K}' . Let $\mathbf{K}^s (4s = n+1)$ be a *left* quaternionic euclidean space, so that \mathbf{K} (hence \mathbf{K}' , hence \mathcal{P}^*) acts on \mathbf{K}^s by *left* scalar multiplication and the symplectic group $\text{Sp}(s)$ acts on the right. The action of $\text{Sp}(s)$ commutes with that of \mathcal{P}^* , and $\text{Sp}(s)$ is transitive on the unit sphere of \mathbf{K}^s . By Lemma 9 we can assume that \mathcal{P}^* goes onto Γ , and $\text{Sp}(s)$ goes into \mathbf{G} , under the inclusions $\mathbf{K}' \subset O(n+1)$ and $\text{Sp}(s) \subset O(n+1)$ induced by an isometry of \mathbf{K}^s onto \mathbb{R}^{n+1} which sends the unit sphere of \mathbf{K}^s onto S^n . Hence \mathbf{G} is transitive on S^n . Q.E.D.

Proof of Theorem 2. By Lemmas 7 and 9, all that remains to be shown is that the images of the representations of Theorem 2 are actually groups of

CLIFFORD translations. Let $\Gamma \subset O(n+1)$ be the image of one of those representations. In the proof of Theorem 3, we saw that the centralizer G of Γ in $O(n+1)$ is transitive on S^n . Now let $\gamma \in \Gamma$, let $x, y \in S^n$, and let δ be the distance function on S^n determined by its RIEMANNIAN metric. There is an element $g \in G$ with $g(x) = y$. Hence

$$\delta(x, \gamma x) = \delta(gx, g\gamma x) = \delta(y, \gamma g x) = \delta(y, \gamma y)$$

so γ is a CLIFFORD translation of S^n . Q.E.D.

IV. Homogeneous space-forms

We will prove Theorem 4. Theorem 2 establishes the equivalence of (1) and (2), Theorem 3 shows that (1) implies (3), and the proof of Theorem 3 shows that (3) implies (1). It is obvious that (3) implies (4): the centralizer of Γ induces a transitive group of isometries of S^n/Γ . Finally, (4) implies (3) is known [3, Théorème 1]. Q.E.D.

We remark that Theorems 3 and 4 provide a proof of a result [3, Théorème 6] previously announced without proof in the *Comptes rendus*, and that Theorems 1 and 4 provide an alternative proof of the classification [3, Théorème 5] of the RIEMANNIAN homogeneous spherical space-forms.

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