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On the Geometric Definition for Quasiconformal Mappings

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Introduction

1. Geometric definition. Let Q denote a (topological) quadrilateral, that is a JORDAN domain in the complex plane with four distinguished boundary points which divide the boundary curve into four arcs, the sides of Q. Q can be mapped conformally onto a rectangle Q' with sides a and b so that the vertices correspond. Consider the pair of opposite sides of Q which correspond to the sides of length a. The modulus of Q with respect to this pair of sides is defined as a/b; the modulus of Q with respect to the other pair is then b/a. We denote either of these moduli by $\operatorname{mod} Q$.

An inequality due to RENGEL [14] allows us to obtain convenient estimates for this modulus. Let $A(Q)$ denote the area of Q, let $L_1(Q)$ denote the distance in Q between the pair of sides with respect to which the modulus is taken, and let $L_2(Q)$ denote the corresponding distance between the other pair of sides. RENGEL's inequality states that

$$\frac{L_2(Q)^2}{A(Q)} \leqq \operatorname{mod} Q \leqq \frac{A(Q)}{L_1(Q)^2}\,. \tag{1}$$

The geometric definition for K-quasiconformal mappings, due to PFLUGER [11] and AHLFORS [1], is as follows. A sense-preserving homeomorphism $w(z)$ of a domain G onto a domain G' is said to be K-quasiconformal if

$$\operatorname{mod} Q' \leqq K \operatorname{mod} Q \tag{2}$$

for each quadrilateral Q whose closure $\overline{Q}$ lies in G, where the moduli are taken with respect to corresponding pairs of sides[1]). A quasiconformal mapping is defined as one which is K-quasiconformal for some K.

It is natural to ask what happens if we replace the quadrilaterals Q in the above definition by a family of other figures possessing conformal moduli. For example, we might consider ring domains or alternatively some subclass of quadrilaterals such as rectangles, oriented rectangles, or squares[2]). We devote the first part of this paper to such questions.

In the second part we present a proof for the measurability of quasicon-

[1]) G and G' will always denote finite plane domains, K a number satisfying $1 \leqq K < \infty$ and Q' the image of Q. Given any set $E \subset G$ we let E' denote its image under $w(z)$.

[2]) A rectangle is oriented if a pair of its sides are parallel to some fixed line.

formal mappings based directly on the geometric definition given above. This argument is very simple and illustrates the advantages of using this definition when proving some of the fundamental properties of quasiconformal mappings.

1. Modifications of the geometric definition

2. Analytic definition. Let us first recall the analytic definition for quasiconformal mappings. We say that a complex-valued function $w(z)$ is ACL (absolutely continuous on lines) in a domain G if for each oriented rectangle $Q, \overline{Q} \subset G$, $w(z)$ is absolutely continuous on almost all lines in Q which are parallel to the sides of Q.

By the analytic definition a sense-preserving homeomorphism $w(z)$ of a domain G is K-quasiconformal if $w(z)$ is ACL in G and

$$\max_{\varphi} | D_\varphi w(z) |^2 \leqq K J(z) \tag{3}$$

a.e. in G. Here $D_\varphi w(z)$ denotes the directional derivative of $w(z)$,

$$D_\varphi w(z) = w_x(z) \cos \varphi + w_y(z) \sin \varphi ,$$

and $J(z)$ the JACOBIan of the mapping.

The equivalence of the geometric and analytic definitions has been proved by MORI [8], BERS [2], PFLUGER [12] and in the above form by GEHRING and LEHTO [6].

3. Rectangles. We begin by proving that it suffices to consider only rectangles in the geometric definition.

Theorem 1. *Let $w(z)$ be a sense-preserving homeomorphism of a domain G and let*

$$\operatorname{mod} Q' \leqq K \operatorname{mod} Q \tag{4}$$

for each rectangle $Q, \overline{Q} \subset G$. Then $w(z)$ is a K-quasiconformal mapping.

In the proof we will make use of the following three lemmas.

Lemma 1. *Let $w(z)$ be a homeomorphism of a domain G and let* (4) *hold for each oriented rectangle $Q, \overline{Q} \subset G$. Then $w(z)$ is* ACL *in G.*

Lemma 2. *Let $w(z)$ be a homeomorphism of a domain G and let $w(z)$ possess finite partial derivatives a.e. in G. Then $w(z)$ is differentiable a.e. in G.*

Lemma 3. *Let $w(z)$ be a sense-preserving homeomorphism of a domain G and let* (4) *hold for each square $Q, \overline{Q} \subset G$. Then* (3) *holds at each point where $w(z)$ is differentiable.*

For Lemma 1 see either PFLUGER [12] or STREBEL [16], and for Lemma 2 see GEHRING and LEHTO [6]. We give here a complete proof for Lemma 3 because we will want to refer to the argument later on.

Proof for Lemma 3. Let z_0 be a point of differentiability for $w(z)$ and assume, for convenience of notation, that $z_0 = 0$, $w(z_0) = 0$. We must show that (3) holds for $z = 0$. There are two cases to consider depending on whether or not the JACOBIan vanishes at $z = 0$.

First assume that $J(0) = 0$. If (3) does not hold, there exists an angle α for which

$$D_\alpha w(0) \neq 0 , \quad D_{\alpha+\pi/2} w(0) = 0 ,$$

and, by first performing a preliminary rotation, we can assume that

$$w_x(0) = a > 0 , \quad w_y(0) = 0 .$$

Next for $h > 0$ let Q denote the square with vertices at $0, h, ih, h + ih$. Then for $0 < \varepsilon < a/2$ we can find a $\delta > 0$ so that

$$|\, w(z) - ax \,| \leqq \varepsilon h \tag{5}$$

for $z \in \bar{Q}$, $0 < h < \delta$. We now use the first part of (1) to estimate the modulus of Q' taken with respect to the "horizontal" sides. By (5)

$$A(Q') \leqq 2\varepsilon h(ah + 2\varepsilon h) , \quad L_2(Q') \geqq ah - 2\varepsilon h ,$$

and hence

$$\operatorname{mod} Q' \geqq \frac{(a - 2\varepsilon)^2}{2\varepsilon(a + 2\varepsilon)}$$

for $0 < h < \delta$. Thus $\lim_{h\to 0} \operatorname{mod} Q' = \infty$. On the other hand, by hypothesis $\operatorname{mod} Q' \leqq K \operatorname{mod} Q = K$ for all $h > 0$. We obtain a contradiction and hence (3) holds for $z = 0$.

Now suppose that $J(0) > 0$. By performing a preliminary similarity mapping, we may assume that $w_x(0) = D \geqq 1$, $w_y(0) = i$. Next let Q be an arbitrary square and let Q'_n be the image of Q under the mapping $w_n(z) = nw(z/n)$. The Q'_n are defined for sufficiently large n and the sides of Q'_n converge uniformly[3]) to the sides of Q'_0, the image of Q under the affine mapping $w_0(z) = Dx + iy$. Hence $\operatorname{mod} Q'_0 = \lim_{n\to\infty} \operatorname{mod} Q'_n$. (See, for example, PFLUGER [12].) Since the mappings $w_n(z)$ all satisfy the condition (4), we have $\operatorname{mod} Q'_n \leqq K \operatorname{mod} Q$ for sufficiently large n and we conclude that $\operatorname{mod} Q'_0 \leqq K \operatorname{mod} Q = K$ for all squares Q. Thus to establish (3) it is suffi-

[3]) A sequence of sets $\{E_n\}$ is said to *converge uniformly* to a set E if for each $\varepsilon > 0$ there exists an N such that $n > N$ implies each point of E_n lies within distance ε of E and each point of E lies within distance ε of E_n.

cient to consider the special case where $w(z)$ is itself the affine mapping $w(z) = Dx + iy$ and show that $D \leqq K$. This follows immediately from applying (4) to any square with sides parallel to the coordinate axes and the proof for Lemma 3 is complete.

We turn now to the proof for Theorem 1.

Proof for Theorem 1. From Lemma 1 it follows that $w(z)$ is ACL in G. Thus $w(z)$ possesses finite partial derivatives a. e. in G and hence is differentiable a. e. by Lemma 2. Next applying Lemma 3 we see that $w(z)$ satisfies the dilatation condition (3) a. e. in G. Hence $w(z)$ is K-quasiconformal by the analytic definition as desired.

4. Oriented rectangles. If the rectangles in Theorem 1 are supposed to be oriented, the theorem does not hold and must be replaced by the following result.

Theorem 2. *Let* $w(z)$ *be a sense-preserving homeomorphism of a domain* G *and let*

$$\operatorname{mod} Q' \leqq K \operatorname{mod} Q \tag{6}$$

for each oriented rectangle Q, $\overline{Q} \subset G$. *Then* $w(z)$ *is a* $(K + \sqrt{K^2 - 1})$-*quasiconformal mapping. The bound is best possible.*

Proof. We begin with the special case where $w(z)$ is the affine mapping

$$w(z) = Dx + iy\,, \quad D \geqq 1\,, \tag{7}$$

and show that the modulus condition (6) implies

$$D \leqq K + \sqrt{K^2 - 1}\,. \tag{8}$$

For $h > 0$ let Q be the rectangle with vertices at the points $0, e^{i\varphi}, ihe^{i\varphi}, e^{i\varphi} + ihe^{i\varphi}$, and fix $|\varphi| \leqq \pi/4$ so that Q is one of the oriented rectangles for which (6) is assumed to hold. The modulus of Q with respect to the sides of length 1 is $1/h$. Estimating the corresponding modulus for Q' we have

$$A(Q') = Dh, \;\; L_2(Q') \geqq \sqrt{D^2 \cos^2\varphi + \sin^2\varphi} - \varepsilon,$$

where $\varepsilon = \varepsilon(h) \to 0$ as $h \to 0$, whence by (1)

$$\operatorname{mod} Q' \geqq \frac{D^2 \cos^2\varphi + \sin^2\varphi - \varepsilon}{D} \operatorname{mod} Q\,.$$

Applying (6) and then letting $h \to 0$ yields $D^2 \cos^2\varphi + \sin^2\varphi \leqq KD$. Since $|\varphi| \leqq \pi/4$, this means $D^2 + 1 \leqq 2KD$ which in turn implies (8) as desired.

Now let $w(z)$ be an arbitrary sense-preserving homeomorphism of G which satisfies (6) for oriented rectangles $Q, \bar{Q} \subset G$. By Lemmas 1 and 2 $w(z)$ is ACL and a.e. differentiable. Hence to complete the proof it is sufficient to show that

$$\max_{\varphi} |D_{\varphi} w(z_0)|^2 \leqq (K + \sqrt{K^2 - 1}) J(z_0) \tag{9}$$

at each point of differentiability z_0.

Fix such a point z_0 and assume, for convenience of notation, that $z_0 = 0$, $w(z_0) = 0$. As in the proof of Lemma 3, we consider two cases depending on whether or not the Jacobian vanishes at z_0.

Suppose first that $J(0) = 0$ and that (9) does not hold. Then as in Lemma 3 we can assume that

$$w_x(0) = a > 0, \quad w_y(0) = 0.$$

For $h > 0$ let Q be the rectangle with vertices at $0, 2he^{i\varphi}, ihe^{i\varphi}, 2he^{i\varphi} + ihe^{i\varphi}$, and fix $|\varphi| \leqq \pi/4$ so that Q is one of the rectangles for which (6) holds. Then for $0 < \varepsilon < a/2\sqrt{2}$ we can find a $\delta > 0$ such that $|w(z) - ax| \leqq \varepsilon h$ for $z \in \bar{Q}$, $0 < h < \delta$. The modulus of Q with respect to the longer sides is 2 and estimating the corresponding modulus for Q' by (1) yields

$$\operatorname{mod} Q' \geqq \frac{(a/\sqrt{2} - 2\varepsilon)^2}{2\varepsilon(\sqrt{5}a + 2\varepsilon)}$$

for $0 < h < \delta$. Hence $\lim_{h \to 0} \operatorname{mod} Q' = \infty$.

But this contradicts the inequality $\operatorname{mod} Q' \leqq K \operatorname{mod} Q = 2K$, and hence (9) holds for $z_0 = 0$. Now suppose that $J(0) > 0$ and assume, as we may, that $w_x(0) = D \geqq 1$, $w_y(0) = i$. Then arguing as in the proof of Lemma 3 we conclude that $\operatorname{mod} Q_0' \leqq K \operatorname{mod} Q$ for all oriented rectangles Q, where Q_0' is the image of Q under the affine mapping $w_0(z) = Dx + iy$. Hence (8) holds, we again obtain (9) for $z_0 = 0$, and $w(z)$ is $(K + \sqrt{K^2 - 1})$-quasiconformal.

To show that the bound $K + \sqrt{K^2 - 1}$ is sharp we consider the mapping $w(z) = Dx + iy$, $D = K + \sqrt{K^2 - 1}$, and prove that (6) holds for all rectangles Q whose sides meet the coordinate axes at an angle $\pi/4$. For this it suffices to consider, for $h > 0$, a rectangle Q with vertices at 0, $1 + i$, $-h + ih$, $1 - h + i + ih$, where the modulus is taken with respect to the sides of length $\sqrt{2}$. Then for Q' we have $A(Q') = 2Dh$, $L_1(Q') \geqq 2Dh/\sqrt{D^2 + 1}$ and (1) yields

$$\operatorname{mod} Q' \leqq \frac{D^2 + 1}{2D} \frac{1}{h} = K \operatorname{mod} Q$$

as desired.

5. Squares. In view of these two results it is natural to ask what can be said about the quasiconformality of a homeomorphism which satisfies the modulus condition (2) for all squares Q. From Lemma 3 it follows that such a mapping will satisfy the dilatation condition $\max_{\varphi} | D_{\varphi} w(z) |^2 \leqq K J(z)$ at every point of differentiability. Hence, in view of Lemma 2, the problem of deciding whether or not such a mapping is K-quasiconformal is reduced to the following open question:

Let $w(z)$ *be a homeomorphism of a domain* G *and let* (2) *hold for all squares* $Q, \overline{Q} \subset G$. *Is* $w(z)$ ACL *in* G?

Unfortunately the methods used in the proof of Lemma 1 do not help to settle this question, for both PFLUGER and STREBEL require that the modulus condition hold for long thin rectangles. However, it is easy to see that the answer must be in the negative if we restrict our attention to *oriented* squares Q. For let $f(x)$ be a continuous singular function which is strictly increasing for all x and consider the mapping $w(z) = f(x) + iy$. If Q is any square whose sides meet the coordinate axes at an angle of $\pi/4$, then Q' will be symmetric in its horizontal diameter and hence $\operatorname{mod} Q' = \operatorname{mod} Q = 1$. On the other hand it is clear that the mapping is not ACL.

6. Rings. A doubly-connected domain is called a ring. An annulus is a ring whose boundary components are two concentric circles, possibly degenerate. It is well known that each ring R can be mapped conformally onto an annulus $0 \leqq a < |z| < b \leqq \infty$, and the conformal invariant $\operatorname{mod} R = \log \frac{b}{a}$ is called the modulus of R.

By means of extremal lengths we obtain inequalities for the modulus of a ring which correspond to RENGEL's inequality (1) for quadrilaterals. Let $\varrho(z)$ be any continuous non-negative function in R and let

$$A(R, \varrho) = \iint_R \varrho^2 d\sigma, \quad L_1(R, \varrho) = \inf_{\gamma_1} \int_{\gamma_1} \varrho ds, \quad L_2(R, \varrho) = \inf_{\gamma_2} \int_{\gamma_2} \varrho ds, \tag{10}$$

where γ_1 is any curve in R which separates the boundary components of R and where γ_2 is any curve in R which joins these components. Then

$$2\pi \frac{L_2(R, \varrho)^2}{A(R, \varrho)} \leqq \operatorname{mod} R \leqq 2\pi \frac{A(R, \varrho)}{L_1(R, \varrho)^2} \tag{11}$$

for each such function $\varrho(z)$. (See, for example, JENKINS [7], pp. 17–19.) When $\varrho(z) \equiv 1$ we denote the quantities in (10) by $A(R)$, $L_1(R)$ and $L_2(R)$ respectively.

Now it is well known that under a K-quasiconformal mapping $w(z)$ the image of a ring R is a ring R' for which $(1/K) \operatorname{mod} R \leqq \operatorname{mod} R' \leqq K \operatorname{mod} R$. We show that the converse is true by establishing the following result.

Theorem 3. *Let* $w(z)$ *be a sense-preserving homeomorphism of a domain* G *and let*

$$\operatorname{mod} R' \leqq K \operatorname{mod} R \tag{12}$$

for all rings R, $\bar{R} \subset G$. *Then* $w(z)$ *is a* K*-quasiconformal mapping.*

We consider first the following preliminary result.

Lemma 4. *Let* $w(z)$ *be a sense-preserving homeomorphism of a domain* G *and let*

$$\operatorname{mod} R' \geqq \frac{1}{K} \operatorname{mod} R \tag{13}$$

for all annuli R, $\bar{R} \subset G$. *Then* $w(z)$ *is a quasiconformal mapping.*

Proof. It is sufficient to show that the inverse mapping $z(w)$ is quasiconformal. For this fix $w \in G'$ and, for sufficiently small $r > 0$, let

$$M(w, r) = \max_{|w-w'|=r} |z(w) - z(w')|,$$

$$m(w, r) = \min_{|w-w'|=r} |z(w) - z(w')|.$$

Then from an argument due to Mori (Lemma 4 of [8]) it follows that

$$H(w) = \limsup_{r \to 0} \frac{M(w, r)}{m(w, r)} \leqq e^{\pi K}$$

at each point $w \in G'$. This, in turn, implies that $z(w)$ is $e^{\pi K}$-quasiconformal as desired. (See Gehring [4].)

We turn now to the proof for Theorem 3.

Proof for Theorem 3. We consider first the special case where $w(z)$ is the affine mapping

$$w(z) = Dx + iy, \quad D \geqq 1$$

and show that the modulus condition (12) implies that $D \leqq K$.

For each $h > 0$, $d > 0$ let R denote a ring bounded by two concentric rectangles with horizontal sides of h and $h + 2d$ and with vertical sides of 1 and $1 + 2Dd$, respectively. Then

$$A(R) = 2d(1 + Dh + 2Dd), \qquad L_1(R) = 2(1 + h) > 2,$$

$$A(R') = 2Dd(1 + Dh + 2Dd), \qquad L_2(R') = Dd,$$

and hence (11) yields

$$\operatorname{mod} R \leqq \pi d(1 + Dh + 2Dd), \qquad \operatorname{mod} R' \geqq \frac{\pi Dd}{1 + Dh + 2Dd}.$$

Condition (12) implies that $D \leqq (1 + Dh + 2Dd)^2 K$, and letting h, $d \to 0$ yields the desired result $D \leqq K$.

Now consider the general case where $w(z)$ is an arbitrary homeomorphism satisfying (12). Since the inverse mapping satisfies (13) we conclude from Lemma 4 that $w(z)$ is quasiconformal. Hence, by virtue of the analytic definition, it suffices to prove that

$$\max_{\varphi} | D_\varphi w(z) |^2 \leqq K J(z) \tag{14}$$

a.e. in G.

Pick a point z_0 where $w(z)$ is differentiable and where $J(z_0) > 0$. Since $w(z)$ is quasiconformal, $J(z) > 0$ a.e. in G[4]) and it will be sufficient to establish (14) for such points z_0. We can thus assume without loss of generality that $z_0 = 0$, $w(z_0) = 0$ and that $w_x(0) = D \geqq 1$, $w_y(0) = i$. Now let R be the ring bounded by concentric rectangles described above and consider its images R'_n under $w_n(z) = n w(z/n)$. The R'_n are defined for large n and their boundary components converge uniformly to those of R'_0, the image of R under the affine mapping $w_0(z) = Dx + iy$. Arguing directly it is easy to verify that

$$\operatorname{mod} R'_0 = \lim_{n \to \infty} \operatorname{mod} R'_n \text{ }^{5)} \tag{15}$$

and, since the R'_n satisfy (12), we conclude that $\operatorname{mod} R'_0 \leqq K \operatorname{mod} R$. By the first part of the proof, this means $D \leqq K$ and hence (14) holds for $z_0 = 0$ as desired.

7. Annuli. If $w(z)$ is a homeomorphism which satisfies the inequality

$$\operatorname{mod} R' \geqq \frac{1}{K} \operatorname{mod} R \tag{16}$$

for all annuli R, then Lemma 4 tells us that $w(z)$ is $e^{\pi K}$-quasiconformal. We now replace $e^{\pi K}$ by the best possible bound.

Theorem 4. *Let $w(z)$ be a sense-preserving homeomorphism of a domain G and let (16) hold for all annuli R, $\overline{R} \subset G$. Then $w(z)$ is a $(K + \sqrt{K^2 - 1})$-quasiconformal mapping. The bound is best possible.*

The proof requires the following estimate for the modulus of an elliptical ring.

Lemma 5. *Let $D \geqq 1$, $h > 1$ and let R be the ring bounded by the two ellipses*

$$\frac{x^2}{D^2} + y^2 = 1 , \quad \frac{x^2}{D^2} + y^2 = h^2 .$$

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[4]) This is an immediate consequence of Theorem 6. Alternatively see, for example, Lemma 6 of [4].

[5]) See also the following paper [5].

Then

$$\operatorname{mod} R \geqq \frac{2D}{D^2+1} \log h \tag{17}$$

and

$$\lim_{h \to 1} \frac{\operatorname{mod} R}{\log h} = \frac{2D}{D^2+1}. \tag{18}$$

Proof. We obtain these estimates by choosing a special function $\varrho(z)$ and applying the inequality (11). For this let

$$u(z) = \tfrac{1}{2}\log\left(\frac{x^2}{D^2} + y^2\right)$$

and set

$$\varrho(z) = |\nabla u(z)| = \frac{\left(\frac{x^2}{D^4} + y^2\right)^{\frac{1}{2}}}{\frac{x^2}{D^2} + y^2}.$$

Making the change of variables $x = Dr\cos\varphi$, $y = r\sin\varphi$ we see that

$$A(R, \varrho) = \int_0^{2\pi}\int_1^h \left(\frac{1}{D}\cos^2\varphi + D\sin^2\varphi\right)\frac{dr}{r}\,d\varphi = \pi\frac{D^2+1}{D}\log h. \tag{19}$$

Next since $u(z)$ is 0 on the inner boundary component and equal to $\log h$ on the outer,

$$\int_{\gamma_2} \varrho\, ds = \int_{\gamma_2} |\nabla u|\, ds \geqq \log h$$

for each curve γ_2 in R which joins these components. Hence

$$L_2(R, \varrho) = \log h, \tag{20}$$

and combining (19), (20) and the first half of (11) yields

$$\operatorname{mod} R \geqq 2\pi \frac{L_2(R,\varrho)^2}{A(R,\varrho)} = \frac{2D}{D^2+1}\log h.$$

This is the inequality (17).

For (18) we must obtain an asymptotic estimate for $L_1(R, \varrho)$ as $h \to 1$. For this let Γ denote the inner boundary component. Setting $x = D\cos\varphi$, $y = \sin\varphi$ yields

$$\int_\Gamma \varrho\, ds = \int_0^{2\pi}\left(\frac{1}{D}\cos^2\varphi + D\sin^2\varphi\right)d\varphi = \pi\frac{D^2+1}{D}.$$

Next it is not difficult to show that for each $\varepsilon > 0$ there exists a $\delta > 1$ with the following property: If $1 < h < \delta$ and if γ_1 is a curve in R which separates the boundary components, then

$$\int_{\gamma_1} \varrho ds \geqq (1-\varepsilon) \int_{\Gamma} \varrho ds \, .$$

Hence,

$$L_1(R, \varrho) \geqq (1-\varepsilon)\pi \frac{D^2+1}{D} \tag{21}$$

for $1 < h < \delta$ and combining (19), (21) and the second half of (11) we have

$$\operatorname{mod} R \leqq 2\pi \frac{A(R, \varrho)}{L_1(R, \varrho)^2} \leqq (1-\varepsilon)^{-2} \frac{2D}{D^2+1} \log h$$

for $1 < h < \delta$. This together with (17) yields (18).

We turn now to the proof for Theorem 4.

Proof for Theorem 4. The mapping is already quasiconformal by Lemma 4. Hence the argument in the proof for Theorem 3 shows it is sufficient to establish the theorem for the special case where $w(z)$ is the affine mapping

$$w(z) = Dx + iy \, , \qquad D \geqq 1 \, .$$

Now let R be the annulus $1 < |z| < h$. Then its image under $w(z)$ is the ring described in Lemma 5, and (16) and (18) yield

$$\frac{2D}{D^2+1} = \lim_{h \to 1} \frac{\operatorname{mod} R'}{\log h} = \lim_{h \to 1} \frac{\operatorname{mod} R'}{\operatorname{mod} R} \geqq \frac{1}{K} \, .$$

Hence $D \leqq K + \sqrt{K^2-1}$ and $w(z)$ is $(K + \sqrt{K^2-1})$-quasiconformal.

To show that the bound is sharp, consider the above affine mapping with $D = K + \sqrt{K^2-1}$. We want to show that (16) holds for all annuli R. It is clear we need only consider annuli R of the form $1 < |z| < h$. Then R' is again the ring of Lemma 5 and (17) yields $\operatorname{mod} R' \geqq \frac{2D}{D^2+1} \log h = \frac{1}{K} \operatorname{mod} R$ as desired.

8. Remarks. Since the modulus of a quadrilateral taken with respect to one pair of sides is the inverse of that taken with respect to the other pair, Theorems 1 and 2 are valid if the modulus conditions are replaced by $\operatorname{mod} Q' \geqq (1/K) \operatorname{mod} Q$. We see also in Theorem 3 that the condition (12) can be replaced by $\operatorname{mod} R' \geqq (1/K) \operatorname{mod} R$ since the inverse mapping $z(w)$ will then satisfy (12).

It is therefore reasonable to ask if Theorem 4 is valid with (12) in place of (16). The answer here is no and it turns out that requiring a homeomorphism to satisfy (12) for annuli R says nothing about the quasiconformality of the mapping. We have, for example, the following result.

Theorem 5. *Let* $w(z)$ *be a continuously differentiable homeomorphism of a simply connected domain* G *and let the absolute value of the* JACOBIAN *be superharmonic in* G. *Then*

$$\operatorname{mod} R' \leqq \operatorname{mod} R \tag{22}$$

for all annuli R, $\bar{R} \subset G$.

Proof. The proof is based on the following important estimate, due to CARLEMAN, for the modulus of a ring.

Lemma 6. *Let* R *be a ring bounded by two disjoint rectifiable curves* B_0 *and* B_1. *Choose concentric circles* B_0^* *and* B_1^* *so that* B_0^* *and* B_1^* *bound the same areas as* B_0 *and* B_1, *respectively, and let* R^* *be the annulus bounded by* B_0^* *and* B_1^*. *Then*

$$\operatorname{mod} R \leqq \operatorname{mod} R^* . \tag{23}$$

For a direct proof of this lemma see CARLEMAN [3]. Lemma 6 can also be established by a symmetrization argument, using the fact that

$$\frac{2\pi}{\operatorname{mod} R} = \iint_R |\nabla u|^2 d\sigma ,$$

where $u(z)$ is harmonic in R with boundary values 0 and 1 on B_0 and B_1, respectively. (See, for example, PÓLYA and SZEGÖ [13].)

Now for Theorem 5 let R be the annulus $0 < a < |z - z_0| < b < \infty$ with $\bar{R} \subset G$. Since G is finite and simply connected, G contains the closed disk $|z - z_0| \leqq b$ and we can pick $0 < \alpha < \beta < \infty$ so that

$$\pi\alpha^2 = \iint_{|z-z_0|<a} |J(z)|\, d\sigma , \qquad \pi\beta^2 = \iint_{|z-z_0|<b} |J(z)|\, d\sigma .$$

Since $|J(z)|$ is superharmonic in G,

$$I(r) = \frac{1}{\pi r^2} \iint_{|z-z_0|<r} |J(z)|\, d\sigma$$

is non-increasing in $a \leqq r \leqq b$. Hence $\beta/\alpha \leqq b/a$ and with Lemma 6 we conclude that $\operatorname{mod} R' \leqq \log \beta/\alpha \leqq \log b/a = \operatorname{mod} R$ as desired.

From Theorem 5 it is clear we can conclude nothing about the quasiconformality of a sense-preserving homeomorphism which satisfies (12) for all annuli. For example, all affine mappings have constant JACOBIANS and hence satisfy this condition. Alternatively consider the mapping $w(z) = x^2 + iy$ of $x > 0$ onto $u > 0$. Here $J(z) = 2x$ is harmonic and (12) holds for all relevant annuli R. On the other hand the dilatation $D(z) = \max(2x, 1/2x)$ is unbounded in $x > 0$ and hence $w(z)$ is not quasiconformal.

2. An elementary proof for the measurability of quasiconformal mappings

9. A homeomorphism of a plane domain is said to be measurable if the image of every (plane) measurable set is itself measurable. It is well known that this is the same as asking that every closed set of measure zero map onto a set of measure zero.

We conclude this paper with an elementary proof of the following result.

Theorem 6. *A quasiconformal mapping is measurable.*

This theorem is usually proved with the aid of the analytic definition and a general form of GREEN's theorem due to MORREY [9]. (See, for example GEHRING [4].) Another argument, due to PESIN [10], uses both definitions to prove that the JACOBIan of a quasiconformal mapping is a. e. positive. This fact, taken in conjunction with a familiar distortion theorem (MORI [8], Lemma 4) and the DE LA VALLÉE POUSSIN Decomposition theorem (SAKS [15], p. 125), implies that the mapping is measurable.

We give here a direct proof for Theorem 6 which was suggested by PESIN's argument but which is based on the geometric definition and uses only the Density theorem and RENGEL's inequality.

Proof. Let $w(z)$ be a K-quasiconformal mapping of a domain G. In order to prove that the inverse mapping $z(w)$ is measurable, we assume the antithesis that there exists a closed set $F' \subset G'$ with $m(F') = 0$ whose image $F \subset G$ is of positive measure. (We let m denote plane measure.) Then F has a point of density and, for each $\varepsilon > 0$, we can find a square S, $\overline{S} \subset G$, such that

$$m(F \cap S) > (1 - \varepsilon^2) m(S) . \tag{24}$$

By conformal mapping, we may assume that the image S' of S is itself a rectangle, that S and S' have horizontal and vertical sides which correspond under the homeomorphism $w(z)$, that S has side 1 and that S' has height 1 and base M. We can further assume that $F \subset S$, in which case (24) becomes just

$$m(F) > 1 - \varepsilon^2 . \tag{25}$$

For each pair of positive integers i and j we divide S by horizontal and vertical lines into ij disjoint open rectangles Q with height $1/i$ and base $1/j$. We let F_1 denote the union of the rectangles Q for which $\overline{Q} \cap F$ is not empty and G_1 the union of the rectangles which do not have this property. Then $\overline{F}_1 \supset F$ and, by (25),

$$m(G_1) < \varepsilon^2 . \tag{26}$$

Now let B' be any open set which contains F'. Since F' is closed, it follows from the uniform continuity of $w(z)$ that F'_1 will lie in B' for sufficiently large i and j. Hence we see that

$$\lim_{i,j\to\infty} m(F'_1) = 0 . \tag{27}$$

Next for each rectangle Q let b denote the distance between the images of the endpoints of the base of Q and l the distance between the images of the vertical sides. Again by continuity given any j we can pick an i so that

$$b \leqq l + \frac{M}{2j} \tag{28}$$

for all rectangles Q.

Fix i and j so that (28) holds and let $i_1 = [(1-\varepsilon)i]$, the integral part of $(1-\varepsilon)i$. By (26) there exist i_1 rows in S, each of which contains no more than $[\varepsilon j]$ rectangles Q of G_1. Consider such a row. Its image is a strip which connects the vertical sides of S' and hence from (28)

$$M \leqq \sum_Q b \leqq \frac{M}{2} + \sum_Q l ,$$

where the sum is taken over the j rectangles in the row. Appealing to the SCHWARZ inequality we have

$$\begin{aligned} \frac{M^2}{8} &\leqq \Big(\sum_{Q\subset F_1} l\Big)^2 + \Big(\sum_{Q\subset G_1} l\Big)^2 \\ &\leqq \Big(\sum_{Q\subset F_1} 1\Big)\Big(\sum_{Q\subset F_1} l^2\Big) + \Big(\sum_{Q\subset G_1} 1\Big)\Big(\sum_{Q\subset G_1} l^2\Big) \\ &\leqq j\Big(\sum_{Q\subset F_1} l^2\Big) + \varepsilon j\Big(\sum_{Q\subset G_1} l^2\Big) . \end{aligned} \tag{29}$$

Here the first sums are taken over the rectangles which lie in F_1, the second sums over those which lie in G_1.

The modulus of each Q with respect to the horizontal sides is i/j. Estimating the corresponding modulus for Q' yields

$$\operatorname{mod} Q' \geqq \frac{L_2(Q')^2}{A(Q')} \geqq \frac{l^2}{m(Q')} ,$$

and, since the mapping is K-quasiconformal, we obtain $l^2 \leqq K\frac{i}{j}m(Q')$. This, together with (29) yields

$$\frac{M^2}{8} \leqq Ki\Big(\sum_{Q\subset F_1} m(Q') + \varepsilon \sum_{Q\subset G_1} m(Q')\Big) \tag{30}$$

and, summing over the i_1 rows for which (30) holds, we have

$$\frac{M^2}{8} \leqq K \frac{i}{i_1} (m(F_1') + \varepsilon M) .$$

Finally, let $i, j \to \infty$ so that (28) holds. Then $i_1/i \to 1 - \varepsilon$ and with (27) we conclude that

$$M \leqq \frac{8 \varepsilon K}{1 - \varepsilon} .$$

But $1/M$ is the conformal modulus for the rectangle S' taken with respect to the vertical sides. Hence $1/M \leqq K$,

$$K^2 \geqq \frac{1 - \varepsilon}{8 \varepsilon} ,$$

and we obtain the desired contradiction for $0 < \varepsilon < (8K^2 + 1)^{-1}$.

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