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# KOEBE Arcs and FATOU Points of Normal Functions

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Let $C$ be the unit circle and $D$ be the open unit disk in the complex $z$-plane, and $C_w$, $D_w$ be the corresponding entities in the complex $w$-plane. The closure of a point set $S$ will be denoted by $\overline{S}$, and the LEBESGUE measure of a measurable set $E$ by $m(E)$.

We begin by setting down some definitions.

**Definition 1.** Let $A$ be an open arc of $C$, possibly $C$ itself. A KOEBE *sequence of arcs* (*relative to* $A$) is a sequence of JORDAN arcs $\{J_n\}$ in $D$ such that (a) for some sequence $\{\varepsilon_n\}$ satisfying the conditions $0 < \varepsilon_n < 1$ $(n = 1, 2, 3, \ldots)$ and $\varepsilon_n \to 0$ as $n \to \infty$, $J_n$ lies in the $\varepsilon_n$-neighborhood of $A$ $(n = 1, 2, 3, \ldots)$, and (b) every open sector $\Delta$ of $D$ subtending an arc of $C$ that lies strictly interior to $A$ has the property that, for all values of $n$ except at most a finite number, the arc $J_n$ contains at least one JORDAN subarc lying wholly in $\Delta$ except for its two end points which lie on distinct sides of $\Delta$.

The terminology in Definition 1 is suggested by the appearance of such arcs in KOEBE's lemma [2, p. 19].

**Definition 2.** *A strong* KOEBE *sequence of arcs* is a KOEBE sequence of arcs $\{J_n\}$ with the property that, to every $\zeta \in C$, there corresponds a rectilinear segment extending from $\zeta$ to a point of $D$, which is intersected by infinitely many of the arcs $J_n$ $(n = 1, 2, 3, \ldots)$.

It is easily verified that a strong KOEBE sequence of arcs is a KOEBE sequence of arcs relative either to $C$ itself or to $C$ minus a single point of $C$.

**Definition 3.** If $f(z)$ is a meromorphic function in $D$ and $c$ is a constant, finite or $\infty$, we say that $f(z) \to c$ *along a* KOEBE *sequence of arcs* $\{J_n\}$, provided that, for some sequence of positive numbers $\{\eta_n\}$, where $\eta_n \to 0$ as $n \to \infty$, we have, for every $z \in J_n (n = 1, 2, 3, \ldots)$, $| f(z) - c | < \eta_n$ or $|f(z)| > 1/\eta_n$, according as $c$ is finite or infinite.

**Definition 4.** If $f(z)$ is a meromorphic function in $D$, we say that $f(z)$ *is bounded by* $M$ *on a* KOEBE *sequence of arcs* $\{J_n\}$, provided that there

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exists a finite positive constant $M$ such that $|f(z)| < M$ for every $z \in J_n$ $(n = 1, 2, 3, \ldots)$.

**Definition 5.** Let $z' = S(z)$ denote an arbitrary one-to-one conformal mapping of $D$ onto itself. A function $f(z)$, meromorphic in $D$, is said to be *normal* in $D$ [5, p. 53], if the family of functions $\{f(S(z))\}$ is normal in $D$ in the sense of MONTEL, where convergence is defined in terms of the spherical metric.

**Definition 6.** A FATOU *point* of a meromorphic function in $D$ is a point $\zeta \in C$ such that, for some complex number $c$ (possibly $\infty$), as $z \to \zeta$ in any STOLZ angle at $\zeta$, $f(z) \to c$; $c$ is then called a FATOU *value* of $f(z)$.

We show first (Theorem 1) that a normal meromorphic function that tends to a constant along a KOEBE sequence of arcs is identically constant. This generalizes a result due to GROSS [4, pp. 35–36] as well as a result due to the present authors [1, Corollary 1, p. 266]. Next we prove (Theorem 2) that a normal holomorphic function that is bounded on a strong KOEBE sequence of arcs must be a bounded function. This generalizes [1, Corollary 2, p. 266]. (The two results in [1] alluded to involve "boundary paths" instead of KOEBE sequences of arcs.)

Theorem 3 asserts that if the set of FATOU points of a normal holomorphic function in $D$ is of measure zero on an arc of $C$, then that arc contains an everywhere dense set of FATOU points of the function at each of which the corresponding FATOU value is $\infty$. This generalizes [1, Theorem 5, p. 267]. It follows immediately that the set of FATOU points of a normal holomorphic function in $D$ is everywhere dense on $C$, which sharpens [1, Theorem 4, p. 267]. This result is to be contrasted with one given in [5, p. 58], according to which there exist normal *meromorphic* functions in $D$ possessing no FATOU points. (*Cf.* also [1, Remark 4, p. 267].) Theorem 4 shows that a normal holomorphic function in $D$ can have its set of FATOU points of arbitrarily small positive measure without having $\infty$ as a FATOU value. This leads us to pose the following problem, which we have not solved.

**Problem.** Let $f(z)$ be a normal holomorphic function in $D$. Suppose that an arc $A$ of $C$ exists such that the measure of the set of FATOU points of $f(z)$ on every subarc of $A$ is less than the length of that subarc. Does $A$ contain a FATOU point of $f(z)$ at which the corresponding FATOU value is $\infty$?

We proceed now to the proofs of our theorems.

**Theorem 1.** *Let $f(z)$ be a normal meromorphic function in $D$. If $f(z) \to c$ along a* KOEBE *sequence of arcs $\{J_n\}$, then $f(z) \equiv c$.*

*Proof.* We may assume that $c = 0$, for otherwise we can replace the normal meromorphic function $f(z)$ by the normal meromorphic function $f(z) - c$ if $c$ is finite, or $1/f(z)$ if $c \neq \infty$.

Let the given sequence $\{J_n\}$ be a KOEBE sequence relative to the arc $A$ (see Definition 1), and consider an arc $B = \{z : |z| = 1,\ q_1 < \operatorname{arc} z < q_2\}$ strictly interior to $A$. Denote by $\Delta$ the open sector of $D$ with vertex at the origin and vertex angle $\beta$, subtending the arc $B$. The sides of $\Delta$ will be called $s_1, s_2$, where these segments terminate in $e^{iq_1}$, $e^{iq_2}$, respectively. In view of (b) in Definition 1, there is no loss of generality in asserting now that for every $n$ the arc $J_n$ contains a JORDAN subarc $\Gamma_n$ lying wholly in $\Delta$ except for its endpoints $P_n^{(1)}$, $P_n^{(2)}$ which lie on $s_1, s_2$, respectively. It is obvious that $\{\Gamma_n\}$ is a KOEBE sequence of arcs relative to $B$.

Set

$$r_n = \min_{z \in \Gamma_n} |z|, \quad R_n = \max_{z \in \Gamma_n} |z| \qquad (n = 1, 2, 3, \ldots).$$

It follows from (a) in Definition 1 that

$$\lim_{n \to \infty} r_n = \lim_{n \to \infty} R_n = 1. \tag{1}$$

For $n = 1, 2, 3, \ldots$, we now define a JORDAN curve $K_n$. Let the circle $|z| = R_n$ intersect $s_1$ and $s_2$ in the respective points $Q_n^{(1)}$, $Q_n^{(2)}$, and denote the radial segments $P_n^{(1)} Q_n^{(1)}$, $P_n^{(2)} Q_n^{(2)}$ by $t_n^{(1)}$, $t_n^{(2)}$, respectively (these segments may reduce to single points). Then, if $B_n$ is the open arc of the circle $|z| = R_n$ which lies in $\Delta$ and $B_n^*$ is the complementary arc, we put

$$K_n = t_n^{(1)} \cup B_n^* \cup t_n^{(2)} \cup \Gamma_n.$$

The interior of $K_n$ will be called $\Omega_n$, and we set $G_n = \{z : |z| < R_n\}$.

CARLEMAN's Extension Principle for harmonic measure implies [7, p. 70] that

$$\omega(0, t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)}, \Omega_n) \geqq \omega(0, B_n, G_n) = \frac{\beta}{2\pi}.$$

We have [7, p. 26]

$$\omega(0, t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)}, \Omega_n) = \omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) + \omega(0, \Gamma_n, \Omega_n).$$

An inequality due to OSTROWSKI [3, p. 42] shows that

$$\omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) \leqq \frac{4}{\pi} \operatorname{arc\,sin} \frac{2\sqrt{\dfrac{R_n - r_n}{2} \cdot \dfrac{R_n + r_n}{2}}}{\dfrac{R_n - r_n}{2} + \dfrac{R_n + r_n}{2}} = \frac{4}{\pi} \operatorname{arc\,sin} \frac{\sqrt{R_n^2 - r_n^2}}{R_n},$$

and (1) implies that $\lim_{n\to\infty} \omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) = 0$. Hence

$$\liminf_{n\to\infty} \omega(0, \Gamma_n, \Omega_n) \geqq \frac{\beta}{2\pi}.$$

Consequently, if $D_w$ is mapped conformally onto $\Omega_n$ by means of the function $z = \psi_n(w)$, where $\psi_n(0) = 0$ and the point $w = e^{iq_1}$ corresponds to the point $z = P_n^{(1)}$, then each arc $\Gamma_n$, for $n$ sufficiently large, is the image of an arc of $C_w$ of length at least $\beta/2$ with its end point of smaller argument at $e^{iq_1}$.

If we set

$$g_n(w) = f(\psi_n(w)) \qquad (n = 1, 2, 3, \ldots), \tag{2}$$

then [5, p. 57] $g_n(w)$ is a normal meromorphic function in $D_w$. Since $f(z)$ is normal in $D$, there exists [5, p. 56] a finite positive constant $\gamma$ such that for every $z \in D$,

$$\frac{|f'(z)|}{1 + |f(z)|^2}(1 - |z|^2) \leqq \gamma. \tag{3}$$

Now from (2) we obtain

$$\frac{|g'_n(w)|}{1 + |g_n(w)|^2}(1 - |w|^2) = \frac{|f'(\psi_n(w))| \cdot |\psi'_n(w)|}{1 + |f(\psi_n(w))|^2}(1 - |w|^2). \tag{4}$$

According to [9, p. 133], if $D_1(z)$ denotes the radius of univalence at the point $z = \psi_n(w)$ of the region $\Omega_n$, we have

$$(1 - |w|^2)\,|\psi'_n(w)| \leqq 4D_1(z), \tag{5}$$

and since $\Omega_n$ lies in $D$,

$$D_1(z) \leqq 1 - |z| \leqq 1 - |z|^2. \tag{6}$$

Combining (3) to (6), we find that

$$\frac{|g'_n(w)|}{1 + |g_n(w)|^2}(1 - |w|^2) \leqq \frac{4|f'(z)|}{1 + |f(z)|^2} \leqq 4\gamma. \tag{7}$$

Let $S$ denote the subarc of $C_w$ whose end point of smaller argument is $e^{iq_1}$ and whose length is $\beta/2$. The hypothesis that $f(z) \to 0$ along the KOEBE sequence $\{J_n\}$ implies that $\lim_{n\to\infty} g_n(w) = 0$ uniformly on $S$. This together with (7) shows, in view of [5, p. 64], that the sequence $\{g_n(w)\}$ tends uniformly to zero on every compact subset of $D_w$.

We shall now show that $f(z) \equiv 0$. Suppose that, on the contrary, $f(z_0) \neq 0$ for some $z_0 \in D$. By (a) in Definition 1, $z_0 \in \Omega_n$ for all sufficiently large values of $n$. Let $w = \varphi_n(z)$ be the inverse of the function $z = \psi_n(w)$. Then, according to (2),

$$g_n(\varphi_n(z_0)) = f(z_0)$$

for all sufficiently large values of $n$. Since $\{g_n(w)\}$ tends uniformly to zero on every compact subset of $D_w$, but $f(z_0) \neq 0$, we must have $\lim_{n \to \infty} |\varphi_n(z_0)| = 1$. But this is impossible; for if we fix $\varrho$ so that $|z_0| < \varrho < 1$, then SCHWARZ's lemma yields

$$|\varphi_n(z_0)| \leqq \frac{|z_0|}{\varrho} < 1$$

for all sufficiently large values of $n$. Our supposition has thus led to a contradiction, and the theorem is proved.

**Theorem 2.** *Let $f(z)$ be a normal holomorphic function in $D$. If $f(z)$ is bounded by $M$ on a strong* KOEBE *sequence of arcs $\{J_n\}$, then $f(z)$ is bounded by $M$ throughout $D$.*

*Proof.* If $f(z)$ is bounded in $D$, then Definition 2 implies that none of its radial limits, except perhaps one, is greater than $M$ in modulus, and the representation of $f(z)$ by its POISSON integral shows immediately that $|f(z)| < M$ throughout $D$.

We shall now suppose that $f(z)$ is unbounded in $D$, and show that this leads to a contradiction of the hypothesis that $\{J_n\}$ is a strong KOEBE sequence. The set of all points $z \in D$ at which $|f(z)| > M + 1$ is open and not empty; let $R_1$ be some component of this set. At all boundary points of $R_1$ that lie in $D$, we have $|f(z)| = M + 1$, and the maximum principle implies that $R_1$ cannot lie wholly in some disk $|z| < \varrho < 1$. Hence, the boundary of $R_1$ contains at least one point of $C$. The region $R_1$ cannot have more than one accessible boundary point on $C$, for if it had two such points $\zeta_1$ and $\zeta_2$, they could be connected by a JORDAN arc $\Gamma$ lying, except for its end points $\zeta_1$ and $\zeta_2$, in $R_1$, and $\Gamma$ would decompose $D$ into two subregions. But $R_1$, and hence $\Gamma$, meets none of the arcs $J_n$ $(n = 1, 2, 3, \ldots)$, and therefore infinitely many of these arcs would have to lie in one of the two subregions of $D$, contradicting the remark following Definition 2 and (b) in Definition 1.

We now map $D_w$ conformally onto the universal covering surface $R_1^*$ of $R_1$ by means of the single-valued function $z = \varphi(w)$, and set

$$g(w) = f(\varphi(w))$$

in $D_w$. We have $|\varphi(w)| < 1$ in $D_w$. The FATOU values of $\varphi(w)$ are of modulus 1 on at most a subset of measure zero of $C_w$; this follows from the RIESZ uniqueness theorem [7, p. 209] and the fact that $R_1$ has at most one accessible boundary point on $C$. Since $R_1^*$ is unbranched over $R_1$, almost all the FATOU values of $\varphi(w)$ are points in $D$ that lie on the boundary of $R_1$. Hence, $g(w)$ possesses limits of modulus $M + 1$ along almost all radii of $C_w$. It follows that $f(z)$ is unbounded in $R_1$, because otherwise we should have $|g(w)| < M + 1$ throughout $D_w$, contradicting the definition of $R_1$.

The set of all points $z \in R_1$, at which $|f(z)| > M + 2$ is open and not empty; let $R_2$ be some component of this set. Then $R_2 \subset R_1$, and if we apply to $R_2$ the foregoing argument for $R_1$, we arrive at the conclusion that $f(z)$ is unbounded in $R_2$. Proceeding in this manner, we obtain a sequence of nested regions

$$R_1 \supset R_2 \supset R_3 \supset \dots$$

such that, for $n = 1, 2, 3, \dots$,

$$|f(z)| > M + n \qquad (z \in R_n). \tag{8}$$

Now take

$$z_1 \in R_1,\ z_2 \in R_2 - \{z_1\},\ z_3 \in R_3 - \{z_1, z_2\}, \dots, z_n \in R_n - \{z_1, z_2, \dots, z_{n-1}\}, \dots,$$

and join $z_1$ to $z_2$ by means of a JORDAN arc $K_1$ lying in $R_1$, join $z_2$ to $z_3$ by means of a JORDAN arc $K_2$ lying in $R_2$ and having no point except $z_2$ in common with $K_1, \dots,$ join $z_n$ to $z_{n+1}$ by means of a JORDAN arc $K_n$ lying in $R_n$ and having no point except $z_n$ in common with $K_1 \cup K_2 \cup \dots \cup K_{n-1}, \dots$. We thus obtain a path

$$P = \bigcup_{n=1}^{\infty} K_n$$

in $D$. Its initial point is $z_1$, and its "end" lies on $C$ because, due to (8) and the fact that $K_n \subset R_n$ $(n = 1, 2, 3, \dots)$,

$$\lim_{n \to \infty} \min_{z \in K_n} |f(z)| = \infty,$$

and $f(z)$, by hypothesis, is holomorphic in $D$. The path $P$ then is a "boundary path" in $D$ along which $f(z) \to \infty$. According to [1, Corollary 1, p. 266], the end of $P$ is a single point $\zeta \in C$. Since $f(z)$ is normal in $D$, $\zeta$ is a FATOU point of $f(z)$ with $\infty$ as the corresponding FATOU value [5, p. 53]. But, in view of Definition 2, this contradicts the hypothesis that $\{J_n\}$ is a strong KOEBE sequence, because $f(z)$ is bounded on $\{J_n\}$; and the theorem is proved.

**Theorem 3.** *Let $f(z)$ be a normal holomorphic function in $D$ and $A$ be an open subarc of $C$. If the set of* FATOU *points of $f(z)$ on $A$ is of measure zero, then $A$ contains a* FATOU *point of $f(z)$ at which the corresponding* FATOU *value is $\infty$.*

*Proof.* Take a point $\zeta \in A$. The function $f(z)$ cannot be bounded in any neighborhood of $\zeta$, because otherwise, by a simple extension of FATOU's theorem, the set of FATOU points of $f(z)$ on $A$ would be of positive measure, contrary to hypothesis. Hence, there exists a number $\delta > 0$ such that the region $H = D \cap \{z : |z - \zeta| < \delta\}$ satisfies the conditions that $\overline{H} \cap C \subset A$ and $f(z)$ is unbounded in $H$. Consequently there exists a sequence of points $\{z_n\}$ in $D$ such that $z_n \to \zeta$ and $M_n = |f(z_n)| \to \infty$ as $n \to \infty$, where $1 < M_1 < M_2 < \cdots < M_n < \cdots$. For $n = 1, 2, 3, \ldots$, let $V_n$ be the open set of all points of $D$ at which $|f(z)| > M_n - 1$, and denote by $R_n$ that component of $V_n$ which contains the point $z_n$. Evidently $|f(z)| = M_n - 1$ *at all boundary points of $R_n$ that lie in $D$.* The maximum principle implies that $\overline{R}_n \cap C$ is not empty. As $n \to \infty$, the diameter of $R_n$ tends to zero. For if $r_n = \min\limits_{z \in \overline{R}_n} |z|$, the hypothesis that $f(z)$ is holomorphic in $D$ implies that $\lim\limits_{n \to \infty} r_n = 1$, so that if the diameter of $R_n$ did not tend to zero as $n \to \infty$, one could obtain a KOEBE sequence of arcs along which $f(z) \to \infty$, which is impossible in view of Theorem 1. Thus there exists a natural number $N$ such that $R_N \subset H$, and we set $G_1 = R_N$.

We shall show that $f(z)$ is unbounded in $G_1$. Let $G_1^*$ be the smallest simply connected region containing $G_1$, and $z = \varphi(w)$ be a function that maps $D_w$ conformally onto $G_1^*$. The set $B^* = \overline{G_1^*} \cap C$ is not empty; we denote by $B_1^*$ the set of all points of $B^*$ that are accessible from the region $G_1^*$. According to FATOU's theorem, $\varphi(w)$ has a radial limit at almost all points of $C_w$; we put

$$\varphi^*(e^{i\mu}) = \lim_{r \to 1} \varphi(re^{i\mu})$$

for every $\mu$ for which the limit exists. The set

$$E_1 = \{e^{i\mu} : |\varphi^*(e^{i\mu})| = 1\}$$

is a BOREL set, and is therefore measurable, and we have

$$B_1^* = \{\varphi^*(e^{i\mu}) : e^{i\mu} \in E_1\} .$$

Consider the function

$$g(w) = f(\varphi(w))$$

in $D_w$. We are going to show that $g(w)$ is unbounded in $D_w$. Assume that $g(w)$ is bounded in $D_w$. We have either $m(E_1) > 0$ or $m(E_1) = 0$.

Suppose first that $m(E_1) > 0$. Let $E_0$ be the BOREL subset of positive measure of $E_1$ at each point of which $g(w)$ possesses a radial limit, and $B_0^*$ be the image of $E_0$ under the mapping $z = \varphi(w)$. An application of an extension of LÖWNER's theorem [10, p. 322] shows that $B_0^*$ is a measurable subset of $B_1^*$ with $m(B_0^*) > 0$. Let $\zeta_0 \in B_0^*$. Then there is a path in $G_1^*$ terminating in $\zeta_0$, and this path is the image, under the mapping $z = \varphi(w)$, of a path in $D_w$ that terminates in a point $e^{i\mu_0} \in E_0$. Now $\varphi^*(e^{i\mu_0}) = \zeta_0$, and $g(w)$ has a radial limit at the point $e^{i\mu_0}$; therefore $f(z)$ tends to a limit along a path in $G_1^*$ terminating in $\zeta_0$. By hypothesis, $f(z)$ is normal in $D$, and consequently [5, p. 53] $\zeta_0$ is a FATOU point of $f(z)$. Since $\zeta_0$ was an arbitrary point of $B_0^*$, and $m(B_0^*) > 0$, we have arrived at a contradiction of the hypothesis that the set of FATOU points of $f(z)$ on $A$ is of measure zero.

Suppose next that $m(E_1) = 0$. Since every boundary point of $G_1^*$ is a boundary point of $G_1$, the italicized remark in the first paragraph of the proof implies that the FATOU values of $g(w)$ are equal to $M_N - 1$ in modulus almost everywhere on $C_w$. The representation of $g(w)$ by its POISSON integral shows that $|g(w)| \leqq M_N - 1$ throughout $D_w$, which implies that $|f(z)| \leqq M_N - 1 = L$ throughout $G_1 = R_N$, contrary to the definition of $R_N$.

Thus $g(w)$ is unbounded in $D_w$, which implies that $f(z)$ is unbounded in $G_1^*$ and hence in $G_1$. It follows that the open set of all points of $G_1$ at which $|f(z)| > L + 1$ is not empty, and letting $G_2$ denote a component of this set, we conclude as above that $f(z)$ is unbounded in $G_2$. Continuing in this manner, we obtain a sequence of nested subregions $G_1 \supset G_2 \supset G_3 \supset \ldots$ of $H$, and now an argument employed in the proof of Theorem 2 enables us to infer the existence of a FATOU point of $f(z)$ on $A$ at which the corresponding FATOU value is $\infty$, thus completing the proof of the theorem.

**Corollary 1.** *The set of* FATOU *points of a normal holomorphic function in* $D$ *is everywhere dense on* $C$.

**Theorem 4.** *Given* $\varepsilon > 0$, *there exists a normal holomorphic function* $f(z)$ *in* $D$ *whose set of* FATOU *points is of measure less than* $\varepsilon$ *but for which* $\infty$ *is not a radial limit.*

*Proof.* Consider first the function $\varphi(w) = g(w) + h(w)$ in $D_w$, where $g(w)$ is the elliptic modular function, holomorphic and normal in $D_w$, whose set of FATOU points $E$ is enumerable and whose FATOU values are $0, 1, \infty$, and $h(w)$ is bounded and holomorphic in $D_w$ and possesses a radial limit at every point of $C_w - E$ but no radial limit at any point of $E$ [6, Theorem 6,

p. 14]. Now $\varphi(w)$ is holomorphic and normal in $D_w$ [5, p. 53]; its set $E_0$ of FATOU points is enumerable, and $\infty$ is its only FATOU value.

Choose a positive number $\delta$ so small that, if $\varrho = \cos\frac{\delta}{2}$, then

$$\frac{L}{\left|\log\frac{\delta}{\varrho}\right| + 1} < \varepsilon\,, \tag{9}$$

where $L$ is a certain positive absolute constant to be specified later. Let $P$ be a perfect nowhere dense set on $C_w$ that contains no point of $E_0$ and for which $m(P) > 2\pi - \delta$, and set $H = C_w - P$. Denote by $R$ the simply connected subregion of $D_w$ whose boundary consists of the points of $P$ and the open chords of $C_w$ that subtend the components of the open set $H$. The boundary of $R$ is evidently a rectifiable JORDAN curve of length less than $2\pi$. Since each component of $H$ is of length less than $\delta$, the region $R$ contains the disk $|w| < \varrho$. Let the function $w = \lambda(z)$ map $D$ conformally onto $R$ so that $\lambda(0) = 0$, and let $S$ be the set of all points on $C$ that correspond under this mapping to points on the chords of $C_w$ subtending components of $H$. Since the sum of the lengths of these chords is less than $\delta$, we have, by a theorem of LAVRENTIEV [8, p. 125],

$$m(S) < \frac{L}{\left|\log\frac{\delta}{\varrho}\right| + 1}\,. \tag{10}$$

Now consider the function $f(z) = \varphi(\lambda(z))$ in $D$. It is holomorphic and normal in $D$ [5, p. 57], does not have $\infty$ as a FATOU value, and its set of FATOU points is $S$. According to (9) and (10), $m(S) < \varepsilon$, and this completes the proof of the theorem.

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