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Homogeneous Manifolds of Constant Curvature[1])

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Contents

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1. Introduction

In a recent note [6] we classified the RIEMANNian homogeneous manifolds of constant sectional curvature; this paper extends the techniques of that note to the classifications (up to global isometry) of the homogeneous pseudo-RIEMANNian (indefinite metric) manifolds of constant nonzero curvature, the symmetric pseudo-RIEMANNian manifolds of constant curvature, the isotropic pseudo-RIEMANNian manifolds of constant curvature, the strongly isotropic pseudo-RIEMANNian manifolds, and the complete pseudo-RIEMANNian manifolds in which parallel translation is independent of path.

Chapter I begins with a review of the necessary material on pseudo- RIEMANNian manifolds. Just as an n-dimensional RIEMANNian manifold M is denoted by M^n, so an n-dimensional pseudo-RIEMANNian manifold M, with metric everywhere of signature $-\sum_1^h x_j^2 + \sum_{h+1}^n x_j^2$, is denoted by M_h^n. We then give euclidean space the structure of a pseudo-RIEMANNian manifold R_h^n of constant zero curvature, and give the quadric $-\sum_1^h x_j^2 + \sum_{h+1}^{n+1} x_j^2 = 1$ in R^{n+1} both the structure of a pseudo-RIEMANNian manifold S_h^n of constant positive curvature and that of a manifold H_{n-h}^n of constant negative curvature. For S_h^n, this is essentially due to S. HELGASON ([2] and private communications). We then define $\tilde{S}_h^n$ and $\tilde{H}_h^n$ to be the respective connected, simply connected manifolds corresponding to S_h^n and H_h^n, and prove that a connected complete pseudo-RIEMANNian manifold M_h^n of constant curvature admits $R_h^n, \tilde{S}_h^n$ or $\tilde{H}_h^n$ as universal metric covering manifold. Together with our criterion of homogeneity for manifolds covered by a homogeneous manifold (Theorem 2.5), this is the basis of our investigations.

Chapter II is the classification of homogeneous pseudo-RIEMANNian manifolds of constant nonzero curvature. We define 14 families of homogeneous manifolds covered by S_h^n (§ 7 and § 9), prove that every homogeneous manifold covered by S_h^n lies in one of those families (§ 10), and then take care of the signature $h = n - 1$, where S_h^n is quite different from $\tilde{S}_h^n$ (§ 11). The basic new technique is consideration of the behavior of a quadric under field extension. We also rely on the elementary theory of associative algebras and on the version of SCHUR's Lemma which says that, if $\mathcal{E}(V)$ is the algebra of linear endomorphisms of a vectorspace V over a field F, and if $G \subset \mathcal{E}(V)$ acts irreducibly on V, then the centralizer of G in $\mathcal{E}(V)$ is a division algebra over F.

Chapter III consists of classifications which are easy or which follow easily from Chapter II, and is best described by the table of contents.

References are as follows: Lemma 8.3 refers to the lemma in § 8.3, Theorem 5 refers to the theorem in § 5, etc.

I wish to thank Professors S. HELGASON and A. BOREL for several helpful discussions. HELGASON suggested that I work on these classification problems; he proved Lemma 4.3 for $h = 1$ and $h = n - 1$ [2], and the proof given here is essentially the same; he communicated the proof of Lemma 4.2, as well as another proof of Lemma 4.3, to me. BOREL suggested that I look at the group $f(GL(s, R))$ of Lemma 8.4 to find a reducible linear subgroup of $O^s(2s)$ transitive on the manifold S_s^{2s-1} in R^{2s}.

Chapter I. The covering theorem for space forms

2. Pseudo-Riemannian manifolds

2.1. We recall a few basic definitions and theorems on pseudo-RIEMANNian manifolds, primarily in order to establish notation and terminology.

A *pseudo-*RIEMANN*ian metric* on a differentiable manifold M is a differentiable field Q of nonsingular real-valued symmetric bilinear forms Q_p on the tangentspaces M_p of M; (M, Q) is then a *pseudo-*RIEMANN*ian manifold.* The pseudo-RIEMANNian metric Q on M defines a unique linear connection, the LEVI-CIVITA *connection* on the tangentbundle of M, with torsion zero and such that parallel translation preserves the inner products Q_p on tangentspaces. (M, Q) is *complete* if its LEVI-CIVITA connection is complete, i. e., if geodesics can be extended to arbitrary values of the affine parameter.

2.2. If $X, Y \in M_p$ span a nonsingular 2-plane $S \subset M_p$ (Q_p is nondegenerate on S), then the *sectional curvature* of (M, Q) along S is defined to be[3])

$$K(S) = -Q_p(\mathcal{R}(X, Y)X, Y)/\{Q_p(X, X)\cdot Q_p(Y, Y) - Q_p(X, Y)^2\}$$

where $\mathcal{R}$ is the curvature tensor of the LEVI-CIVITA connection. (M, Q) is said to have *constant curvature* K if $K(S) = K$ for every nonsingular 2-dimensional subspace S of a tangentspace of M. Let $x \to (x_1, \ldots, x_n)$ be a local coordinate system valid in an open subset U of M; the coefficients of $\mathcal{R}$ are given by

$$\mathcal{R}\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) \cdot \frac{\partial}{\partial x_l} = \sum_i R_l{}^i{}_{jk} \frac{\partial}{\partial x_i}$$

and we make the usual definitions $q_{ij}(p) = Q_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ and $R_{ijkl} = \sum_m q_{jm} R_i{}^m{}_{kl}$.

[3]) This was brought to my attention by S. HELGASON.

Then (M, Q) has constant curvature K if and only if[4])

$$R_{ijkl}(p) = K\,(q_{jk}(p)q_{il}(p) - q_{ik}(p)q_{jl}(p))$$

for every $p \in U$ and every (i, j, k, l), as U runs through a covering of M by coordinate neighborhoods.

2.3. An *isometry* $f:(M, Q) \to (M', Q')$ is a diffeomorphism $f: M \to M'$ such that $Q_p(X, Y) = Q'_{f(p)}(f_* X, f_* Y)$ for every $p \in M$ and all $X, Y \in M_p$, where f_* denotes the induced maps on tangentspaces. The collection of all isometries of (M, Q) onto itself is a LIE group $I(M, Q)$; (M, Q) is *homogeneous* if $I(M, Q)$ is transitive on the points of M.

2.4. We will now consider only those pseudo-RIEMANNian manifolds where the metric has the same signature at every point. If each Q_p has signature $-\sum_1^h x_j^2 + \sum_{h+1}^n x_j^2$, then (M, Q) will be denoted M_h^n.

2.5. A *pseudo-*RIEMANN*ian covering* is a covering $\pi: N_h^n \to M_h^n$ of connected pseudo-RIEMANNian manifolds where π is a local isometry. A deck transformation of the covering (homeomorphism $d: N_h^n \to N_h^n$ such that $\pi \cdot d = \pi$) is an isometry of N_h^n. Our main tool is:

Theorem. *Let $\pi: N_h^n \to M_h^n$ be a pseudo-*RIEMANN*ian covering, let D be the group of deck transformations of the covering, and let G be the centralizer of D in $I(N_h^n)$. If M_h^n is homogeneous, then G is transitive on N_h^n. If G is transitive on N_h^n and the covering is normal, then M_h^n is homogeneous.*

The proof is identical to the proof of the RIEMANNian case [6, Th. 1].

2.6. Our definitions agree with the usual ones in the RIEMANNian case, where each Q_p is positive definite.

3. The model spaces R_h^n

Given integers $0 \leqq h \leqq n$, we have a pseudo-RIEMANNian metric $Q^{(h)}$ on the n-dimensional real number space R^n, given by $Q_p^{(h)}(X, Y) = -\sum_1^h x_j y_j + \sum_{h+1}^n x_j y_j$ where $X, Y \in (R^n)_p$ correspond respectively to $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in R^n$ under the identification of $(R^n)_p$ with R^n by parallel translation. $(R^n, Q^{(h)})$ is denoted R_h^n.

Let $O^h(n)$ denote the orthogonal group of the bilinear from $Q_0^{(h)}$ on R^n; let $NO^h(n)$ be the corresponding inhomogeneous group, semidirect product

[4]) This was brought to my attention by S. HELGASON.

$O^h(n)\cdot R^n$, consisting of all pairs (A, a) with $A \in O^h(n)$ and $a \in R^n$, and acting on R^n by $(A, a): x \to Ax + a$. It is easy to see $I(R^n_h) = NO^h(n)$.

The geodesics of R^n_h are just the straight lines. R^n_h has constant curvature zero, by a result of K. NOMIZU [4, formula 9.6], because it is a reductive homogeneous coset space of an *abelian* group R^n, with an R^n — invariant affine connection.

4. The model spaces S^n_h and H^n_h.

4.1. Given integers $0 \leqq h \leqq n$, let Q^n_h denote the quadric $-\sum_1^h x_j^2 + \sum_{h+1}^{n+1} x_j^2 = 1$ in R^{n+1}. Q^n_h is homeomorphic to $R^h \times S^{n-h}$ ($S^m = m$-sphere). The orthogonal group of the bilinear form $-\sum_1^h x_j y_j + \sum_{h+1}^{n+1} x_j y_j$, denoted $O^h(n+1)$, is transitive on Q^n_h. The isotropy subgroup at $p_0 = (0, \ldots, 0, 1)$ is $O^h(n)$. We will regard Q^n_h as a symmetric coset space $O^h(n+1)/O^h(n)$, the symmetry at p_0 being given by $(x_1, \ldots, x_n, x_{n+1}) \to (-x_1, \ldots, -x_n, x_{n+1})$.

4.2. Lemma. *Let* $\mathfrak{O}^h(n+1)$ *and* $\mathfrak{O}^h(n)$ *be the* LIE *algebras of* $O^h(n+1)$ *and* $O^h(n)$, and *let* E_{ij} *be the* $(n+1) \times (n+1)$ *matrix whose only nonzero entry is* 1 *in the* (i, j)*-place. Then* $\mathfrak{O}^h(n+1)$ *has basis consisting of all* $X_{ij} = E_{ij} - E_{ji}$ *for* $h < i < j \leqq n+1$, *all* $Y_{uv} = E_{uv} - E_{vu}$ *for* $1 \leqq u < v \leqq h$, *and all* $Z_{uj} = E_{uj} + E_{ju}$ *for* $1 \leqq u \leqq h < j \leqq n+1$. $\mathfrak{O}^h(n+1)$ *has* KILLING *form*

$$B(X, Y) = -2(n-1)\{-\sum x_{ij}y_{ij} - \sum x_{uv}y_{uv} + \sum x_{uj}y_{uj}\}$$

where $X = \sum x_{ij}X_{ij} + \sum x_{uv}Y_{uv} + \sum x_{uj}Z_{uj}$ *and* $Y = \sum y_{ij}X_{ij} + \sum y_{uv}Y_{uv} + \sum y_{uj}Z_{uj}$. $\mathfrak{O}^h(n)$ *has basis* $\{X_{ij}, Y_{uv}, Z_{uj}\}_{j \leqq n}$ *and is the eigenspace of* $+1$ *for the symmetry at* p_0; *the eigenspace* $\mathfrak{P}^n_h$ *of* -1 *for that symmetry has basis consisting of all* $X_u = Z_{u,n+1}$ *for* $1 \leqq u \leqq h$ *and all* $X_j = X_{j,n+1}$ *for* $h < j \leqq n$.

Proof. The symmetry at p_0 acts on $\mathfrak{O}^h(n+1)$ as conjugation by $\begin{pmatrix} -I_n & \\ & 1 \end{pmatrix}$; now the statements on eigenspaces and bases are clear. The proof of the statement on the KILLING form is essentially due to S. HELGASON [2, Lemma 6, p. 256], but is given here for the convenience of the reader. Let $\mathfrak{G}$ be the complexification of $\mathfrak{O}^h(n+1)$; we have an isomorphism f of $\mathfrak{G}$ onto the LIE algebra $\mathfrak{O}(n+1, C)$ of the complex orthogonal group given by $X_{ij} \to X_{ij}$, $Y_{uv} \to Y_{uv}$ and $Z_{uj} \to \sqrt{-1}\,(E_{uj} - E_{ju})$. $\mathfrak{O}(n+1, C)$ has KILLING form $B'(X, Y) = (n-1)$ trace (XY), and trace $(XY) =$ trace $(f(X)f(Y))$, whence $\mathfrak{G}$ has KILLING form $B_c(X, Y) = (n-1)$ trace (XY). As B is the restriction of B_c, our assertion follows. *Q.E.D.*

4.3. Identify $\mathfrak{P}^n_h$ with the tangentspace to Q^n_h at p_o; this gives an ad $(O^h(n))$-invariant bilinear form $K^{-1}Q_{p_0}(X, Y) = K^{-1}(-\sum_1^h x_u y_u + \sum_{h+1}^n x_j y_j)$ on that tangentspace, where $0 \neq K \in R$, and $X = \sum_1^n x_b X_b$ and $Y = \sum_1^n y_b X_b$ are elements of $\mathfrak{P}^n_h$. The invariance follows from the fact (Lemma 4.2) that $K^{-1}Q_{p_0}$ is proportional to the restriction of the KILLING form of $\mathfrak{O}^h(n+1)$. Thus $K^{-1}Q_{p_0}$ defines an $O^h(n+1)$-invariant pseudo-RIEMANNian metric $K^{-1}Q$ on Q^n_h.

Lemma. *The pseudo-*RIEMANNIAN *manifold* $(Q^n_h, K^{-1}Q)$ *has constant curvature* $K \neq 0$.

Proof. The manifold is homogeneous, so we need only prove our statement at p_o. According to K. NOMIZU [4, Th. 15.6], the LEVI-CIVITA connection is the canonical affine connection on Q^n_h induced by its structure as a symmetric coset space, whence [4, Th. 10.3 and 15.1] the curvature tensor is given by $\mathcal{R}(X, Y)\cdot Z = -[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{P}^n_h$. We choose local coordinates $\{u_b\}$ at p_o such that $\left.\frac{\partial}{\partial u_b}\right|_{p_0} = X_b$, $1 \leqq b \leqq n$. In these coordinates, $q_{bc}(p_o) = K^{-1}Q_{p_0}(X_b, X_c) = -K^{-1}\delta_{bc}$ if $b \leqq h$, $K^{-1}\delta_{bc}$ if $b > h$, where δ_{bc} is the KRONECKER symbol. Also, $R_b{}^c{}_{cb} = 1 = -R_b{}^c{}_{bc}$ for $c \neq b > h$, $R_b{}^c{}_{bc} = 1 = -R_b{}^c{}_{cb}$ for $c \neq b \leqq h$, and all other $R_d{}^e{}_{bc}$ are zero, at p_o. Thus $R_{bcbc} = K^{-1} = -R_{bccb}$ for $c \leqq h < b$ and for $b \leqq h < c$, and $R_{bcbc} = -K^{-1} = -R_{bccb}$ if $b \neq c$ and either $b, c > h$ or $b, c \leqq h$, and all other R_{debc} are zero. This gives $R_{debc}(p_o) = K\{q_{eb}(p_o)q_{dc}(p_o) - q_{db}(p_o)q_{ec}(p_o)\}$, whence $(Q^n_h, K^{-1}Q)$ has constant curvature K. *Q.E.D.*

4.4. We define $S^n_h = (Q^n_h, Q)$ and $H^n_{n-h} = (Q^n_h, -Q)$. Note that the signatures of metric are correct. S^n_h has constant curvature $+1$; it is the indefinite metric analogue of the sphere $S^n = S^n_o$. H^n_h has constant curvature -1; it is the indefinite metric analogue of the hyperbolic space $H^n = H^n_o$. We will sometimes speak loosely and refer to $(Q^n_h, K^{-1}Q)$ as S^n_h if $K > 0$ or H^n_{n-h} if $K < 0$.

$\tilde{S}^n_h$ and $\tilde{H}^n_{n-h}$ are the connected, simply connected pseudo-RIEMANNian manifolds corresponding to S^n_h and H^n_{n-h}. $\tilde{S}^n_h = S^n_h$ and $\tilde{H}^n_{n-h} = H^n_{n-h}$ if $h < n-1$; $\tilde{S}^n_{n-1}$ and $\tilde{H}^n_1$ are the respective universal pseudo-RIEMANNian covering manifolds of S^n_{n-1} and H^n_1; $\tilde{S}^n_n$ and $\tilde{H}^n_o$ are the respective components of p_o in S^n_n and H^n_o.

4.5. Lemma. $I(S^n_h) = O^h(n+1) = I(H^n_{n-h})$.

Proof. The isotropy subgroup at p_o is maximal for conservation of the metric. *Q. E. D.*

4.6. We mention an alternative description of S^n_h and H^n_h. The alternative description is more direct, but the description in § 4.1 to § 4.3 simplifies the proofs of Lemmas 4.3 and 4.5.

One can prove that the manifold S^n_h of curvature $K > 0$ is isometric to the metric submanifold of R^{n+1}_h given by $-\sum_1^h x_j^2 + \sum_{h+1}^{n+1} x_j^2 = K^{-\frac{1}{2}}$, and that the manifold H^n_h of curvature $-K < 0$ is isometric to the metric submanifold of R^{n+1}_{h+1} given by $-\sum_1^{h+1} x_j^2 + \sum_{h+2}^{n+1} x_j^2 = -(K^{-\frac{1}{2}})$.

5. The universal covering theorem

Theorem. *Let M^n_h be a complete connected pseudo-*RIEMANN*ian manifold of constant curvature K. Then, assuming the metric normalized such that K is $-1, 0$ or $+1$, the universal pseudo-*RIEMANN*ian covering manifold of M^n_h is $\tilde{H}^n_h$ if $K < 0$, R^n_h if $K = 0$, and $\tilde{S}^n_h$ if $K > 0$.*

Proof. The universal pseudo-RIEMANNian covering manifold N^n_h of M^n_h is complete, connected and simply connected; we must show N^n_h is isometric to $\tilde{H}^n_h, R^n_h$ or $\tilde{S}^n_h$. Let A^n_h be the one of $\tilde{H}^n_h, R^n_h, \tilde{S}^n_h$ of constant curvature K.

Let $a \in A^n_h, b \in N^n_h$ and let $f_* : (A^n_h)_a \to (N^n_h)_b$ be a linear isometry of tangentspaces; f_* exists because the metrics have the same signature. Given a broken geodesic α of A^n_h emanating from a, let β be the broken geodesic on N^n_h emanating from b which corresponds (see [3]) to α under f_*. Let S be a 2-dimensional subspace of $(A^n_h)_a$; the curvature tensors, hence the curvature forms, take the same values on corresponding parallel translates of S and f_*S along α and β, as they are determined by the sectional curvatures. The torsion forms are identically zero because we are dealing with LEVI-CIVITA connections. It follows from a theorem of N. HICKS [3, Th. 1] that f_* extends to a connection-preserving diffeomorphism $g: A^n_h \to N^n_h$. Let $a' \in A^n_h$, let s be an arc in A^n_h from a' to a, and let s_* be the parallel translation along s from a' to a; let t_* be the parallel translation along $g(s)$ from b to $b' = g(a')$. The tangential map $(g_*)_{a'}$, equals $t_* \cdot f_* \cdot s_*$ because g is connection-preserving; as $t_* \cdot f_* \cdot s_*$ is a linear isometry of tangentspaces; it follows that g is an isometry of pseudo-RIEMANNian manifolds. *Q.E.D.*

Chapter II. Homogeneous manifolds of nonzero curvature

The goal of this Chapter is the global classification of the homogeneous pseudo-RIEMANNian manifolds of constant nonzero curvature. The main tools are SCHUR's Lemma, real division algebras, bilinear invariants of group representations, and the technique of changing the basefield of a quadric.

To a manifold M^n_h of constant curvature k, there corresponds a manifold N^n_{n-h} of constant curvature $-k$, obtained from M^n_h on replacing the metric by its negative. For complete manifolds, this correspondence is given by $\tilde{S}^n_h/D \leftrightarrow \tilde{H}^n_{n-h}/D$. It suffices, then, to do our classification for the case of positive curvature.

6. The real division algebras

We assemble some facts on real division algebras, for the most part well known, which will be useful in the sequel. All algebras will be finite dimensional.

6.1. A real division algebra F is one of the fields R (real), C (complex) or K (quaternion). Let $x \to \bar{x}$ denote the conjugation of F over R, let F^* denote the multiplicative group of nonzero elements of F, and let F' denote the subgroup of unimodular ($x\bar{x} = 1$) elements of F. dim. F is understood to be the dimension of F over R. A *standard R-basis* of F is an *R-basis* $\{a_1, \ldots, a_{\dim. F}\}$ such that $a_1 = 1 = -a_i^2$ for $i > 1, a_i a_j = -a_j a_i$ for $i \neq j, i > 1, j > 1$ and $\pm a_i a_j \in \{a_k\}$ for each i, j. Given $x \in F, |x|$ is the positive square root of $x\bar{x}$.

6.2. F' is a compact topological group, so any discrete subgroup is finite. The finite subgroups of R' are 1 and $\{\pm 1\}$. The finite subgroups of C' are the cyclic groups Z_q of any finite order $q > 0$; Z_q is generated by $\exp(2\pi\sqrt{-1}/q)$. The finite subgroups of K' are cyclic or binary polyhedral.

We recall the binary polyhedral groups. The *polyhedral groups* are the dihedral groups $\mathfrak{D}_m$, the tetrahedral group $\mathfrak{T}$, the octahedral group $\mathfrak{O}$ and the icosahedral group $\mathfrak{J}$ – the respective groups of symmetries of the regular m-gon, the regular tetrahedron, the regular octahedron and the regular icosahedron. Each polyhedral group has a natural imbedding in $SO(3)$. Let $\pi : K' \to SO(3)$ be the universal covering; the *binary polyhedral groups* are the *binary dihedral groups* $\mathfrak{D}^*_m = \pi^{-1}(\mathfrak{D}_m)$, the *binary tetrahedral group* $\mathfrak{T}^* = \pi^{-1}(\mathfrak{T})$, the *binary octahedral group* $\mathfrak{O}^* = \pi^{-1}(\mathfrak{O})$, and the *binary icosahedral group* $\mathfrak{J}^* = \pi^{-1}(\mathfrak{J})$. We exclude $\mathfrak{D}^*_1$ because it is cyclic. Note that isomorphic finite subgroups of K' are conjugate.

6.3. Lemma. *Let D be a subgroup of F^* and set $D' = D \cap F'$. Then D is a discrete subgroup of F^* if and only if D' is finite and $D = \{D', d\}$ for some $d \in D$ with $|d| \geqq 1$.*

Proof. Suppose D discrete. If $D = D'$, set $d = 1$. Otherwise, note that $D_1 = \{|g| : g \in D\}$ is a discrete subgroup of R^*, for D cannot have an infinite number of elements in a compact neighborhood of F'. It easily follows that D_1 is generated by an element $|d|$ which minimizes $|d| - 1 > 0$. To see $D = \{D', d\}$, we note that every $g \in D$ satisfies $|g| = |d|^m$ for some integer m; thus $gd^{-m} \in D'$, so $g \in \{D', d\}$.

The converse is trivial. *Q.E.D.*

7. The 6 classical families of space forms

7.1. Let $0 \leqq s < m$ be integers, choose a basis $\{z_j\}$ in a left m-dimensional vectorspace V over F, and consider the hermitian form $Q_F(u, v) = - \sum_1^s u_j \bar{v}_j + \sum_{s+1}^m u_j \bar{v}_j$ where $u = \sum u_j z_j$ and $v = \sum v_j z_j$. The real part $Q(u, v) = \mathcal{R}e\,(Q_F(u, v))$ is given by $Q(u, v) = - \sum_1^{sr} x_j y_j + \sum_{sr+1}^{mr} x_j y_j$ where $r = \dim. F$, $\{a_i\}$ is a standard R-basis of F, $\{w_j\}$ is the corresponding R-basis of V, given by $a_i z_j = w_{r(j-1)+i}$, $u = \sum x_j w_j$ and $v = \sum y_j w_j$. As $Q_F(u, u)$ is real, this allows us to identify Q_{rs}^{mr-1} with the hermitian quadric of equation $Q_F(u, u) = 1$. Under this identification, both F' and the unitary group $U^s(m, F)$ of Q_F becomes subgroups of the orthogonal group $O^{sr}(mr)$ of Q. The actions of F' and $U^s(m, F)$ commute because $U^s(m, F)$ is F-linear, $U^s(m, F)$ is transitive on Q_{sr}^{mr-1}, and F' acts freely on Q_{sr}^{mr-1}. If D is a finite subgroup of F', it follows that S_{sr}^{mr-1}/D is a homogeneous pseudo-RIEMANNIAN manifold M_{sr}^{mr-1} of constant curvature $+1$. We will refer to these manifolds as the *classical space forms*; special cases are studied in [5] and [6].

7.2. Let $\mathcal{S}$ denote the family of connected classical space forms for $F = R$. More precisely, to every triple (h, n, t) of integers with $0 \leqq h \leqq n$, $1 \leqq t \leqq 2$, and $t = 2$ if $h = n$, there corresponds precisely one element S_h^n/Z_t of $\mathcal{S}$. Note that S_n^n/Z_2 is isometric to a component of S_n^n. The elements of $\mathcal{S}$ are the *spherical* and *eliptic spaces* of indefinite metric.

Let $\mathcal{Z}$ denote the family of classical space forms for $F = C$ which are not in $\mathcal{S}$. To every triple (h, n, t) of integers with $0 \leqq h \leqq n, t > 2, h$ even and $n + 1$ even, there corresponds precisely one element S_h^n/Z_t of $\mathcal{Z}$. The elements of $\mathcal{Z}$ are the *cyclic spaces* of indefinite metric; they correspond to the *Linsenräume* of THRELFALL and SEIFERT [5].

Let $\mathcal{D}$ denote the family of classical space forms for $F = K$ and D binary dihedral. To every triple (h, n, t) of integers with $0 \leqq h \leqq n, t > 1$, and both h and $n + 1$ divisible by 4, there corresponds exactly one element $S_h^n/\mathfrak{D}_t^*$ of $\mathcal{D}$. The elements of $\mathcal{D}$ are the *dihedral spaces* of indefinite metric; they correspond to the *Prismaräume* of THRELFALL and SEIFERT [5].

Let $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{J}$ denote the families of classical space forms for $F = K$, with D respectively binary tetrahedral, binary octahedral and binary icosahedral. To every pair (h, n) of integers with $0 \leqq h \leqq n$ and both h and $n + 1$ divisible by 4, there corresponds precisely one element $S_h^n/\mathfrak{T}^*$ of $\mathcal{T}$, one element $S_h^n/\mathfrak{O}^*$ of $\mathcal{O}$, and one element $S_h^n/\mathfrak{J}^*$ of $\mathcal{J}$. These are the *tetrahedral octahedral* and *icosahedral spaces* of indefinite metric; they correspond to the *Tetraederraum*, *Oktaederraum* and *Dodekaederraum* of THRELFALL and SEIFERT [5].

8. Behavior of quadric under field extension

We will consider material converse to that of § 7.1 and discuss reducible groups which are transitive on quadrics.

8.1. Lemma. *Let* Q *be the bilinear form* $Q(u, v) = -\sum_{1}^{h} u_j v_j + \sum_{h+1}^{n+1} u_j v_j$ *on* $V = R^{n+1}$, *so* Q_h^n *is given by* $Q(u, u) = 1$, *and* $O^h(n + 1)$ *is the orthogonal group of* Q. *Let* $G \subset O^h(n + 1)$ *be transitive on* Q_h^n, *let* F *be a real division subalgebra of* $\mathcal{E}(V)$ *which centralizes* G, *and let* $r = \dim. F$. *Then* $F' \subset O^h(n + 1)$, $s = h/r$ *and* $m = (n + 1)/r$ *are integers, and there is a* Q-*orthonormal* R-*basis* $\{w_j\}$ *of* V *and a standard* R-*basis* $\{a_j\}$ *of* F *such that* $w_{rj+i} = a_i w_{rj+1}$, $\{w_1, w_{r+1}, \ldots, w_{r(m-1)+1}\}$ *is an* F-*basis of* V, *and* $Q(u, v) = \mathcal{R}e\left(-\sum_{1}^{s} x_j \bar{y}_j + \sum_{s+1}^{m} x_j \bar{y}_j\right)$ *where* $u = \sum x_j w_{r(j-1)+1}$ *and* $v = \sum y_j w_{r(j-1)+1}$ *with* $x_j, y_j \in F$.

Proof. Let $a \in F'$. Given $u, v \in Q_h^n$ we have some $g \in G$ with $g(u) = v$, so $Q(av, av) = Q(agu, agu) = Q(gau, gau) = Q(au, au)$; thus $f(a) = Q(av, av)$ for any $v \in Q_h^n$ is well defined. Let $w \in V$ with $Q(w, w) > 0$. There is a real $x > 0$ with $xw \in Q_h^n$, so $Q(aw, aw) = x^{-2}Q(axw, axw) = x^{-2}f(a) = f(a)Q(w, w)$. Note that $f(a) > 0$: this is clear for $F = R$; F' is connected if $F \neq R$, so we need only check that $f(a) \neq 0$. As $f(a) = 0$ would imply that a is a homeomorphism of the open $(n + 1)$-dimensional set $P = \{w \in V : Q(w, w) > 0\}$ into the lower dimensional set $L = \{w \in V : Q(w, w) = 0\}$, this is clear. Now let $a, b \in F'$, and note that $(v \in Q_h^n)$ $f(ab) = Q(abv, abv) = f(a)Q(bv, bv) = f(a)f(b)$; thus f is a homomorphism of the compact group F' into the multiplicative group of positive real numbers; it follows that $f(F') = 1$.

We now have $Q(w, w) = Q(aw, aw)$ for $a \in F'$ and $w \in P$. By continuity of $Q(aw, aw) - Q(w, w)$, this also holds if w is in the boundary L of P. Polarization shows $Q(au, av) = Q(u, v)$ if u, v and $u + v$ lie in $P \cup L$. Now let $\{v_j\}$ be a Q-orthonormal R-basis of V, set $u_j = v_j$ for $j > h$, and set $u_j = v_j + \sqrt{2}v_{n+1}$ for $j \leqq h$. One easily checks that $\{u_j\}$ is a basis of V such that u_i, u_j and $u_i + u_j \in P \cup L$ for every (i, j); it follows that $Q(au, av) = Q(u, v)$ for $a \in F'$ and $u, v \in V$. Thus $F' \subset O^h(n + 1)$.

Let $\{a_i\}$ be any standard R-basis of F, and choose $w_1 \in V$ such that $Q(w_1, w_1) = -1$ if $h > 0, Q(w_1, w_1) = 1$ if $h = 0$, set $w_i = a_i w_1$, and let V_1 be the Q-orthogonal complement of $\{w_1, \ldots, w_r\}$. Choose $w_{r+1} \in V_1$ such that $Q(w_{r+1}, w_{r+1}) = -1$ if $h > r, Q(w_{r+1}, w_{r+1}) = 1$ if $h = r$, set $w_{r+i} = a_i w_{r+1}$, and let V_2 be the Q-orthogonal complement of $\{w_1, \ldots, w_{2r}\}$. Continuing this process, we obtain the desired basis and see that s and m are integers. *Q. E. D.*

8.2. In the following, we will not always mention Q explicitly. However, a totally isotropic subspace will mean a totally Q-isotropic subspace (one on which Q vanishes identically), orthonormal will mean Q-orthonormal, and orthogonal «complements» will be taken relative to Q unless we state otherwise.

Lemma. *Let $G \subset O^h(n + 1)$ be transitive on Q_h^n, let U be a proper G-invariant subspace of $V = R^{n+1}$, set $W = U \cap U^\perp$, and let $k = n + 1 - h$. Then $2h > n$, W is a G-invariant maximal totally isotropic subspace of V, G acts irreducibly on W, and V has an orthonormal basis $\{w_j\}$ such that $(e_j = w_{h+j} + w_j$ for $1 \leqq j \leqq k)$ $\{e_1, \ldots, e_k\}$ is a basis of W and $\{w_{k+1}, \ldots, w_h, e_1, \ldots, e_k\}$ is a basis of $W^\perp$. Finally, if F is a real division subalgebra of $\mathcal{E}(V)$ which centralizes G and preserves U, then W and $W^\perp$ are F-invariant and we may take $\{w_j\}$ to satisfy the conclusion of Lemma* 8.1.

Proof. If F is not given, set $F = R$. As U and $U^\perp$ are proper G-invariant subspaces of V, and as Q_h^n contains a basis of V, it follows that Q is negative semi-definite on U and on $U^\perp$. Were Q nondegenerate on U, we would have $V = U \oplus U^\perp$ with Q negative definite on each summand, so Q would be negative definite on V. Thus Q is degenerate on U, so $W \neq 0$. Now W is a proper G-invariant totally isotropic subspace of V, F-invariant because $F' \subset O^h(n + 1)$ (by Lemma 8.1) implies that U and $U^\perp$ are F-invariant. Thus we may assume $W = U$. W and V are left vectorspaces over F. Set $Q(v) = Q(v, v)$ for $v \in V$.

Lemma 8.1 gives an F-invariant orthogonal decomposition $V = V_1 \oplus V_2$ with Q negative definite on V_1 and positive definite on V_2. Choosing an F-basis $\{b_1, \ldots, b_t\}$ of W, we have $b_i = r_i + s_i$ with $r_i \in V_1, s_i \in V_2$.

Set $e_1 = p_1 + q_1 = Q(s_1)^{-\frac{1}{2}} b_1$ with $p_1 \in V_1, q_1 \in V_2$. We construct $e_u = p_u + q_u$ with $p_u \in V_1, q_u \in V_2$ from $\{e_1, \ldots, e_{u-1}, b_u\}$ by setting $z_u = b_u - \sum_{j<u} Q(q_j, s_u) e_j = x_u + y_u$ with $x_u \in V_1, y_u \in V_2$, and by $e_u = Q(y_u)^{-\frac{1}{2}} z_u$. Now $\{p_1, \ldots, p_t; q_1, \ldots, q_t\}$ is linearly independent over F, $Q(p_i, q_j) = 0$, and $Q(q_i, q_j) = \delta_{ij} = -Q(p_i, p_j)$. By Lemma 8.1, we can complete it to an orthonormal F-basis $\{p_1, \ldots, p_t, c_1, \ldots, c_a; q_1, \ldots, q_t, d_1, \ldots, d_b\}$ of V. $W^\perp$ then has F-basis $\{e_1, \ldots, e_t, c_1, \ldots, c_a, d_1, \ldots, d_b\}$ and Q is negative semi-definite on $W^\perp$, so $b = 0$. Now let $r = \dim. F$ and let $\{a_i\}$ be a standard R-basis of F. Set $w_{r(j-1)+1} = p_j$ for $1 \leqq j \leqq t$, $w_{r(t+j-1)+1} = c_j$ for $1 \leqq j \leqq a$, $w_{r(t+a+j-1)+1} = q_j$ for $1 \leqq j \leqq t$, and $w_{rj+i} = a_i w_{rj+1}$. The Lemma is now easily verified. *Q. E. D.*

8.3. Lemma. *Let $G \subset O^h(n+1)$ be transitive on Q^n_h, let U and W be proper G-invariant subspaces of $V = R^{n+1}$, and suppose $U \cap W = 0$. Then $2h = n+1$, U and W are maximal totally isotropic subspaces of V, and $V = U \oplus W$.*

Proof. Let $k = n + 1 - h$, $U' = U \cap U^\perp$ and $W' = W \cap W^\perp$. By Lemma 8.2, both U' and W' are maximal totally isotropic and are k-dimensional, so we need only show $V = U' \oplus W'$. Were $Q(U', W') = 0$, $U' \oplus W'$ would be totally isotropic, contradicting the fact that U' is *maximal* totally isotropic. It follows that $U' \oplus W'$ has an element v with $Q(v, v) = 1$, so $Q^n_h \subset U' \oplus W'$ because G is transitive on Q^n_h and preserves $U' \oplus W'$. Thus $U' \oplus W' = V$. *Q. E. D.*

We will now consider the existence of a $G \subset O^h(n+1)$ which is transitive on Q^n_h and preserves two linearly disjoint proper subspaces of V.

8.4. Given integers $0 \leqq s < m$, we consider the hermitian form $Q_F(u, v) = -\sum_1^s u_j \bar{v}_j + \sum_{s+1}^m u_j \bar{v}_j$ on a left m-dimensional vectorspace V over F. As before, $r = \dim. F$, $Q(u, v) = \mathcal{R}e\,(Q_F(u, v))$, and Q^{mr-1}_{sr} is given by $Q_F(u, u) = 1$.

Let U be an F-subspace of V. If U is totally Q_F-isotropic, it clearly is totally Q-isotropic. Suppose, conversely, that U is totally Q-isotropic, let $u, v \in U$, and choose a standard R-basis $\{a_i\}$ of F. $Q_F(u, v) = \Sigma_i a_i x_i$ with $x_i \in R$; $0 = Q(a_i u, v) = -x_i$ shows $Q_F(u, v) = 0$; thus U is totally Q_F-isotropic.

The unitary group $U^s(m, F)$ of Q_F is the centralizer of F' in $O^{sr}(mr)$. One inclusion is obvious and has already been noted, so we need only show that an element $g \in O^{sr}(mr)$ which centralizes F' lies in $U^s(m, F)$. Centralizing F', g is F-linear. Let $u, v \in V$ and note $\mathcal{R}e\,(Q_F(a_i u, v)) = Q(a_i u, v) =$

$= Q(ga_i u, gv) = Q(a_i gu, gv) = \mathcal{R}e(Q_F(a_i gu, gv))$. It follows that $Q_F(u,v) = Q_F(gu, gv)$, so $g \in U^s(m, F)$.

Combining these two facts with Lemma 8.3, we see the usefulness of:

Lemma. *Consider the hermitian form* $Q_F(u,v) = -\sum_1^s u_j\bar{v}_j + \sum_{s+1}^{2s} u_j\bar{v}_j$ *on a left* $2s$*-dimensional vectorspace* V *over* F, *let* U *and* W *be totally* Q_F*-isotropic* F*-subspaces of* V *such that* $V = U \oplus W$. *Then there are* F*-bases* $\{e_i\}$ *of* U *and* $\{f_j\}$ *of* W *with* $Q_F(e_i, f_j) = 2\delta_{ij}$. *Let* f *represent the general linear group* $GL(s, F)$ *on* V *by* $f(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\bar{\alpha}^{-1}\end{pmatrix}$ *in the basis* $\{e_i, f_j\}$ *of* V, *and let* $G = \{g \in U^s(2s, F) : Ug = U, Wg = W\}$. *Then* G *is the image of* f, *and* G *is transitive on* $Q_F(u,u) = 1$ *if and only if* $F = R$.

Proof. The bases $\{e_i\}$ and $\{f_j\}$ are easily constructed by the method of Lemma 8.2. It is easy to check that G is the image of f, and the last statement is obvious for $s = 1$. Now assume $s > 1$, and we'll prove the last assertion. For $1 \leq j \leq s$, set $w_j = \frac{1}{2}(e_j - f_j)$ and $w_{s+j} = \frac{1}{2}(e_j + f_j)$, so $\{w_j\}$ is a Q_F-orthonormal basis of V over F. As $Q_F(w_{s+1}, w_{s+1}) = 1$, we see that G is transitive if and only if, given $x \in V$ with $Q_F(x,x) = 1$, there is an $\alpha \in GL(s, F)$ with $f(\alpha) : w_{s+1} \to x$. Let $x = \sum x_j w_j$; it is easy to check that $f(\alpha) : w_{s+1} \to x$ if and only if

$$\alpha = \begin{pmatrix} x_{s+1} + x_1, \ldots, x_{2s} + x_s \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{pmatrix} \text{ and } {}^t\bar{\alpha}^{-1} = \begin{pmatrix} x_{s+1} - x_1, \ldots, x_{2s} - x_s \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{pmatrix}.$$

As $\alpha\alpha^{-1} = I$, this would give $1 = \sum_1^s (x_{s+j} + x_j)(\overline{x_{s+j} - x_j}) = -\sum_1^s x_j\bar{x}_j + \sum_1^s x_{s+j}\overline{x_{s+j}} + \sum_1^s (x_j\overline{x_{s+j}} - x_{s+j}\bar{x}_j)$. $1 = Q_F(x,x) = -\sum_1^s x_j\bar{x}_j + \sum_1^s x_{s+j}\overline{x_{s+j}}$ then gives $\sum_1^s x_j\overline{x_{s+j}} = \sum_1^s x_{s+j}\bar{x}_j$, so $\sum_1^s x_j\overline{x_{s+j}} \in R \subset F$. If $F \neq R$ we can always find an $x \in V$ with $Q_F(x,x) = 1$ and $\sum_1^s x_j\overline{x_{s+j}} \notin R$, so G will not be transitive on $Q_F(u,u) = 1$.

We now assume $F = R$, write $Q = Q_F$, identify Q_s^{2s-1} with $Q_F(u,u) = 1$, and will show G transitive on Q_s^{2s-1}. We set $z = x_{s+1} + x_1$, $y = x_{s+1} - x_1$, $r = (x_{s+2} + x_2, \ldots, x_{2s} + x_s)$, and $t = (x_{s+2} - x_2, \ldots, x_{2s} - x_s)$. We wish to find $\alpha = \begin{pmatrix} z & r \\ {}^tu & A\end{pmatrix} \in GL(s, R)$ with $\alpha^{-1} = \begin{pmatrix} y & v \\ {}^tt & B\end{pmatrix}$. If $y \neq 0$, we set $\alpha = \begin{pmatrix} z & r \\ -y^{-1}\cdot{}^tt & I\end{pmatrix}$ and have $\alpha^{-1} = \begin{pmatrix} y & -y\cdot r \\ {}^tt & I - {}^tt\cdot r\end{pmatrix}$. If $z \neq 0$, we set $\alpha^{-1} = \begin{pmatrix} y & -z^{-1}\cdot r \\ {}^tt & I\end{pmatrix}$ and have $\alpha = \begin{pmatrix} z & r \\ -z^t\cdot t & I - {}^tt\cdot r\end{pmatrix}$. Now suppose $x = z = 0$, note that a solution would give $\alpha\alpha^{-1} = \begin{pmatrix} 1 & r\cdot B \\ A\cdot{}^tt & {}^tu\cdot v + A\cdot B\end{pmatrix}$, put the usual euclidean inner product (a, b) on the space R^{s-1} of real $(s-1)$-tuples, and observe that $r, t, u, v \in R^{s-1}$. Let X be an $(s-3)$-dimensional subspace of $r^\perp \cap t^\perp$, $\perp$ taken relative to $(\ ,\ )$, and let $\{y_3, \ldots, y_{s-1}\}$ be an orthonormal base of X. $(r, t) = Q(x, x) = 1$, so we can extend $\{y_j\}$ to bases $\{e, y_3, \ldots, y_{s+1}\}$ of $t^\perp$ and $\{d, y_3, \ldots, y_{s-1}\}$ of $r^\perp$ with $(d, e) = 1$ and $(d, y_j) = 0 = (e, y_j)$. Now pick $u_i, v_i \in R$ with $u_1 v_1 + u_2 u_2 = 1$. We set $B = (u_1{}^t d, u_2{}^t d, {}^t y_3, \ldots, {}^t y_{s-1})$ and ${}^t A = (v_1{}^t e, v_2{}^t e, {}^t y_3, \ldots, {}^t y_{s-1})$, so $r \cdot B = 0$ and $A \cdot {}^t t = 0$ by construction. Set $u = (u_2, -u_1, 0, \ldots, 0) \in R^{s-1}$ and $v = (v_2, -v_1, 0, \ldots, 0) \in R^{s-1}$; $1 = u_1 v_1 + u_2 v_2$ gives ${}^t u \cdot v + A \cdot B = I$, so the desired α and α^{-1} in $GL(s, R)$ are constructed. *Q.E.D.*

8.5. We end § 8 with two more lemmas, placed here for lack of a better location.

Lemma. *Let $G \subset O^h(2h)$ be transitive on Q_h^{2h-1}, let U and W be proper G-invariant subspaces of $V = R^{2h}$ with $U \cap W = 0$, and let G_U be the restriction of G to U. Then there is no nonzero G_U-invariant bilinear form on U.*

Proof. Let $B(u, v)$ be a G_U-invariant bilinear form. If B is not skew, we may replace it with $B'(u, v) = B(u, v) + B(v, u)$, which is symmetric. But the proof of Lemma 8.4 and the irreducibility of G_U shows that G_U has no nonzero symmetric bilinear invariant; we conclude that B is skew. Assume $B \neq 0$, so B is nondegenerate by the irreducibility of G_U. Lemma 8.4 gives us an orthonormal basis $\{v_i\}$ of V such that ($e_i = v_{h+i} + v_i$ and $f_i = v_{h+i} - v_i$ for $1 \leqq i \leqq h$) $\{e_i\}$ is a basis of U and $\{f_i\}$ a basis of W, and every $g \in G$ is of the form $g = \begin{pmatrix} g_1 & 0 \\ 0 & {}^t g_1{}^{-1} \end{pmatrix}$ in the basis $\{e_i, f_j\}$ of V. We replace G by $G' = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & {}^t g_1{}^{-1} \end{pmatrix} : g_1 \text{ preserves } B \right\}$ which, containing G, is transitive on Q_h^{2h-1}. Let G'' be the subgroup of G' given by taking g_1 to be of the form $g_1 = \begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & \\ \vdots & g_1' \\ 0 & \end{pmatrix}$. One checks $\dim. G' = \frac{1}{2}h(h+1)$ and $\dim. G'' = \frac{1}{2}h(h-1) - 1$; observing that G'' is the isotropy subgroup of G' at $v_{h+1} \in Q_h^{2h-1}$, it follows that $h - 1 = \dim. G' - \dim. G'' = \dim. Q_h^{2h-1} = 2h - 1$, so $h = 0$. This contradicts $B \neq 0$. *Q.E.D.*

8.6. Lemma. *Let D be a subgroup of $O^h(n+1)$, let G be the centralizer of D in $O^h(n+1)$, suppose G transitive Q_h^n, and let U be a maximal totally isotropic G-invariant subspace of $V = R^{n+1}$. Then U is D-invariant.*

Proof. Let $d \in D$. Then dU is G-invariant and, as G is irreducible on U, either $dU = U$ or $dU \cap U = 0$. In the latter case, $2h = n + 1$ and

$V = U \oplus dU$ by Lemma 8.3; $B(u, v) = Q(u, dv)$ is a G_U-invariant bilinear form on U, so $B = 0$ by Lemma 8.5. Thus $Q(U, dU) = 0$ so $Q = 0$, contradicting $dU \frown U = 0$. Thus $dU = U$. *Q.E.D.*

9. The 8 exceptional families of space forms

9.1. In an orthonormal basis $\{v_j\}$ of $V = R^{2h}$, we have

$$r_H(s) = \begin{pmatrix} \cosh(s) I_h & \sinh(s) I_h \\ \sinh(s) I_h & \cosh(s) I_h \end{pmatrix} \in O^h(2h),$$

where $s > 0$ is a real number and I_h is the $h \times h$ identity matrix. To check this, we set $e_j = v_{h+j} + v_j$ and $f_j = v_{h+j} - v_j$ for $1 \leqq j \leqq h$, observe that the hyperbolic rotation $r_H(s) = \begin{pmatrix} \exp(-s) I_h & 0 \\ 0 & \exp(s) I_h \end{pmatrix}$ in the basis $\{f_i, e_j\}$ of V, and refer to the first part of Lemma 8.4. Lemma 8.4 also tells us that the centralizer G of $r_H(s)$ in $O^h(2h)$ is transitive on Q_h^{2h-1}. Note that $r_H(s)$ is conjugate to $r_H(t)$ if and only if $s = t$, t also being taken positive, and that for a given s the subgroups $\mathfrak{H}(s, +) = \{r_H(s)\}$, $\mathfrak{H}(s, -) = \{-r_H(s)\}$ and $\mathfrak{H}(s, \pm) = \{-I_{2h}, r_H(s)\}$ are discrete subgroups of $O^h(2h)$. It follows that these groups are free and properly discontinuous on Q_h^{2h-1}, and that the quotients $S_h^{2h-1}/\mathfrak{H}(s, +)$, $S_h^{2h-1}/\mathfrak{H}(s, -)$ and $S_h^{2h-1}/\mathfrak{H}(s, \pm)$ are homogeneous manifolds of constant positive curvature. The class $\mathcal{H}_r$ of these *hyperbolic-rotation spaces* is our first class of exceptional space forms. To every triple $(h, s, +$ or $-$ or $\pm)$ where $h > 1$ is an integer and $s > 0$ is real, there corresponds exactly one element $S_h^{2h-1}/\mathfrak{H}(s, +$ or $-$ or $\pm)$ of $\mathcal{H}_r$, and distinct elements of $\mathcal{H}_r$ are not isometric. Note that $\mathfrak{H}(s, +) \cong \mathbf{Z} \cong \mathfrak{H}(s, -)$ and $\mathfrak{H}(s, \pm) \cong \mathbf{Z} \times \mathbf{Z}_2$.

9.2. In an orthonormal basis $\{v_j\}$ of $V = R^{n+1}$ with $2h > n$ and $k = n + 1 - h$ even, we choose a skew nonsingular $k \times k$ matrix d and have

$$t_H(d) = \begin{pmatrix} I_k - d & 0 & -d \\ 0 & I_{h-k} & 0 \\ d & 0 & I_k + d \end{pmatrix} \in O^h(n+1).$$

This is best seen by setting $e_j = v_{h+j} + v_j$ and $f_j = v_{h+j} - v_j$ for $1 \leqq j \leqq k$ and observing that $t_H(d) = \begin{pmatrix} I_k & 0 & 2d \\ 0 & I_{h-k} & 0 \\ 0 & 0 & I_k \end{pmatrix}$ in the basis $\beta = \{f_1, \ldots, f_k; v_{k+1}, \ldots, v_h; e_1, \ldots, e_k\}$ of V.

If a linear transformation of V commutes with $t_H(d)$, it must preserve the subspace U of basis $\{e_i\}$. Thus the centralizer G of $t_H(d)$ in $O^h(n+1)$ preserves both U and $U^\perp$, and consequently is in upper triangular block form $g = \begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{pmatrix}$ relative to β. For a matrix of that form, $g \cdot t_H(d) = t_H(d) \cdot g$

is expressed by $dg_6 = g_1 d$, and $g \in O^h(n+1)$ is given by $g_6 = {}^t g_1^{-1}$, $g_4 \in O(h-k)$ ordinary orthogonal group, $2\langle (g_1)_i, (g_3)_j\rangle + 2\langle (g_3)_i, (g_1)_j\rangle = \langle (g_2)_i, (g_2)_j\rangle$ and $2\langle (g_1)_i, (g_5)_j\rangle = \langle (g_2)_i, (g_4)_j\rangle$ where $(g_u)_v$ denotes the v^{th} row of g_u and $\langle \, , \, \rangle$ is the ordinary euclidean pairing of m-tuples.

We first note that, if d' is another skew nonsingular $k \times k$ matrix, then $t_H(d')$ is conjugate to $t_H(d)$ by an element of $O^h(n+1)$ which preserves U. For the associated skew bilinear forms on R^k are each equivalent to $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, so there is a nonsingular $k \times k$ matrix g_1 with $g_1 d\, {}^t g_1 = d'$. Now set $g = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & {}^t g_1^{-1} \end{pmatrix}$ and notice $g \cdot t_H(d) g^{-1} = t_H(d')$, so the subgroups $\mathfrak{T}(+) = \{t_H(d)\}$, $\mathfrak{T}(-) = \{-t_H(d)\}$ and $\mathfrak{T}(\pm) = \{-I_{n+1}, t_H(d)\}$ of $O^h(n+1)$ are defined up to conjugacy without reference to any particular d. These subgroups are free and properly discontinuous on Q^n_h; we will see that the corresponding quotients of S^n_h are homogeneous.

We must show G transitive on Q^n_h. We take any $x = \sum_1^k a_j v_j + \sum_1^{h-k} b_j v_{k+j} + \sum_1^k c_j v_{h+j} \in Q^n_h$, and wish to find $g \in G$ with $g(v_{h+1}) = x$. Calculating $g(v_{h+1})$, we see that this is equivalent to $c - a = (g_1)_1$, $c + a = (g_3)_1 + (g_6)_1$ and $2b = (g_2)_1$, where $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_{h-k})$ and $c = (c_1, \ldots, c_k)$, and we add coordinatewise. As $H = \{g_1 \in GL(k, R) : {}^t g_1 d g_1 = d\}$ is the real sympletic group, transitive on the nonzero elements of R^k, we may choose $g_1 \in H$ with $c - a = (g_1)_1$. $g_6 = {}^t g_1^{-1}$ is then determined, and $(g_2)_1$ and $(g_3)_1$ are given by $gv_{h+1} = x$. We must fill out g_2 and g_3, and then construct g_4 and g_5. Suppose we have the first $u - 1$ rows of g_2 and g_3, $2 \leqq u \leqq k$, and $2\langle (g_1)_i, (g_3)_j\rangle + 2\langle (g_3)_i, (g_1)_j\rangle = \langle (g_2)_i, (g_2)\rangle$ for $i, j < u$, with $(g_2)_i = 0$ for $1 < i < u$. Set $(g_2)_u = 0$; the conditions on $(g_3)_u$ are then $\langle (g_1)_i, (g_3)_u\rangle + \langle (g_3)_i, (g_1)_u\rangle = 0$ for $1 \leqq i \leqq u$, so we have linear forms p_i on R^k and constants q_i such that these conditions are $p_i((g_3)u) + q_i = 0$ for $1 \leqq i \leqq u$. The rows of g_1, hence the p_i, are independent; thus we have a solution $(g_3)_u$. Thus g_2 and g_3 are constructed. Set $g_4 = I_{h-k}$, and the existence of g_5 follows, as above, from the independence of the rows of g_1. Thus G is transitive on Q^n_h, so $S^n_h/\mathfrak{T}(+)$, $S^n_h/\mathfrak{T}(-)$ and $S^n_h/\mathfrak{T}(\pm)$ are homogeneous manifolds of constant positive curvature. The class $\mathcal{H}_t$ of these *hyperbolic-translation spaces* is our second class of exceptional space forms. To every triple $(h, n, + \text{ or } - \text{ or } \pm)$, where h and n are integers with $2h > n > 1$ and $k = n + 1 - h$ even, there corresponds exactly one element $S^n_h/\mathfrak{T}(+ \text{ or } - \text{ or } \pm)$ of $\mathcal{H}_t$. Notice that $\mathfrak{T}(+) \cong Z \cong \mathfrak{T}(-)$ and $\mathfrak{T}(\pm) \cong Z \times Z_2$.

9.3. Assume $2h > n > 1$ and that $k = n + 1 - h$ is divisible by 4; for every complex number $u \notin R$, we have groups $\mathfrak{L}(u, + \text{ or } - \text{ or } \pm)$ which give homogeneous space forms. If $\mathfrak{T}(+)$ is viewed as the additive group of integers, then the free abelian part of $\mathfrak{L}$ is the additive lattice in C generated by 1 and u.

We retain the notation of § 9.2, set $q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and define $J = \begin{pmatrix} q & & \\ & \ddots & \\ & & q \end{pmatrix} \in GL(k, R)$. Now let d be any skew nonsingular element of $GL(k, R)$ such that $dJ + Jd = 0$. As $J^2 = -I_k$, we may identify C with the subalgebra of $\mathcal{E}(U)$ consisting of all $aI + bJ$ with $a, b \in R$. Given $u = aI + bJ \in C$ with $b \neq 0$, we define subgroups

$$\begin{aligned} \mathfrak{L}(u, +) &= \{t_H(d), t_H(du)\} \\ \mathfrak{L}(u, -) &= \{t_H(d), -t_H(du)\} \\ \mathfrak{L}(u, \pm) &= \{-I_{n+1}, t_H(d), t_H(du)\} \end{aligned}$$

of $O^h(n + 1)$. As dv is skew and nonsingular for $v \in C^*$ and $t_H(dv_1) \cdot t_H(dv_2) = t_H(d(v_1 + v_2))$ for $v_i \in C$, these *lattice groups* are free and properly discontinuous on Q_h^n. Let $\mathcal{L}$ be the collection of space forms $S_h^n/\mathfrak{L}(u, + \text{ or } - \text{ or } \pm)$; we must check that the elements of $\mathcal{L}$ are homogeneous.

Let $g = \begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{pmatrix} \in O^h(n + 1)$ in the basis β of V; g centralizes $\mathfrak{L}(u, + \text{ or } - \text{ or } \pm)$ if and only if $g_6 J = J g_6$ and $dg_6 = g_1 d$. We now may proceed as in the proof that the elements of $\mathcal{H}_t$ are homogeneous, except that H must be replaced by $H' = \{g_1 \in GL(k, R) : {}^t g_1 d g_1 = d, g_1 J = J g_1\}$; we need only check that H' is transitive on the nonzero elements of R^k. Let $2s = k$; we give R^k the complex structure defined by J; H' becomes $\{g_1 \in GL(s, C) : {}^t\bar{g}_1 \cdot d_c \cdot g_1 = d_c\}$ where d_c is the skew C-linear map $x \to \overline{d(x)}$ and conjugation is with respect to a C-basis of the form $\{e_{2j-1}\}$, whence the transitivity of H' on $R^k - \{O\} = C^s - \{O\}$ is clear. Thus the *lattice space forms* $S_h^n/\mathfrak{L}(u, + \text{ or } - \text{ or } \pm)$ are homogeneous. One can check that $\mathfrak{L}(u, + \text{ or } - \text{ or } \pm)$ is defined up to conjugacy in $O^h(n + 1)$ without reference to d or J, and that $(x, y = + \text{ or } - \text{ or } \pm)$ $\mathfrak{L}(u, x)$ is conjugate to $\mathfrak{L}(v, y)$ if and only if $x = y$ and the lattices $\{1, u\}$ and $\{1, v)$ in C are equal.

9.4. Assume $2h > n > 1$, h is even and $k = n + 1 - h$ is divisible by 4; we then have an infinite dihedral group $\mathfrak{D}_\infty^*$, extension of $\mathfrak{T}(\pm)$ by an automorphism of order 2, and, for $m = 3, 4, 6$ or 8, some extensions $\mathfrak{L}_m(u)$ of the lattice groups.

We retain the notation of § 9.2 and, as before, set $q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} q & & \\ & \ddots & \\ & & q \end{pmatrix} \in GL(k, R)$, and choose a skew nonsingular $k \times k$ matrix d with $Jd + dJ = 0$. Let $J' = \begin{pmatrix} q & & \\ & \ddots & \\ & & q \end{pmatrix} \in GL(h-k, R)$ and let $J_{n+1} = \begin{pmatrix} J & 0 & 0 \\ 0 & J' & 0 \\ 0 & 0 & J \end{pmatrix}$ relative to β; $J_{n+1} \in O^h(n+1)$. Finally, $u = aI_k + bJ, b \neq 0$, represents a non-real complex number. $\mathfrak{D}^*_\infty = \{J_{n+1}, t_H(d)\}$ is an infinite dihedral group $(x, y : x^4 = 1, xyx^{-1} = y^{-1})$.

R_m is the rotation $\cos(2\pi/m) I_{h+1} + \sin(2\pi/m) J_{n+1}$ of V. $R_m \in O^h(n+1)$. Note that $R_4 = J_{n+1}$. Let $m = 3, 4, 6$ or 8, and suppose that the lattice $\{1, u\}$ in C is invariant under left multiplication by $\cos(4\pi/m) + \sqrt{-1}\sin(4\pi/m)$, where $\sqrt{-1}$ is chosen such that $u = a + \sqrt{-1}b$. We then set $\mathfrak{L}_m(u) = \{R_m, t_H(d), t_H(ud)\}$. $\mathfrak{L}_m(u)$ is free on Q^n_h, and acts properly discontinuously because, as

$$R_m \cdot t_H(vd) \cdot R_m^{-1} = t_H\{(\cos(4\pi/m) + \sqrt{-1}\sin(4\pi/m))\, vd\}$$

for $v \in C, \mathfrak{L}(u, +)$ has index m or $m/2$ in $\mathfrak{L}_m(u)$. Similarly, $\mathfrak{D}^*_\infty$ is free on Q^n_h, and acts properly discontinuously because $\mathfrak{T}(+)$ is a subgroup of index 4. Let $\mathcal{L}_m$ be the collection of space forms $S^n_h/\mathfrak{L}_m(u)$ $(m = 3, 4, 6, 8)$ and let $\mathcal{D}_\infty$ be the collection of space forms $S^n_h/\mathfrak{D}^*_\infty$; we must show that the elements of $\mathcal{L}_m$ and of $\mathcal{D}_\infty$ are homogeneous.

Let $D = \mathfrak{L}_m(u)$ or $\mathfrak{D}^*_\infty$; in the basis β, the centralizer of D in $O^h(n+1)$ is the collection of all $g = \begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{pmatrix} \in O^h(n+1)$ with $gJ_{n+1} = J_{n+1}g$ and $g_1 d = dg_6$. We give V the complex structure defined by J_{n+1}, so Q is the real part of a hermitian form Q_c and $\{v_{2j-1}\}$ is a Q_c-orthonormal basis of V. We set

$$(k = 2s, h = 2t)\ \beta_c = \{f_1, f_3, \ldots, f_{2s-1}; v_{c(s+1)-1}, \ldots, v_{2t-1}; e_1, \ldots, e_{2s-1}\},$$

C-basis of V. Let $d_c \in GL(s, C)$ be the matrix of the C-linear transformation $x \to \overline{d(x)}$ of U, relative to $\{e_{2j-1}\}$. The centralizer of D in $O^h(n+1)$ is then the collection of $g \in U^t(s+t, C)$ such that $g = \begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{pmatrix}$ in the C-basis β_c of V, with $g_1 d_c = d_c g_6$. The condition $g \in U^t(s+t, C)$ is expressed $g_6 = {}^t\bar{g}_1^{-1}, g_4 \in U(t-s)$ ordinary unitary group, $2\langle (g_1)_i, (g_3)_j\rangle + 2\langle (g_3)_i, (g_1)_j\rangle = \langle (g_2)_i, (g_2)_j\rangle$ and $2\langle (g_1)_i, (g_5)_j\rangle = \langle (g_2)_i, (g_4)_j\rangle$ where $(g_u)_v$ is the v^{th} row of g_u and $\langle\ ,\ \rangle$ is the ordinary hermitian pairing of complex

m-tuples. We take any $x = \sum_1^s a_j v_{2j-1} + \sum_1^{t-s} b_j v_{2(s+j)-1} + \sum_1^s c_j v_{2(s+t+j)-1}$ in Q_h^n, which is given by $Q_c(v, v) = 1$, and wish to find $g \in O^h(n+1)$ which centralizes D and gives $g(v_{2t+1}) = x$; this last is equivalent to $c - a = (g_1)_1$, $c + a = (g_3)_1 + (g_6)_1$ and $2b = (g_2)_1$. We may choose g_1 with $g_1 d_c = d_c {}^t\bar{g}_1^{-1}$ and $(g_1)_1 = c - a$, set $g_6 = {}^t\bar{g}_1^{-1}$, let $(g_2)_1 = 2b$ and let $(g_3)_1 = c + a - (g_6)_1$. The proof is now identical to that of § 9.2, but that some condition are conjugate linear.

Now $\mathcal{D}_\infty$ and $\mathcal{L}_m$ $(m = 3, 4, 6, 8)$ consist of homogeneous manifolds of constant positive curvature. For every pair (h, n) of integers with $2h > n > 1$, h even and $n + 1 - h$ divisible by 4, there is exactly one element $S_h^n/\mathfrak{D}_\infty^*$ in $\mathcal{D}_\infty$. For every quadruple $(h, n; m; \lambda)$ with h, n and m integers such that $2h > n > 1, h$ is even, $n + 1 - h$ is divisible by 4 and $m = 3, 4, 6$ or 8, and where $\lambda = \{1, u\}$ is a lattice in C with $u \notin R$ and $\exp(4\pi\sqrt{-1}/m)\lambda = \lambda$, there is exactly one element $S_h^n/\mathfrak{L}_m(u)$ of $\mathcal{L}_m$.

10. The classification theorem for homogeneous manifolds covered by quadrics

The **main result** of this paper is

10.1. Theorem. *Let D be a subgroup of $O^h(n+1)$ such that S_h^n/D is a connected homogeneous manifold. If D is finite, then is it cyclic or binary polyhedral, and S_h^n/D is a classical space form, element of $\mathcal{S}, \mathcal{Z}, \mathcal{D}, \mathcal{T}, \mathcal{O}$ or $\mathcal{J}$. If D is infinite but fully reducible, then it is isomorphic to Z or $Z \times Z_2$ and $S_h^n/D \in \mathcal{H}_r$. If D is abelian but not fully reducible, then either it is isomorphic to Z or $Z \times Z_2$ with $S_h^n/D \in \mathcal{H}_t$, or it is isomorphic to $Z \times Z$ or $Z \times Z \times Z_2$ with $S_h^n/D \in \mathcal{L}$. If D is neither abelian nor fully reducible, then either it is an extension of Z by an element of order 4 and $S_h^n/D \in \mathcal{D}_\infty$, or it is an extension $Z \times Z$ by an element of order m, with $m = 3, 4, 6$ or 8, and $S_h^n/D \in \mathcal{L}_m$.*

Remark. For some (h, n), certain families of space forms are excluded. Thus $S_h^n/D \notin \mathcal{L}_m$ or $\mathcal{D}_\infty$ if $2h \leqq n$, if $n + 1 - h \not\equiv 0 \pmod 4$, or if h is odd; $S_h^n/D \notin \mathcal{L}$ if $2h \leqq n$ or if $n + 1 - h \not\equiv 0 \pmod 4$; $S_h^n/D \notin \mathcal{H}_t$ if $2h \leqq n$ or if $n + 1 - h$ is odd; $S_h^n/D \notin \mathcal{H}_r$ if $2h \neq n + 1$; $S_h^n/D \notin \mathcal{D}, \mathcal{T}, \mathcal{O}$ or $\mathcal{J}$ if $h \not\equiv 0 \pmod 4$ or if $n + 1 \not\equiv 0 \pmod 4$; $S_h^n/D \notin \mathcal{Z}$ if h is odd or if $n + 1$ is odd.

Proof. Let $V = R^{n+1}$, let G be the centralizer of D in $O^h(n+1)$, and let B be the subalgebra of $\mathcal{E}(V)$ generated by D. Theorem 2.5 says that G is transitive on Q_h^n. We will divide the proof into several cases depending on whether G is irreducible on V, whether D is fully reducible on V and whether B is a division algebra. Note that B is a division algebra if G is irreducible on V, by SCHUR's Lemma.

10.2. Suppose that B is a division algebra. We view V as a left vector-space over B, note that D is a subgroup of the multiplicative group B^*, and (Lemma 8.1) choose a Q-orthonormal R-basis $\{w_j\}$ of V and a standard R-basis $\{a_i\}$ of B such that $(r = \dim. B) w_{rj+i} = a_i w_{rj+1}$. Then $s = h/r$ and $m = (n+1)/r$ are integers, $v_j = w_{r(j-1)+1}$ gives us a B-basis $\{v_j\}$ of V, and the hermitian form $Q_B(x, y) = -\sum_1^s x_j \overline{y_j} + \sum_{s+1}^m x_j \overline{y_j}$ $(x = \Sigma x_j v_j$ and $y = \Sigma y_j v_j$ with $x_j, y_j \in B)$ has the property that $Q(x, y) = \mathcal{R}e(Q_B(x, y))$ and Q_h^n is given by $Q_B(x, x) = 1$.

D is a subgroup of the unimodular group B' of B because D preserves Q_h^n, and D is discrete because it is properly discontinuous on Q_h^n; it follows from compactness of B' that D is finite. Thus D is a finite subgroup of B' acting by left scalar multiplication on $S_h^n = \{x \in V : Q_B(x, x) = 1\}$. Now S_h^n/D is a classical space form.

10.3. Suppose that D is fully reducible on V. If B has an element whose characteristic polynomial is not a power of an irreducible polynomial, then the rational cononical decomposition $V = \Sigma V_i$ of V by that element is nontrivial. G preserves each summand V_i, so Lemma 8.3 says that $2h = n + 1$ and the decomposition is of the form $V = U \oplus W$ with U and W maximal totally isotropic in V. U and W are D-invariant by Lemma 8.6, and G is irreducible on each by Lemma 8.2; it follows that the restrictions $B|_U$ and $B|_W$ are division algebras. These division algebras are each isomorphic to R by Lemma 8.4, whence D is scalar on U and on W. We choose an orthonormal basis $\{v_i\}$ of V such that $(e_i = v_{h+i} + v_i, f_i = v_{h+i} - v_i$ for $1 \leqq i \leqq h)$ $\{f_i\}$ is a basis of W and $\{e_i\}$ is a basis of U. Then every $d \in D$ is of the form $d = \begin{pmatrix} aI_h & 0 \\ 0 & a^{-1}I_h \end{pmatrix}$ in the basis $\{f_j, e_i\}$. The real numbers a form a discrete subgroup of R^*; Lemma 6.3 then says that D is generated by some one $\pm\begin{pmatrix} aI_h & 0 \\ 0 & a^{-1}I \end{pmatrix}$ and perhaps also $-I$, with $a > 1$, in $\{f_j, e_i\}$. Changing to the basis $\{v_i\}$, we see $S_h^n/D \in \mathcal{H}_r$.

Now suppose that every $b \in B$ has characteristic polynomial which is a power of an irreducible polynomial; we will see that B is a division algebra, whence D is finite, G is irreducible and S_h^n/D is classical. This is clear if G is irreducible on V; now assume G reducible. We then have a G-invariant D-invariant maximal totally isotropic subspace U of V by Lemmas 8.2

and 8.6; the restriction $B \mid_U$ is a division algebra, by SCHUR's Lemma, because G is irreducible on U. Let A be the kernel of the restriction $f: B \to B|_U$. By our hypothesis on characteristic polynomials, A is the set of all elements of B whose every eigenvalue is zero; it follows that A is contained in the radical of B. B is fully reducible on V, consequence of the full reducibility of D; it follows that B is semisimple [1, Th. 4, p. 118]. Thus $A = 0$, so $f: B \cong B \mid_U$ and B is a division algebra.

10.4. Suppose that D is not fully reducible on V; we will see $S_h^n/D \in \mathcal{H}_t$, $\mathcal{L}$, $\mathcal{D}_\infty$ or $\mathcal{L}_m$. First note that D is infinite and G is reducible. Lemmas 8.2 and 8.6 give us a G-invariant D-invariant proper maximal totally isotropic subspace U of V, of dimension $k = n + 1 - h$, on which G is irreducible, and gives us a Q-orthonormal R-basis $\gamma = \{v_1, \ldots, v_{h+k}\}$ such that ($e_j = v_{h+j} + v_j$ and $f_j = v_{h-j} - v_j$ for $1 \leqq j \leqq k$) $\{e_j\}$ is a basis of U. $U^\perp$ is also G-invariant and D-invariant, and has basis $\{v_{k+1}, \ldots, v_h; e_1, \ldots, e_k\}$; it follows that every element of G, D or B has block form

$$a = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix}$$

in the basis $\beta = \{f_1, \ldots, f_k; v_{k+1}, \ldots, v_h; e_1, \ldots, e_k\}$ of V. Note that $a \in G$ or $a \in D$ gives $a \in O^h(n+1)$, whence $a_6 = {}^t a_1^{-1}$ and $a_4 \in O(h-k)$.

Let r_1, r_4 and r_6 be the matric representations of respective degrees $k, h-k$ and k of G, given by $a \to a_1$, $a \to a_4$ and $a \to a_6$. r_6 is the restriction $G \to G_6 = G \mid_U$, hence irreducible; r_1 is irreducible because it is contragredient to r_6; r_4 is an orthogonal representation of G. Let $a \in G$, $av_{h+1} = \Sigma x_j v_j$. A short calculation shows that the first row of a_1 is $(x_{h+1} - x_1, \ldots, x_{h+k} - x_k)$; it follows from the transitivity of G on Q_h^n that neither r_1 nor r_6 has a nonzero symmetric bilinear invariant.

Let t_1, t_4 and t_6 be the matric representations of respective degrees $k, h-k$ and k of B, given by $a \to a_1$, $a \to a_4$ and $a \to a_6$, and let s_1, s_4 and s_6 be their restrictions to D. $t_6(B) = B_6$ is a division algebra by SCHUR's Lemma because it centralizes the irreducible group G_6; thus s_6 is fully reducible. s_1 is fully reducible because it is contragredient to s_6, and s_4 is fully reducible because it is an orthogonal representation. Every element of D has characteristic polynomial which is a power of an irreducible polynomial: if not, Lemma 8.3 would give $h = k$ and show that we could also assume $W = \{f_i\}$ to be G-invariant and D-invariant; it would follow that $D = (s_1 \oplus s_6)(D)$ so D would be fully reducible on V. Thus s_1, s_4 and s_6 have the same kernel Δ and every $d \in \Delta$ is of the form

$$d = \begin{pmatrix} I & d_2 & d_3 \\ 0 & I & d_5 \\ 0 & 0 & I \end{pmatrix} \text{ relative to } \beta .$$

s_6 represents by unimodular elements of the division algebra B_6, for $s_6(d)$ was just seen to have the same characteristic equation as its contragredient ${}^ts_6(d)^{-1}$, for every $d \in D$. Now s_6 is self-contragredient, hence equivalent to s_1. It follows that t_1 and t_6 are equivalent, hence have the same kernel A. We have seen that there is no nontrivial G-invariant direct sum decomposition of V, so the characteristic polynomial of any element of B is a power of an irreducible polynomial. Thus every element of the kernel Ker. t_4 has all characteristic roots zero, hence lies in A because $t_6(B)$ is a division algebra. On the other hand, $t_4(A)$ is easily seen to be the radical of $t_4(B)$, and $t_4(B)$ is semisimple because t_4 is fully reducible [1, Th. 4, p. 118], consequence of the fact that s_4 is fully reducible; it follows that $A = \text{Ker.}\, t_4$ unless $h - k = 0$. Now we see that every $a \in A$ is of the form $a = \begin{pmatrix} 0 & a_2 & a_3 \\ 0 & 0 & a_5 \\ 0 & 0 & 0 \end{pmatrix}$, so A is a nilpotent ideal of B and the difference algebra (see [1], p. 27) $B - A \cong t_6(B)$, being a real division algebra, is separable [1, Th. 21, p. 44]. This shows that A is the radical of B and that [1, Th. 23, p. 47] B has a subalgebra B_U with $t_6 : B_U \cong t_6(B)$. By Lemma 8.2, we may take a standard R-basis $\{\alpha_i\}$ of B_U and assume that $v_{rj+i} = \alpha_i v_{rj+1} (r = \dim. B_U)$ and that the respective R-linear spans of $\{f_i\}$, $\{v_{k+1}, \ldots, v_h\}$ and $\{e_i\}$ are B_U-invariant. Note that we then have $e_{rj+i} = \alpha_i e_{rj+1}$ and $f_{rj+i} = \alpha_i f_{rj+1}$. In particular, B_U consists of all $\begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_4 & 0 \\ 0 & 0 & b_6 \end{pmatrix}$ for $b \in B$, and A consists of all matrices $\begin{pmatrix} 0 & b_2 & b_3 \\ 0 & 0 & b_5 \\ 0 & 0 & 0 \end{pmatrix}$ for $b \in B$. It follows that G centralizes every matrix $d_U = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_4 & 0 \\ 0 & 0 & d_6 \end{pmatrix}$ and that $d_1 = d_6$, for $d \in D$.

Given $d \in D$, we note that $d_U \in O^h(n+1)$; thus $d_U^{-1} d = \begin{pmatrix} I & d_1^{-1}d_2 & d_1^{-1}d_3 \\ 0 & I & d_4^{-1}d_5 \\ 0 & 0 & I \end{pmatrix} \in O^h(n+1)$. It follows that $d_1^{-1}d_2 = 2\,({}^t(d_4^{-1}d_5))$. Set $d' = d_1^{-1}d_2$. Now let $g \in G$. $g(d_U^{-1}d) = (d_U^{-1}d)g$ gives us $g_1 d' = d' g_4$ and $g_4(2\,{}^td') = (2\,{}^td')g_6$, whence $(d' \cdot {}^td')g_6 = d' g_4\, {}^td' = g_1(d' \cdot {}^td')$. As $g_1 = {}^tg_6^{-1}$ and as $d' \cdot {}^td'$ is symmetric, it follows that $d' \cdot {}^td'$ is the matrix of a G_6-invariant symmetric bilinear form on U. But r_6 has no nonzero symmetric bilinear invariant; thus $d' \cdot {}^td' = 0$. This gives $d_1^{-1}d_2 = 0$ and $d_4^{-1}d_5 = 0$, whence $d_2 = 0$ and $d_5 = 0$. We conclude that every element of D is of the form $d = \begin{pmatrix} d_1 & 0 & d_3 \\ 0 & d_4 & 0 \\ 0 & 0 & d_1 \end{pmatrix}$ relative to β. Notice that some $d \in D$ has $d_3 \neq 0 : D$ would be fully reducible if this were not the case.

Choose $d \in D$ with $d_3 \neq 0$ and note that $dg = gd$ gives ${}^t g_6 d_3 g_6 = d_3$ for every $g \in G$. Let F be the centralizer of $r_6(G)$ in $\mathcal{E}(U)$. F is a division algebra which contains $t_6(B)$ as a subalgebra. Given $u \in F$, we have ${}^t g_6 (d_3 u) g_6 = d_3 u$ for every $g \in G$. Thus $d_3 u$ is a bilinear invariant of r_6. It follows that $d_3 u$ is a nonsingular skew matrix. This gives $d_3 u = - {}^t(d_3 u) = - {}^t u \cdot {}^t d_3 = {}^t u d_3$. In particular, given $u, v \in F$, we have $uv = d_3^{-1} \cdot {}^t(uv) \cdot d_3 = d_3^{-1} \cdot {}^t v \cdot d_3 \cdot d_3^{-1} \cdot {}^t u \cdot d_3 = uv$, so F is commutative.

Given $b \in B$ and $g \in G$, $dg^{-1} = g^{-1} d$ and $bg = gd$ gives $d_3 g_6^{-1} = {}^t g_6 d_3$ and $b_3 g_6 = {}^t g_6^{-1} b_3$; thus $d_3^{-1} b_3 \cdot g_6 = d_3^{-1} \cdot {}^t g_6^{-1} \cdot b_3 = g_6 d_3^{-1} b_3$, so $d_3^{-1} b_3 \in F$; it follows that $b_3 \in d_3 \cdot F$. Let B_3 be the collection of all matrices b_3 for $b \in B$; we have just seen that the linear space B_3 has dimension at most dim. F. Let Δ_3 be the collection of matrices d'_3 for $d' \in \Delta$. Given $d', d'' \in \Delta$, we have

$$d'd'' = \begin{pmatrix} I & 0 & d'_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & d''_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & d'_3 + d''_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = d''d' .$$

Thus Δ is free abelian, isomorphic to the additive group Δ_3. Δ is discrete, as D is discrete, so Δ_3 is a lattice in B_3. This shows Δ free abelian on at most dim. F generators. If $t_6(B) \cong R$ and $\Delta = \{I\}$, then $s_6 : D \cong s_6(D) \subset R'$, contradicting the hypothesis that D be infinite; thus $\Delta \neq \{I\}$ if $t_6(B) \cong R$. If $t_6(B) \cong C$ and $\Delta = \{I\}$, then $s_6 : D \cong s_6(D) \subset C'$ shows D abelian, so B is abelian; then $b_3 b'_1 = b'_1 b_3$ for every $b, b' \in B$, whence $b'_1 = {}^t b'_1$ for $b' \in B$. This gives $t_6(B) \cong R$, contrary to our hypothesis; thus $\Delta \neq \{I\}$ if $t_6(B) \cong C$. We conclude that $\Delta \neq \{I\}$ and Δ is free abelian on at most dim. F generators.

Suppose $t_6(B) \cong R$ and Δ is infinite cyclic. Let $\delta = \begin{pmatrix} I & 0 & \delta_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ generate Δ. k is even because δ_3 is skew and nonsingular. If $s_6(D) = \{I\}$, then $D = \Delta$, so $D = \mathfrak{T}(+)$ and $S^n_h/D \in \mathcal{H}_t$. If $s_6(D) \neq \{I\}$, then $s_6(D) = \{\pm I\}$, and we choose $d = \begin{pmatrix} -I & 0 & d_3 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix} \in D$. $D = \{d, \Delta\}$ and $d^2 \in \Delta$, so $-2d_3$ is an integral multiple of δ_3. If $-2d_3$ is an even multiple of δ_3, then $d_3 = m\delta_3$ for some integer m and $-I = d\delta^m \in D$; we then have $D = \{-I, \Delta\} = \mathfrak{T}(\pm)$ and $S^n_h/D \in \mathcal{H}_t$. If $-2d_3$ is an odd multiple of $\delta_3, -2d_3 = (2m+1)\delta_3$, then $d\delta^m = \begin{pmatrix} -I & 0 & \frac{1}{2}\delta_3 \\ 0 & -I & 0 \\ 0 & .0 & -I \end{pmatrix}$ generates D; we then have $D = \mathfrak{T}(-)$ and $S_h^n/D \in \mathcal{H}_t$.

Suppose $t_6(B) \cong R$ and $\varDelta$ is not infinite cyclic. Then $F \cong C$ and $\varDelta$ is free abelian on two generators δ and δ'; $\delta_3' = \delta_3 u$ for some non-real $u \in F$. k is divisible by 4, for we identify F with C and observe that $x \to \overline{\delta_3 x}$ is a C-linear nonsingular transformation of U whose matrix must be skew because it is a bilinear invariant of the C-linear group $r_6(G)$. If $s_6(D) = \{I\}$ then $D = \varDelta$, so $D = \mathfrak{L}(u, +)$ and $S_h^n/D \in \mathcal{L}$. If $s_6(D) \neq \{I\}$, then $s_6(D) = \{\pm I\}$ and, as in the last paragraph, we choose $d \in s_6^{-1}(-I)$ and observe that $d^2 \in \varDelta$ implies $-2d_3 = m\delta_3 + m'\delta_3'$ for integers m, m'. If both m and m' are even, then $D = \{-I, \varDelta\} = \mathfrak{L}(u, \pm)$ and $S_h^n/D \in \mathcal{L}$. If both m and m' are odd, we replace d by an appropriate $d\delta^u\delta'^v$ and assume $m = -1 = m'$, $d = \begin{pmatrix} -I & 0 & -\frac{1}{2}(\delta_3 + \delta'_3) \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix}$, $d^2 = \delta\delta'$.

Now $D = \{\delta, d\}$ and $-d = \begin{pmatrix} I & 0 & \delta_3(u+1)/2 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$; therefore $D = = \mathfrak{L}((u+1)/2, -)$ and $S_h^n/D \in \mathcal{L}$. If m is odd and m' is even, we may assume $d = \begin{pmatrix} -I & 0 & -\frac{1}{2}\delta_3 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix}$; then $D = \{\delta', d\} = \mathfrak{L}(\frac{1}{2}u^{-1}, -)$ and $S_h^n/D \in \mathcal{L}$. Finally, if m is even and m' is odd, then we may assume $d = \begin{pmatrix} -I & 0 & -\frac{1}{2}\delta_3' \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix}$; then $D = \{\delta, d\} = \mathfrak{L}(\frac{1}{2}u, -)$ and $S_h^n/D \in \mathcal{L}$.

Suppose $t_6(B) \cong C$ and $\varDelta$ is infinite cyclic. Let $\delta = \begin{pmatrix} I & 0 & \delta_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ generate $\varDelta$. As $\varDelta$ is a normal subgroup of D, we have $d\delta d^{-1} \in \varDelta$ for every $d \in D$. Using $d_3 d_1^{-1} = {}^t d_1^{-1} d_3 = d_1 d_3$ and $\delta_3 d_1^{-1} = {}^t d_1^{-1}\delta_3 = d_1\delta_3$, this gives $d_1{}^2\delta_3 = m\delta_3$ for some integer m. Similarly, $d^{-1}\delta d \in \varDelta$ gives $d_1{}^{-2}\delta_3 = p\delta_3$ for some integer p; $\delta_3 = d_1{}^{-2}d_1{}^2\delta_3 = mp\delta_3$ gives $\mathrm{p} = m^{-1}$, whence $m = \pm 1$. $m = 1$ givex $d_1 = \pm I$; $m = -1$ gives $d_1{}^2 = -I$. There exists $d \in D$ with $d_1^2 = -I$, for $s_6(D) \neq \{\pm I\}$. Given such an element d, we note that $d^2 = \begin{pmatrix} d_1^2 & 0 & (d_1 + d_1^{-1})d_3 \\ 0 & d_4^2 & 0 \\ 0 & 0 & d_1^2 \end{pmatrix} = -I$, so the subalgebra of B generated by d is mapped isomorphically onto $t_6(B)$ by t_6. We may, then, chose $d \in D$ with $d_1{}^2 = -I$ and asume $d = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_4 & 0 \\ 0 & 0 & d_1 \end{pmatrix}$. Now

$D = \{d, \delta\}, d^2 = -I, d\delta d^{-1} = \delta^{-1}$, the argument of the last paragraph shows k divisible by 4, and h is even because $B_U \cong C$. Thus $D = \mathfrak{D}^*_\infty$ and $S^n_h/D \in \mathcal{D}_\infty$.

Suppose $t_6(B) \cong C$ and Δ is not infinite cyclic. Then h is even and, looking at Δ, k is divisible by 4. Let δ, δ' generate Δ. Given $d \in D, d_1^2 + d_1^{-2}$ is a real scalar matrix; thus $(d\delta d^{-1})(d^{-1}\delta d) = \begin{pmatrix} I & 0 & (d_1^2 + d_1^{-2})\delta_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \in \Delta$ shows $d_1^2 + d_1^{-2}$ integral. Identifying $t_6(B)$ with C, set $d_1 = \cos(t) + \sqrt{-1}\sin(t)$; $d_1^2 + d_1^{-2}$ is then $2\cos(2t)$, so $\cos(2t) = -1, -\frac{1}{2}, 0, \frac{1}{2}$ or 1.

If $\cos(2t) = -1$, then $t = \pm\pi/2$ and $d_1^2 = -I$; it follows that $d^2 = -I$. If $\cos(2t) = 0$, then $t = \pm\pi/4$ or $\pm 5\pi/4$ and $d_1^4 = -I$; it follows that $d^4 = -I$. If $\cos(2t) = 1$, then $d_1 = \pm I$. If $\cos(2t) = -\frac{1}{2}$, then $t = \pm 2\pi/3$ or $\pm\pi/3$, and one easily checks $d_1^2 + I + d_1^{-2} = 0$; it follows that $d^3 = \pm I$. Finally, $\cos(2t) = \frac{1}{2}$ is impossible because it would imply $t = \pm\pi/6$ or $\pm 7\pi/6$, whence $d^2 = \begin{pmatrix} d_1^2 & 0 & \pm\sqrt{3}\, d_3 \\ 0 & d_4^2 & 0 \\ 0 & 0 & d_1^2 \end{pmatrix}$ and $d^6 = \begin{pmatrix} -I & 0 & \pm 2\sqrt{3}\, d_3 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix}$ commute; this would give $d_1^2 d_3 = d_3 d_1^2$, hence $d_3 = 0$, and tell us that $\Delta = I$ because $d = s_6^{-1}s_6(d)$. Thus $s_6(D)$ is cyclic of order $3, 4, 6$ or 8 (for $s_6(D) \not\subset R'$), and we have an element $d^* \in D$ such that $s_6 : \{d^*\} \cong s_6(D)$. The subalgebra of B generated by d^* maps isomorphically onto $t_6(B)$ under t_6, so we may assume B_U chosen with $d^* \in B_U$. This gives $d^*_3 = 0$ for our choice of d^*. Now choose $J \in t_6(B)$ with $J^2 = -I, J_{n+1} \in B_U$ with $t_6(J_{n+1}) = J$; we may assume $d^* = \cos(2\pi/m)I_{n+1} + \sin(2\pi/m)J_{n+1}$ where $m = 3, 4, 6$ or 8. d^* now acts on Δ by $(d^*dd^{*-1})_3 = (\cos(4\pi/m)I + \sin(4\pi/m)J)d_3$. Thus $D = \mathfrak{L}_m(u)$ where $\delta'_3 = u\delta_3, u = aI + bJ$, and $S^n_h/D \in \mathfrak{L}_m$. Q.E.D.

10.5. Corollary. *Let D be a subgroup of $O^h(n+1)$ such that S^n_h/D is a connected homogenous manifold. If $2h \leq n$ or $n + 1 - h$ is odd, then S^n_h/D is a classical space form, element of $\mathcal{S}, \mathcal{Z}, \mathcal{D}, \mathcal{T}, \mathcal{O}$, or $\mathcal{J}$.*

For none of the exceptional families of space forms occur in these signatures. Thus:

10.6. Corollary. *Let M^n be a connected RIEMANNIAN homogeneous manifold of constant negative curvature. Then M^n is isometric to the hyperbolic space $H^n = H^n_0$.*

10.7. Corollary. *Let M^n be a connected RIEMANNian homogeneous manifold* *of constant positive curvature. Then M^n is isometric to S^n/D where S^n is the sphere $||x|| = k^{-\frac{1}{2}}$ in a left hermitian (positive definite) vectorspace over a real division algebra F, D is a finite subgroup of F', and M^n has curvature k. All these* RIEMANN*ian manifolds S^n/D are homogeneous of constant positive curvature. The only cases are* (1) $D = \{I\}$ *or* Z_2, (2) $n \equiv 1$ (*mod* 2) *and* $D \cong Z_q (q > 2)$ *and* (3) $n \equiv 3$ (*mod* 4) *and* $D \cong \mathfrak{D}_t^*, \mathfrak{T}^*, \mathfrak{O}^*$ *or* $\mathfrak{J}^*$.

Corollaries 10.6 and 10.7 follow immediately, and give the classification of the RIEMANNian homogeneous manifolds of constant nonzero curvature. That classification is known [6].

11. The difficult signature

S^h_n and $\tilde{S}^n_h$ are essentially different for $h = n - 1$; $\pi_1(S^n_{n-1}) \cong Z$. We will relate the homogeneous manifolds M^n_{n-1} of constant positive curvature with the homogeneous manifolds S^n_{n-1}/D covered by quadrics. The trick is to use the universal covering group of the non-connected group $O^{n-1}(n+1)$.

11.1. Lemma. *Let* $\pi: \tilde{S}^n_{n-1} \to S^n_{n-1}$ *be the universal pseudo-*RIEMANN*ian covering, let* $\tilde{O}^{n-1}(n+1)$ *be the full group of isometries of* $\tilde{S}^n_{n-1}$, *and let* D_π *be the group of deck transformations of the covering. Then* D_π *is a central subgroup of the identity component* $\tilde{SO}^{n-1}(n+1)$ *of* $\tilde{O}^{n-1}(n+1)$, D_π *is normal in* $\tilde{O}^{n-1}(n+1)$, *and we have an epimorphism* $f: \tilde{O}^{n-1}(n+1) \to O^{n-1}(n+1)$ *of kernel* D_π *defined by* $f(g)\cdot\pi(\tilde{x}) = \pi(g\tilde{x})$.

Proof. Let G be the normalizer of D_π in $\tilde{O}^{n-1}(n+1)$, let $\pi(\tilde{p}) = p$, let $\tilde{K}$ be the isotropy subgroup of $\tilde{O}^{n-1}(n+1)$ at $\tilde{p}$, and let K be the isotropy subgroup of $O^{n-1}(n+1)$ at p. $\tilde{O}^{n-1}(n+1) = G\cdot\tilde{K}$ because homogeneity of S^n_{n-1} implies that G is transitive on $\tilde{S}^n_{n-1}$. f is well defined on G, and $f(G) = O^{n-1}(n+1)$ because every isometry can be lifted; it follows that $f(G \cap \tilde{K}) = K$. Isomorphic to the linear isotropy group of G at $\tilde{p}$, $G \cap \tilde{K}$ is isomorphic to a subgroup of $O^{n-1}(n)$; $K \cong O^{n-1}(n)$ now gives $G \cap \tilde{K} \cong O^{n-1}(n)$; this gives $\tilde{K} \subset G$ because $\tilde{K}$ is isomorphic to a subgroup of $O^{n-1}(n)$. Thus $\tilde{O}^{n-1}(n+1) = G\cdot\tilde{K} = G$. It follows that D_π is normal in $\tilde{O}^{n-1}(n+1)$ and f is well defined on $\tilde{O}^{n-1}(n+1)$. It is clear that $\tilde{SO}^{n-1}(n+1)$ centralizes D_π, being a connected group which normalizes the discrete group D_π, and D_π is the kernel of f by construction.

Now we need only show $D_\pi \subset \tilde{SO}^{n-1}(n+1)$; as f is onto, it suffices to show that $\tilde{O}^{n-1}(n+1)$ and $O^{n-1}(n+1)$ have the same finite number of components. $\tilde{K} \cap \tilde{SO}^{n-1}(n+1)$ is connected because $\tilde{S}^n_{n-1}$ is simply connected, and $\tilde{K}$ meets every component of $\tilde{O}^{n-1}(n+1)$ because $\tilde{S}^n_{n-1}$ is connected; it follows that $\tilde{O}^{n-1}(n+1)$ and $\tilde{K}$ have the same number of components. $\tilde{K} \cong O^{n-1}(n) \cong K$, whence $\tilde{K}$ and K each has 4 components. $O^{n-1}(n+1)$ has 4 components. *Q. E. D.*

11.2. We check that $\pi^{-1}(\{\pm I\})$ is infinite cyclic. This is the case $m=2$ of

Lemma. *Let S^n_{n-1}/Z_m be homogeneous. Then the fundamental group $\pi_1(S^n_{n-1}/Z_m)$ is infinite cyclic.*

Proof. S^n_{n-1} is given by $x_n^2 + x_{n+1}^2 = 1 + \sum_1^{n-1} x_j^2$, so the map $s:[0,1]\to S^n_{n-1}$ given by $s(t) = (0,\ldots,0,\sin(2\pi t),\cos(2\pi t))$ represents a generator of $\pi_1(S^n_{n-1})$. Let $b: S^n_{n-1}\to S^n_{n-1}/Z_m$ be the projection and set $B = b_*\pi_1(S^n_{n-1})$. We may assume Z_m generated by I if $m=1$, $-I$ if $m=2$, and $\begin{pmatrix} R(1/m) & & \\ & \ddots & \\ & & R(1/m)\end{pmatrix}$ if $m>2$, where $R(q) = \begin{pmatrix} \cos(2\pi q) & \sin(2\pi q) \\ -\sin(2\pi q) & \cos(2\pi q)\end{pmatrix}$. Let $v:[0,1]\to S^n_{n-1}/Z_m$ by $v(t) = b(s(t/m))$; $b_*s = v^m \in \pi_1(S^n_{n-1}/Z_m)$. Thus B has index m in the cyclic subgroup $\{v\}$ of $\pi_1(S^n_{n-1}/Z_m)$. The covering b has multiplicity m, whence B has index m in $\pi_1(S^n_{n-1}/Z_m)$; thus $\{v\}$ is all of $\pi_1(S^n_{n-1}/Z_m)$. *Q. E. D.*

11.3. Certain abelian subgroups of $\tilde{O}^{n-1}(n+1)$ will be seen to contain conjugates of every group D such that $\tilde{S}^n_{n-1}/D$ is homogeneous. We will describe these groups.

If n is odd, every rotation matrix $R(s)_{n+1} = \begin{pmatrix} R(s) & & \\ & \ddots & \\ & & R(s)\end{pmatrix}$ lies in $SO^{n-1}(n+1)$, where $R(s) = \begin{pmatrix} \cos(2\pi s) & \sin(2\pi s) \\ -\sin(2\pi s) & \cos(2\pi s)\end{pmatrix}$. The collection A_R of all these rotation matrices is a circle subgroup of $SO^{n-1}(n+1)$. It follows from Lemma 11.2 that the inverse image $\tilde{A}_R = \pi^{-1}(A_R)$ is a closed non-periodic 1-parameter subgroup of $\tilde{SO}^{n-1}(n+1)$.

If $n>2$, we take d to be any nonsingular skew 2×2 matrix, and recall (§ 9.2) that each translation matrix $t_H(sd) = \begin{pmatrix} I_2 - sd & 0 & -sd \\ 0 & I_{n-3} & 0 \\ sd & 0 & I_2 + sd\end{pmatrix}$ $(s\in R)$ lies in $SO^{n-1}(n+1)$. The collection A_T of all such $\pm t_H(sd)$ is a closed subgroup of $SO^{n-1}(n+1)$ isomorphic to $Z_2\times R^1$; Lemmas 11.1 and 11.2 inform us that the inverse image $\tilde{A}_T = \pi^{-1}(A_T)$ is a closed subgroup of $\tilde{O}^{n-1}(n+1)$ isomorphic to $Z\times R^1$, where the Z is $\pi^{-1}(\{\pm I\})$ and the R^1 is the lifting of the 1-parameter group $\{t_H(sd)\}$.

Each hyperbolic rotation $Rh(s) = \begin{pmatrix} \cosh(s)I_2 & \sinh(s)I_2 \\ \sinh(s)I_2 & \cosh(s)I_2 \end{pmatrix}$ lies in $SO^2(4)$. As with the translation matrices, $\tilde{A}_H = \pi^{-1}(\{\pm Rh(s)\})$ is a closed subgroup of $\tilde{SO}^2(4)$ isomorphic to $Z \times R^1$, where the Z is $\pi_1^{-1}(\{\pm I\})$ and the R^1 is the lifting of the 1-parameter group $\{Rh(s)\}$.

Finally, set $\tilde{A}_Z = \pi^{-1}(\{\pm I\}) \subset \tilde{O}^{n-1}(n+1)$, and $A_Z = \{\pm I\}$.

Lemma. *Let D be a subgroup of $\tilde{O}^{n-1}(n+1)$. Then $\tilde{S}^n_{n-1}/D$ is a homogeneous manifold if and only if D is conjugate to a discrete subgroup of $\tilde{A}_Z$, $\tilde{A}_R$, $\tilde{A}_T$ or $\tilde{A}_H$.*

Proof. Suppose $\tilde{S}^n_{n-1}/D$ homogeneous. Let $\tilde{G}'$ be the centralizer of D in $\tilde{O}^{n-1}(n+1)$. The identity component of $\tilde{G}'$ is transitive on $\tilde{S}^n_{n-1}$, and centralizes D_π by Lemma 11.1; it follows that the centralizer $\tilde{G}$ of $D \cdot D_\pi$ is transitive on $\tilde{S}^n_{n-1}$. Thus the centralizer $G = f(\tilde{G})$ of $f(D)$ in $O^{n-1}(n+1)$ is transitive on S^n_{n-1}. We now look at the proof of Theorem 10.1; G acts on $V = R^{n+1}$ and B is the subalgebra of $\mathcal{E}(V)$ generated by $f(D)$. If B is a division algebra then, as in § 10.2, $f(D)$ is conjugate to a subgroup of A_Z (for n even) or A_R (for n odd). If $f(D)$ is fully reducible on V and B is not a division algebra then, as in § 10.3, B has an element whose characteristic polynomial is not a power of an irreducible polynomial, $n = 3$, and $f(D)$ is conjugate to a subgroup of A_H. If $f(D)$ is not fully reducible on V, then, as in § 10.4, $f(D)$ is conjugate to a subgroup of A_T. Thus D is conjugate to a subgroup of $\tilde{A}_Z, \tilde{A}_R, \tilde{A}_H$ or $\tilde{A}_T$. Free and properly discontinuous on $\tilde{S}^n_{n-1}$, D is discrete.

Let D be a discrete subgroup of $\tilde{A}_Z, \tilde{A}_R, \tilde{A}_H$ or $\tilde{A}_T$. The centralizer G of $f(D)$ in $O^{n-1}(n+1)$ contains the centralizer of A_Z, A_R, A_H or A_T, and is thus transitive on S^n_{n-1}; it follows that the centralizer of D in $\tilde{O}^{n-1}(n+1)$ is transitive on $\tilde{S}^n_{n-1}$. As D acts effectively, this shows D free on $\tilde{S}^n_{n-1}$. As D is discrete, it easily follows that $\tilde{S}^n_{n-1}/D$ is a homogeneous manifold.

Q. E. D.

11.4. The discrete subgroups of $\tilde{A}_Z, \tilde{A}_R, \tilde{A}_H$ and $\tilde{A}_T$ are easily described. Any subgroup of $\tilde{A}_Z$ is discrete and infinite cyclic. The discrete subgroups of $\tilde{A}_R$ are the subgroups on one generator; they are discrete. Viewing $\tilde{A}_H$ and $\tilde{A}_T$ as closed subgroups $Z \times R^1$ of a vector group R^2, we see that a discrete subgroup is on 1 or 2 generators; a subgroup on 1 generator is discrete and infinite cyclic; one easily checks that a non-cyclic subgroup on 2 generators is discrete if and only if it does not lie in the identity component of $\tilde{A}_H$ (*or* $\tilde{A}_T$).

12. The classification theorem for homogeneous manifolds of constant non-zero curvature

Theorem. *Let M_h^n be a connected homogeneous pseudo-*RIEMANN*ian manifold of constant positive curvature. If $h < n - 1$, then M_h^n is isometric to a classical or exceptional space form S_h^n/D, element of $\mathcal{S}$, $\mathcal{Z}$, $\mathcal{D}$, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{J}$, $\mathcal{H}_r$, $\mathcal{H}_t$, $\mathcal{D}_\infty$, $\mathcal{L}$, $\mathcal{L}_3$, $\mathcal{L}_4$, $\mathcal{L}_6$ or $\mathcal{L}_8$. If $h = n - 1$, then M_h^n is isometric to a manifold $\tilde{S}_{n-1}^n/D$, where D is a discrete subgroup of a closed subgroup $\tilde{A}_Z, \tilde{A}_R, \tilde{A}_H$* or *$\tilde{A}_T$ of $\tilde{O}^{n-1}(n+1)$. If $h = n$, then M_h^n is isometric to $\tilde{S}_n^n$, a component of S_n^n.*

Proof. This is an immediate consequence of Theorem 5, Theorem 10.1 and Lemma 11.3.

Chapter III. Isotropic and symmetric manifolds of constant curvature

13. The notion of a symmetric or isotropic manifold

Let Q be the metric on a pseudo-RIEMANNian manifold M_h^n and, given $p \in M_h^n$, let $I(M_h^n)_p$ be the isotropy subgroup at p of the group $I(M_h^n)$ of isometries. A *symmetry* at p is an element $s_p \in I(M_h^n)$ of order 2 with p as isolated fixed point; such an element is unique and central in $I(M_h^n)_p$ because the tangent map at p is $-I$. M_h^n is *symmetric* if there is a symmetry at every point. M_h^n is *isotropic* if, given $p \in M_h^n$ and $a \in R$, $I(M_h^n)_p$ is transitive on the set of tangentvectors X at p such that $Q_p(X, X) = a$. M_h^n is *strongly isotropic* if, given $p \in M_h^n$, $I(M_h^n)_p$ is transitive on the Q_p-orthonormal frames at p, i. e., $I(M_h^n) \supseteq O^h(n)$.

Note that a strongly isotropic manifold is an isotropic manifold of constant curvature; we will prove the converse. We'll also see that an isotropic manifold of constant curvature is symmetric.

It is well known that a connected manifold is homogeneous if it is either isotropic or symmetric. For, given a tangentvector X at a point p, we have an element $s_X \in I(M_h^n)_p$ such that $(s_X)_* X = -X$. Now let $x, y \in M_h^n$ and join them by a broken geodesic. If $x_1, \ldots, x_m$ are the successive midpoints of the geodesic segments of this broken geodesic, and if $X_1, \ldots, X_m$ are the respective tangentvectors to the geodesic segments at $x_1, \ldots, x_m$, then $s_{X_m} s_{X_{m-1}} \cdots s_{X_1}(x) = y$.

14. The classification theorem for symmetric manifolds of zero curvature and for nonholonomic manifolds

14.1. Theorem. *Let M_h^n be a connected pseudo-*RIEMANN*ian manifold. Then these are equivalent:*

1. *M_h^n is a symmetric manifold of constant curvature zero.*

2. *M_h^n is a complete manifold with trivial holonomy group, i.e., parallel translation in M_h^n is independent of path.*

3. *M_h^n is isometric to a manifold R_h^n/D where D is a discrete subgroup of the underlying vector group R^n of R_h^n.*

Proof. Suppose first that M_h^n is complete with trivial holonomy. Then M_h^n has constant curvature zero and admits R_h^n as universal pseudo-RIEMANNian covering manifold. Let D be the group of deck transformations of the universal pseudo-RIEMANNian covering $\pi: R_h^n \to M_h^n$. D is a discrete subgroup of $I(R_h^n) = NO^h(n)$; we write every $d \in D$ in the form $d = (R_d, t_d)$ with $R_d \in O^h(n), t_d \in R_h^n$ and $d: x \to R_d(x) + t_d$. $\{R_d : d \in D\}$ is isomorphic to the holonomy group of M_h^n at $\pi(0)$, whence D is a group of pure translations. Thus (2) implies (3). The last part of our argument shows that (3) implies (2).

To see that (3) implies (1), we note that R_h^n/D carries the structure of a connected abelian LIE group, and that $y \to x^2y^{-1}$ is the symmetry at x because $y \to y^{-1}$ is the symmetry at the coset D.

Now suppose M_h^n is symmetric of constant curvature zero, let D be the group of deck transformations of the universal pseudo-RIEMANNian covering $\pi: R_h^n \to M_h^n$, and write every $d \in D$ in the form (R_d, t_d) as above. Let $Q(x, y) = -\sum_1^n x_j y_j + \sum_{h+1}^h x_j y_j$ on R^n, so $O^h(n) \subset I(R_h^n)$ is the orthogonal group of Q. Choose $d \in D$; we must show that d is a pure translation. The space $N_h^n = R_h^n/\{d\}$ ($\{d\}$ is the subgroup of D generated by d) is symmetric because it covers M_h^n and we can lift each symmetry; it follows that the centralizer G of d in $NO^h(n)$ is transitive on R_h^n, and that every symmetry of R_h^n normalizes $\{d\}$.

Given $x \in R_h^n$, choose $g \in G$ with $g(x) = 0$. $Q(d(x) - x, d(x) - x) = Q(gd(x) - g(x), gd(x) - g(x)) = Q(t_d . t_d)$. Thus $Q((R_d - I)x, (R_d - I)x) + 2Q((R_d - I)x, t_d) = 0$ for every $x \in R_h^n$. Replacing x by ux as u runs through R, we see that $Q((R_d - I)x, (R_d - I)x) = 0 = Q((R_d - I)x, t_d)$ for every $x \in R_h^n$. Polarizing the first equality, then expanding and using $R_d \in O^h(n)$, we get $Q(x, (2I - R_d - R_d^{-1})y) = 0$ for $x, y \in R_h^n$. Thus $2I = R_d + R_d^{-1}$, whence $(R_d - I)^2 = 0$. We conclude that $R_d = I + N$ with $N^2 = 0$, $N(R_h^n)$ is totally Q-isotropic, and t_d is Q-orthogonal to $N(R_h^n)$.

In particular, $(R_d)^m = I + mN$, whence either $R_d = I$ or R_d has infinite order. Thus $\{d\}$ is either trivial or infinite cyclic.

We now use the fact that $\{d\}$ is normalized by the symmetry $s_0 = (-I, 0)$ of R^n_h at 0. $s_0 d s_0^{-1} = (R_d, -t_d)$ is either d or d^{-1}, as $\{d\}$ is either trivial or infinite cyclic. $s_0 d s_0^{-1} \neq d$ as $t_d \neq 0$, for d has no fixed point. Thus $s_0 d s_0^{-1} = d^{-1} = (R_d^{-1}, -R_d^{-1} t_d)$. In particular, $R_d^2 = I$. As $R_d^2 = I + 2(R_d - I)$, as seen in the last paragraph, we conclude that $R_d = I$ and d is a pure translation. Thus (1) implies (3). *Q. E. D.*

14.2. Corollary. *Let M^n be a connected* RIEMANNian *manifold. Then these are equivalent:*

1. *M^n is a symmetric manifold of constant curvature zero.*
2. *M^n is a complete manifold with trivial holonomy group.*
3. *M^n is isometric to a product of a euclidean space with a flat torus, i. e., M^n is isometric to a quotient R^n/D of a (positive-definite) euclidean space by a discrete vector subgroup.*
4. *M^n is a homogeneous manifold of constant curvature zero.*

Proof. The theorem establishes the equivalence of (1), (2) and (3), and (3) trivially implies (4). Now assume M^n homogeneous of constant zero curvature, and let $d = (R_d, t_d)$ be a deck transformation of the universal RIEMANNian covering $\pi: R^n \to M^n$. In the proof that (1) imply (3) in the theorem, we saw that $(R_d - I)(R^n)$ must be totally isotropic. Thus $R_d = I$, for R^n is a positive definite euclidean space. *Q. E. D.*

15. The classification theorem for isotropic manifolds of zero curvature

Theorem. *Let M^n_h be a connected pseudo-*RIEMANNian *manifold of constant curvature zero. Then these are equivalent:*

1. *M^n_h is isotropic.*
2. *M^n_h is strongly isotropic.*
3. *M^n_h is isometric to R^n_h.*

Proof. It is clear that (3) implies (2) and that (2) implies (1); now assume M^n_h isotropic. Let D be the group of deck transformations of the universal pseudo-RIEMANNian covering $\pi: R^n_h \to M^n_h$, and write $d = (R_d, t_d) \in NO^h(n)$ for every $d \in D$. The centralizer G of D in $NO^h(n)$ is transitive on R^n_h, and the isotropy subgroup K of G at 0 is irreducible when we view K as a group of linear transformations of the vectorspace $V = R^n_h$. As in the proof of the last theorem, transitivity of G implies that every $(R_d - I)^2 = 0$, whence $R_d - I$ is nilpotent. K centralizes every $R_d - I$ by construction, whence $R_d - I = 0$ by SCHUR's Lemma. Thus D is a group of pure translations (I, t_d). Let W be the subspace of V spanned by the vectors t_d with $d \in D$. Identify V with the tangentspace to R_h^n at O; then a tangentvector $X \in V$ is an element of W if and only if the geodesic $\exp(t\pi_* X)$ in M_h^n lies in a compact set; it follows that W is K-invariant. Thus $W = 0$ or $W = V$, by the irreducibility of K. $W \neq V$ because K cannot carry a closed geodesic of M_h^n onto an open geodesic. We conclude that D consists only of $(I, 0)$.

Q.E.D.

16. The classification theorem for symmetric and isotropic manifolds of constant nonzero curvature

When considering pseudo-RIEMANNian symmetric, isotropic or strongly isotropic manifolds of constant nonzero curvature, we may assume that curvature positive; for $I(\tilde{S}_h^n/D) = I(\tilde{H}_{n-h}^n/D)$.

16.1. Theorem. *Let M_h^n be a connected pseudo-RIEMANNian manifold of constant positive curvature. Then these are equivalent:*

1. *M_h^n is strongly isotropic.*
2. *M_h^n is isotropic.*
3. *M_h^n is symmetric.*
4. *M_h^n is complete, so we can speak of the group D of deck transformations of the universal pseudo-RIEMANNian covering $\beta : \tilde{S}_h^n \to M_h^n$; furthermore, D is a normal subgroup of $I(\tilde{S}_h^n)$.*
5. *If $h < n - 1$, then M_h^n is isometric to S_h^n or $S_h^n/\{\pm I\}$, element of $\mathcal{S}$; if $h = n$, then M_h^n is isometric to $\tilde{S}_h^n$, element of $\mathcal{S}$; if $h = n - 1$, and if $\tilde{A}_Z$ denotes the complete inverse image of $\{\pm I\} \subset O^{n-1}(n+1)$ under the epimorphism $f : \tilde{O}^{n-1}(n+1) \to O^{n-1}(n+1)$ of Lemma 11.1, then M_h^n is isometric to a manifold $\tilde{S}_{n-1}^n/D$ where D is a subgroup of the infinite cyclic group $\tilde{A}_Z$.*
6. *$S_h^n/\{\pm I\}$ admits a pseudo-RIEMANNian covering by M_h^n.*

16.2. *Proof.* If M_h^n satisfies any of the 6 conditions, then it is homogeneous. If $n = h$, this implies that M_h^n is isometric to $S_n^n/\{\pm I\}$, which is isometric to $\tilde{S}_n^n$, whence M_h^n satisfies all 6 conditions. Now we will assume $h < n$. Suppose first that M_h^n admits a pseudo-RIEMANNian covering $\alpha : S_h^n \to M_h^n$ (S_h^n is connected because $h < n$), and let D_α be the group of deck transformations of the covering. α is the universal covering, hence normal, if $h < n - 1$.

α is also normal if $h = n - 1$, for, in the terminology of Lemma 11.1 and of our various conditions, $\beta = \alpha \cdot \pi$ and we need only check that $D_\alpha = f(D)$; the latter is clear because $f^{-1}(D_\alpha)$ is D, being the normalizer of D_π in D. Thus $M_h^n = S_h^n/D_\alpha$. Then, as every discrete normal subgroup of $O^h(n+1)$ is contained in $\{\pm I\}$, conditions (4), (5) and (6) are each equivalent to the condition

7. $D_\alpha \subset \{\pm I\}$.

As (7) implies (1), and (1) implies (2) and (3), we need only check that (2) and (3) each implies (7).

16.3. Suppose that $M_h^n = S_h^n/D_\alpha$ is symmetric. Let $b \in D_\alpha$ and let B be the subgroup of D_α generated by b; the manifold $N_h^n = S_h^n/B$ covers M_h^n, so every symmetry of M_h^n lifts to N_h^n. Thus N_h^n is symmetric. Now every symmetry of N_h^n lifts to a symmetry of S_h^n, whence every symmetry of S_h^n normalizes B. Thus the symmetry $s = \text{diag.}\ \{-1, -1, \ldots, -1, 1\}$ to S_h^n at $p_0 = (0, \ldots, 0, 1)$ normalizes every conjugate of B. We now use homogeneity of M_h^n and the list provided by Theorem 10.1 to prove $D_\alpha \subset \{\pm I\}$, which is the same as $M_h^n \in \mathcal{S}$. If M_h^n is classical but not contained in $\mathcal{S}$, then we have an element $b \in D_\alpha$ which is conjugate to $b' = \text{diag.}\ \{R(v), \ldots, R(v)\}$ where $R(v) = \begin{pmatrix} \cos(2\pi v) & \sin(2\pi v) \\ -\sin(2\pi v) & \cos(2\pi v) \end{pmatrix}$ and v is a rational number such that $\sin(2\pi v) \neq 0$; s normalizes the group B' generated by b', whence $sb's^{-1}b'^{-1} \in B'$; this contradicts $\sin(2\pi v) \neq 0$ because $sb's^{-1}b'^{-1} = \text{diag.}\ \{I_2, \ldots, I_2, S\}$ where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot R(v) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot R(-v).$$

If $M_h^n \in \mathcal{H}_r$, then we have an element $b \in D_\alpha$ which is conjugate to $b' = \pm\ \text{diag.}\ \{Rh(v), \ldots, Rh(v)\}$ where $Rh(v) = \begin{pmatrix} \cosh(v) & \sinh(v) \\ \sinh(v) & \cosh(v) \end{pmatrix}$ and $\sinh(v) \neq 0$; the group B' generated by b' is infinite cyclic and normalized by s, whence $sb's^{-1}b'^{-1} = I$ or b'^{-2}; this contradicts $\sinh(v) \neq 0$ because $sb's^{-1}b'^{-1} = \text{diag.}\ \{I_2, \ldots, I_2, S'\}$ where

$$S' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot Rh(v) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot Rh(-v).$$

Now suppose that M_h^n is neither classical nor in $\mathcal{H}_r$; by Theorem 10.1 and the homogeneity of M_h^n, D_α has an element b which is conjugate to some $t_H(d)$ with d skew, nonsingular and of degree $k = n + 1 - h \leqq h$. If

$\{v_i\}$ is our given $O^h(n+1)$-orthonormal basis of $V = R^{n+1}$, we may set $e_i = v_{h+i} + v_i$ and $f_i = v_{h+i} - v_i$, and

$$t_H(d) = \begin{pmatrix} I_k & 0 & 2d \\ 0 & I_{h-k} & 0 \\ 0 & 0 & I_k \end{pmatrix} \text{ in the basis}$$

$\beta_V = \{f_1, \ldots, f_k; v_{k+1}, \ldots, v_h; e_1, \ldots, e_k\}$ of V. As $p_0 = v_{n+1}$,

$$s = \begin{pmatrix} A & 0 & E \\ 0 & -I_{h-k} & 0 \\ E & 0 & A \end{pmatrix} \text{ in the basis } \beta_V,$$

where $A = \text{diag.}\{-1, \ldots, -1, 0\}$ and $E = I_k + A$. The group B' generated by $t_H(d)$ is infinite cyclic and normalized by s, whence $s \cdot t_H(d) \cdot s^{-1}$ is $t_H(\pm d)$. Using $A^2 + E^2 = I_k$ and $AE = 0 = EA$, we see that

$$s \cdot t_H(d) \cdot s^{-1} = \begin{pmatrix} I_k & 0 & 2AdA \\ 0 & I_{h-k} & 0 \\ 0 & 0 & I_k \end{pmatrix},$$

whence $AdA = \pm d$. As A is singular and d is nonsingular, this is impossible. Thus (3) implies (7).

16.4. Suppose that $M_h^n = S_h^n / D_\alpha$ is isotropic, and let $b \in D_\alpha$. Let G be the normalizer of D_α in $O^h(n+1)$ and let H be the isotropy subgroup at $x \in S_h^n$. H is transitive on the tangentvectors to S_h^n of any given length, whence $H(y)$ is uncountable if $\pm x \neq y \in S_h^n$. If $b(x) \neq \pm x$, this shows that $\{b^{-1} g b g^{-1}(x) : g \in H\}$ is uncountable. On the other hand, $b^{-1} g b g^{-1} \in D_\alpha$ for $g \in H$, and D_α is countable. Thus $b(x) = \pm x$ for every $b \in D_\alpha$ and every $x \in S_h^n$. This shows that every $b \in D_\alpha$ is $\pm I$, whence (2) implies (7).

16.5. We have now seen that our 6 conditions are equivalent if M_h^n admits a pseudo-RIEMANNian covering by S_h^n. Now we may assume $h = n - 1$. We retain the notation of Lemma 11.1. The f-image of a discrete normal subgroup of $\tilde{O}^{n-1}(n+1)$ is a countable normal subgroup of $O^{n-1}(n+1)$, thus contained in $\{\pm I\}$; it follows that (4), (5) and (6) are each equivalent to

8. $D \subset \tilde{A}_z$, i. e., $f(D) \subset \{\pm I\}$.

As (8) implies (1), and (1) implies (2) and (3), the theorem will be proved when we check that (2) and (3) each implies (8), when $h = n - 1$.

16.6. Let M_{n-1}^n be symmetric. As every symmetry of M_{n-1}^n lifts to a symmetry of $\tilde{S}_{n-1}^n$, we see that every symmetry of $\tilde{S}_{n-1}^n$ normalizes D; as every symmetry of S_{n-1}^n lifts to a symmetry of $\tilde{S}_{n-1}^n$, it follows that every symmetry of S_{n-1}^n normalizes $f(D)$. As in § 16.3, we see $f(D) \subset \{\pm I\}$. Thus (3) implies (8).

Let M^n_{n-1} be isotropic, and let G be the normalizer of D in $\tilde{O}^{n-1}(n+1)$. Every isotropy subgroup of G is transitive on the tangentvectors, at the relevant point, of any given length; it follows that the normalizer of $f(D)$ in $O^h(n+1)$ is $f(G)$ and has the same property. As in § 16.4, we see $f(D) \subset \{\pm I\}$. Thus (2) implies (8). *Q. E. D.*

17. Applications

17.1. Theorem. *Let M^n_h be a connected isotropic pseudo-RIEMANNian manifold of constant curvature. Then M^n_h is symmetric.*

Proof. This is an immediate consequence of Theorems 14.1, 15 and 16.1.

17.2. Theorem. *Let M^n_h be a pseudo-RIEMANNian manifold. Then these are equivalent:*

1. *M^n_h is strongly isotropic.*
2. *M^n_h is an isotropic manifold of constant curvature.*

Proof. Note that we may assume M^n_h connected. The theorem is then an immediate consequence of Theorems 15 and 16.1, and of the fact that a strongly isotropic manifold has constant curvature. *Q. E. D.*

17.3. Theorem. *Let $\pi : N^n_h \to M^n_h$ be a pseudo-RIEMANNian covering with N^n_h strongly isotropic, and let D be the group of deck transformations of the covering. Then M^n_h is strongly isotropic of and only if D is a normal subgroup of $I(N^n_h)$; in that case, the covering is normal.*

Proof. N^n_h has constant curvature, and the theorem follows trivially from Theorem 15 if that curvature is zero; we may now assume that N^n_h has constant positive curvature. We have pseudo-RIEMANNian coverings $\alpha : \tilde{S}^n_h \to N^n_h$ and $\beta : \tilde{S}^n_h \to M^n_h$ with respective groups D_α and D_β of deck transforms.

Suppose M^n_h strongly isotropic. Then $D_\alpha \subset D_\beta$, D_α and D_β are normal in $I(\tilde{S}^n_h)$ by Theorem 16.1, and α induces an isomorphism $I(\tilde{S}^n_h)/D_\alpha \to I(N^n_h)$ which carries D_β onto D. Thus D is normal in $I(N^n_h)$, and π is normal.

Suppose D normal in $I(N^n_h)$. We still have an isomorphism $I(\tilde{S}^n_h)/D_\alpha \to I(N^n_h)$ because N^n_h is strongly isotropic, whence the inverse image D_β of D is normal in $I(\tilde{S}^n_h)$. By Theorem 16.1, M^n_h is strongly isotropic. *Q. E. D.*

Added in proof: **1. Remark on Corollary 10.5. If $2h \leq n$, then D is finite even if S^n_h/D is not assumed to be homogeneous. See [7].**

2. Remark on Corollary 14.2. The analogous result is true for manifolds M_h^n provided that $h = 1$ or $h = n-1$ (the Lorenz signatures), or that M_h^n is compact, or that $n \leqq 4$; it is false if these assumptions are not made. See [8].

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Added in proof:

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