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Fibrations and Cocategory[1])

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1. Introduction and results

In a previous paper [3], I have defined a based homotopy type invariant, the cocategory, which appears as dual to LUSTERNIK-SCHNIRELMANN category within the framework of the ECKMANN-HILTON [1] duality in homotopy theory. The cocategory, cocat X, of an arbitrary topological space X with base-point is the (possibly infinite) strictly positive integer given by the following inductive

Definition 1.1. *cocat $X = 1$ if and only if X is contractible; cocat $X \leqq n + 1$ whenever there exists a fibration $Q \to Y \to B$ such that the fibre Q dominates X and cocat $Y \leqq n$. If the phrase cocat $X \leqq n$ is false for all $n \geqq 1$, we put cocat $X = \infty$.*

Evidently, cocat $X = n$ will mean that cocat $X \leqq n$ is true but cocat $X \leqq n - 1$ is false. It is assumed that all homotopies involved in 1.1 keep base-points fixed; the precise sense of the word fibration is explained in the next section.

In the ECKMANN-HILTON setting, fibrations are dual to cofibrations, i. e. sequences $Q \leftarrow Y \leftarrow B$ in which the first arrow results by pinching to a point the image of B in Y, while the second has the lowering homotopy property [5; p. 14]. It follows that cocategory is indeed dual to category since, provided X has the based homotopy type of a connected CW-complex, cat X is equal to the invariant obtained by reversing the arrows and replacing in 1.1 the words *fibration* and *fibre* by *cofibration* and *cofibre* respectively [3; Th. 1.9]. The following relations between cocat X and standard homotopy invariants of X have been established:

1.2. *If X is a 1-connected CW-complex with only n non-trivial Postnikov invariants k^{q+2}, then cocat $X \leqq n + 2$; in particular, if the 1-connected CW-complex X has only n non-trivial homotopy groups, then cocat $X \leqq n + 1$* [3; Th. 2.10 and Cor. 2.11].

1.3. *If X has a non-trivial n-fold WHITEHEAD product, then cocat X $\geqq n + 1$* [3; Cor. 2.13].

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[1]) Presented at the International Colloquium on Differential Geometry and Topology, Zürich, June 1960.

Also, for every $n \geqq 1$ there exists a connected CW-complex X such that cocat $X = n$ [3; Remark 2.16].

The purpose of this paper is to present two further results.

Theorem 1.4. *If X is a $(p-1)$-connected $(p \geqq 2)$ CW-complex such that $\pi_q(X) = 0$ for $q \geqq r+1$, then cocat $X \leqq [(r-1)/(p-1)] + 1$.*

Here $[a/b]$ stands for the largest integer $\leqq a/b$. Theorem 1.4 dualizes a previous result by D. P. GROSSMAN [4] according to which cat $X \leqq [r/p] + 1$ if X is a $(p-1)$-connected complex of dimension $\leqq r$. In fact, GROSSMAN's result may be restated for a 1-connected complex X such that $H^q(X; G) \neq 0$ only if $p \leqq q \leqq r$. The slight difference between the numerical estimation given in 1.4 and that of the GROSSMAN theorem agrees with the relations

$$\dim[\alpha, \beta] = \dim \alpha + \dim \beta - 1 \quad \text{and} \quad \dim u \smile v = \dim u + \dim v,$$

involving WHITEHEAD and cup products which are dual to each other. The proof of 1.4 is based on the extension, given in the next section, of a well known result concerning fibrations with a $K(\pi, n)$ as fibre.

Our next result refers to the cocategory of $(n-1)$-connective spaces (X, n) over X, and to that of spaces (n, X) obtained by attaching cells to X so as to kill its homotopy groups in dimensions $\geqq n$. When X is a CW-complex we shall assume, as we may, that both (X, n) and (n, X) have the based homotopy type of CW-complexes, and state

Theorem 1.5. *Let X be a connected CW-complex. Then, for all $n \geqq 1$, cocat $(X, n) \leqq$ cocat X and cocat $(n, X) \leqq$ cocat X.*

For $n = 2$ we have the

Corollary 1.6. *The simply connected covering space $\tilde{X}$ of a connected CW-complex X satisfies cocat $\tilde{X} \leqq$ cocat X.*

2. A lemma on induced fibrations

All spaces, maps, and homotopies hereafter are assumed to possess, preserve, or keep fixed a base-point, generally denoted by $*$. A sequence $\mathcal{F}: Q \xrightarrow{\eta} Y \xrightarrow{\beta} B$ of spaces and maps is a fibration with fibre $Q = \beta^{-1}(*)$ and inclusion map η if for any space E, any homotopy $h_t: E \to B$ and any map $k: E \to Y$ satisfying $\beta \circ k = h_0$, there is a homotopy $H_t: E \to Y$ such that $H_0 = k$ and $\beta \circ H_t = h_t$. We do not require that β be onto. The space of paths in B emanating from the base-point is denoted by EB, the loop-space by ΩB. Consider the fibration $\mathcal{F}$ above, and let $\Phi: C \to B$ be a map; the sequence $Q \xrightarrow{\zeta} Z \xrightarrow{\gamma} C$ in which

$$Z = \{(c, y) \mid \Phi(c) = \beta(y)\} \subset C \times Y, \quad *_Z = (*_C, *_Y),$$
$$\zeta(q) = (*_C, \eta(q)) \quad \text{and} \quad \gamma(c, y) = c,$$

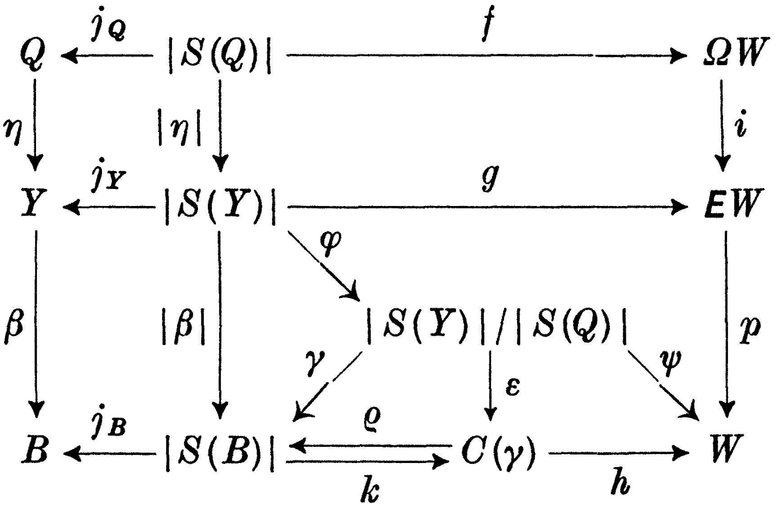
is the familiar fibration induced by $\mathcal{F}$ via Φ. Suppose the rows in the diagram

$$\begin{array}{ccccc} Q_1 & \xrightarrow{\eta_1} & Y_1 & \xrightarrow{\beta_1} & B_1 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Q_2 & \xrightarrow{\eta_2} & Y_2 & \xrightarrow{\beta_2} & B_2 \end{array}$$

are fibrations. The first is algebraically equivalent to the second if there are singular homotopy equivalences f, g, h such that each square commutes. A map $f: X \to Y$ is a singular homotopy equivalence if $f_q: \pi_q(X, x) \to \pi_q(Y, f(x))$ is isomorphic for all $q \geqq 0$ and all $x \in X$; if X and Y are 0-connected, it suffices to take $x = *$. We shall often use the geometric realization $|S(X)|$ of the singular complex of an arbitrary space X and the canonical map $j_X: |S(X)| \to X$ which induces homotopy and homology isomorphisms in all dimensions [7].

Lemma 2.1. *Let* $\mathcal{F}: Q \xrightarrow{\eta} Y \xrightarrow{\beta} B$ *be a fibration with* Y *and* B *both having the based homotopy type of a CW-complex. Suppose that* B *is* $(m-1)$*-connected and that* $\pi_q(Q) \neq 0$ *only if* $n \leqq q \leqq n + m - 2$, *where* $m \geqq 2$ *and* $n \geqq 1$. *Suppose further that there exists a singular homotopy equivalence* $\theta: Q \to \Omega W$, *where* W *is a 1-connected space. Then, there exists a map* $\Phi: B \to W$ *such that* $\mathcal{F}$ *is algebraically equivalent to the fibration induced by* $\mathcal{G}: \Omega W \xrightarrow{i} EW \xrightarrow{p} W$ *via* Φ.

Proof. Introduce the diagram



(1)

The maps $|\eta|$ and $|\beta|$ are induced by η and β respectively, and the two squares on the left commute; $|S(Q)|$ is a subcomplex of $|S(Y)|$ and $|\eta|$ is the inclusion map. Since $|S(*)| = *$ and $\beta \circ \eta(Q) = *$, we have

$$|\beta| \circ |\eta| (|S(Q)|) = * . \tag{2}$$

Let $|S(Y)|/|S(Q)|$ and φ result by pinching the subset $|S(Q)|$ of $|S(Y)|$ to a point, which will serve as base-point in $|S(Y)|/|S(Q)|$. It follows from (2) that $\gamma = |\beta| \circ \varphi^{-1}$ is single-valued, and hence continuous. According to [11; § 8], the space $|S(Y)|/|S(Q)|$ may be given a CW-structure and γ is easily seen to be cellular; therefore, its reduced mapping cylinder $C(\gamma)$ is a CW-complex in which $|S(Y)|/|S(Q)|$ and $|S(B)|$ are embedded as subcomplexes by means of the maps ε and k respectively. The standard retraction ϱ of $C(\gamma)$ onto $|S(B)|$ satisfies the relation $\varrho \circ \varepsilon = \gamma$. Let $f = \theta \circ j_Q$. Since the CW-pair $(|S(Y)|, |S(Q)|)$ has the homotopy extension property and since EW is contractible, there exists a map g such that

$$g \circ |\eta| = i \circ f . \tag{3}$$

Finally, since $p \circ i(\Omega W) = *$, (3) implies that $\psi = p \circ g \circ \varphi^{-1}$ is single-valued, and hence continuous.

We now prove that there is a map h such that

$$h \circ \varepsilon = \psi .$$

Since ε is an inclusion map, this amounts to extending ψ over the complex $C(\gamma)$. The diagram

$$\begin{array}{ccccc}
H_q(Y,Q) & \xleftarrow{j_*} & H_q(|S(Y)|,|S(Q)|) & \xrightarrow{\varphi_*} & H_q(|S(Y)|/|S(Q)|,*) \\
\downarrow \beta_* & & \downarrow |\beta|_* & \swarrow \gamma_* & \downarrow \varepsilon_* \\
H_q(B,*) & \xleftarrow{(j_B)_*} & H_q(|S(B)|,*) & \xleftarrow{\varrho_*} & H_q(C(\gamma),*)
\end{array} \tag{4}$$

in which j_* is induced by the map of pairs defined by j_Y, is obviously commutative. Consideration of the upper left square in (1) and the five lemma show j_* isomorphic in all dimensions; excision in the CW-pair $(|S(Y)|, |S(Q)|)$ implies that so is also φ_*, while $(j_B)_*$ and ϱ_* are standard isomorphisms. Since $\pi_q(Q) = 0$ for $q < n$ and $\pi_q(B) = 0$ for $q < m$, a well known result by SERRE [9; p. 469] implies that β_*, whence $|\beta|_*$ and ε_*, are monomorphic for $q \leqq n + m - 1$ and epimorphic for

$q \leqq n + m$. Passing to cohomology, the universal coefficient theorem yields

$$H^{q+1}(C(\gamma), |S(Y)| / |S(Q)|; G) = 0 \quad \text{for all} \quad q \leqq n + m - 1 \tag{5}$$

and all coefficient groups G. Since $\pi_q(\Omega W) \approx \pi_q(Q)$, we also have

$$\pi_q(W) = 0 \quad \text{for all} \quad q \geqq n + m . \tag{6}$$

From (5) and (6) we finally obtain

$$H^{q+1}(C(\gamma), |S(Y)| / |S(Q)|; \pi_q(W)) = 0 \quad \text{for all} \quad q \geqq 0 ,$$

and a standard obstruction argument now yields the desired map h.

Since Y and B have the based homotopy type of a CW-complex, there exist homotopy inverses e_Y and e_B of j_Y and j_B respectively. Select a homotopy

$$b_t : |S(B)| \to |S(B)| \quad \text{with} \quad b_0 = \mathrm{id} , \quad b_1 = e_B \circ j_B , \quad b_t(*) = * .$$

Notice next that there is a homotopy

$$k_t : |S(Y)| / |S(Q)| \to C(\gamma) \quad \text{with} \quad k_0 = \varepsilon , \quad k_1 = k \circ \gamma , \quad k_t(*) = * .$$

Define a homotopy $H_t : |S(Y)| \to W$ by

$$\begin{aligned} H_t(y) &= h \circ k_{2t} \circ \varphi(y) & \text{if } 0 \leqq t \leqq \tfrac{1}{2} , \\ &= h \circ k \circ b_{2t-1} \circ |\beta| (y) & \text{if } \tfrac{1}{2} \leqq t \leqq 1 . \end{aligned}$$

Taking (2) into account, we obtain

$$H_t(|S(Q)|) = * = p \circ g(|S(Q)|) \quad \text{and} \quad H_0(y) = p \circ g(y) .$$

Therefore, by [6], there is a map $g_1 : |S(Y)| \to EW$ such that

$$g_1 \circ |\eta| = g \circ |\eta| \quad \text{and} \quad h \circ k \circ e_B \circ j_B \circ |\beta| = p \circ g_1 \tag{7}$$

Let $\Phi = h \circ k \circ e_B$ and let $\mathscr{H} : \Omega W \xrightarrow{\zeta} Z \xrightarrow{\lambda} B$ be the fibration induced by $\mathcal{G}$ via Φ. According to (7), a map $d : |S(Y)| \to Z$, satisfying

$$d \circ |\eta| = \zeta \circ f \quad \text{and} \quad j_B \circ |\beta| = \lambda \circ d , \tag{8}$$

is defined by setting $d(y) = (j_B \circ |\beta| (y), g_1(y))$. In the sequence

$$\pi_q(|S(Y)|, |S(Q)|) \xrightarrow{j_q} \pi_q(Y, Q) \xrightarrow{\beta_q} \pi_q(B, *) \xleftarrow{\lambda_q} \pi_q(Z, \Omega W) ,$$

where j_q is induced by the map of pairs defined by j_Y, the first arrow, as in (4), is isomorphic for all $q \geqq 1$; since $\mathcal{F}$ and $\mathcal{H}$ are fibrations, so are also β_q and λ_q. Therefore, (8) and commutativity on the left in (1) imply that the map of pairs defined by d induces isomorphisms

$$\pi_q(|S(Y)|, |S(Q)|) \approx \pi_q(Z, \Omega W)$$

in all dimensions. Since $f = \theta \circ j_Q$ is a singular homotopy equivalence, the first of the relations (8) and the five lemma now imply that d also is a singular homotopy equivalence. As easily seen, the map $\lambda \circ d \circ e_Y : Y \to B$ is homotopic to β. Since $\mathcal{H}$ is a fibration, the covering homotopy theorem yields a map $D : Y \to Z$, which is homotopic to $d \circ e_Y$, and satisfies $\lambda \circ D = \beta$; let $F : Q \to \Omega W$ be the map defined by D. Like $d \circ e_Y$, D is a singular homotopy equivalence; the five lemma implies that so is also F, and the required algebraic equivalence is now provided by the maps id_B, D, F.

Remark 2.2. Letting $m = 2$ in 2.1 we recover the well known result concerning fibrations with a $K(\pi, n)$ as fibre (see for instance [5; Th. 7.1, p. 43]).
Lemma 2.1 has a dual concerning induced cofibrations.

3. Proof of Theorem 1.4

It is well known that any $(n-1)$-connected CW-complex of dimension $< 2n$ has the homotopy type of a suspension. Dually, we have

Lemma 3.1. *Let X be an arbitrary space and let $n \geqq 2$. If X is $(n-1)$-connected and $\pi_q(X) = 0$ for $q \geqq 2n - 1$, then there exists a 1-connected space W and a singular homotopy equivalence $X \to \Omega W$.*

Proof. The space W is obtained by attaching cells to the reduced suspension ΣX so as to kill its homotopy groups in dimensions $\geqq 2n$. Let $\sigma : \Sigma X \to W$ denote the inclusion map and consider the sequence

$$X \overset{e}{\to} \Omega \Sigma X \overset{\Omega\sigma}{\to} \Omega W,$$

in which e is the natural embedding. Evidently, $\Omega\sigma$ induces isomorphisms of homotopy groups in dimensions $\leqq 2n - 2$; by the FREUDENTHAL theorem (see for instance [8; p. 05]), so does also e. Finally, for $q \geqq 2n - 1$ we have $\pi_q(X) = \pi_q(\Omega W) = 0$.

Proof of 1.4. The result is obvious if $1 \leqq r \leqq p - 1$ since X then is contractible. Suppose $r \geqq p$ and let X be an arbitrary $(p-1)$-connected

CW-complex such that $\pi_q(X) = 0$ for $q \geqq r+1$. Let the CW-complex B result by attaching cells to X so as to kill its homotopy groups in dimensions $\geqq r-p+2$. Replace the inclusion map $X \to B$ by a homotopy equivalent fibre map to obtain a fibration $\mathcal{F}: Q \to Y \to B$ such that

$$Y \textit{ has the homotopy type of } X , \tag{9}$$

$$\pi_q(B) \neq 0 \quad \text{only if} \quad p \leqq q \leqq r-p+1 , \tag{10}$$

$$\pi_q(Q) \neq 0 \quad \text{only if} \quad \max(p, r-p+2) \leqq q \leqq r . \tag{11}$$

Since $r-p+1<r$, we may assume as an induction hypothesis that (10) implies

$$\operatorname{cocat} B \leqq [(r-p)/(p-1)]+1 . \tag{12}$$

It follows from (10), (11), 3.1, and Lemma 2.1 that there is a 1-connected space W and a map $\Phi: B \to W$ such that $\mathcal{F}$ is algebraically equivalent to the fibration $\Omega W \to Z \to B$ induced by $\Omega W \to EW \to W$ via Φ. Therefore, Y has the homotopy type of the singular polytope of Z. By (9), [3; Prop. 2.8 and 2.9], and (12) we finally obtain

$$\operatorname{cocat} X = \operatorname{cocat} |S(Z)| \leqq \operatorname{cocat} Z \leqq \operatorname{cocat} B + 1 \leqq [(r-1)/(p-1)]+1 .$$

4. Proof of Theorem 1.5

For any 0-connected space X and any $n \geqq 1$ there is a space (X, n) and a map $p: (X, n) \to X$ such that $\pi_q(X, n) = 0$ if $q < n$ and p_q: $\pi_q(X, n) \approx \pi_q(X)$ if $q \geqq n$. Similarly, there is a space (n, X) and a map $j: X \to (n, X)$ such that $\pi_q(n, X) = 0$ if $q \geqq n$ and $j_q: \pi_q(X) \approx \pi_q(n, X)$ if $q < n$. When X has the homotopy type of a CW-complex, we shall assume, as we may, that the same holds for both (X, n) and (n, X); their homotopy type is then uniquely determined by that of X and n.

Proof of 1.5. If $\operatorname{cocat} X = 1$, then X is contractible and so are both (X, n) and (n, X). Suppose 1.5 is true for any connected CW-complex of cocategory $\leqq m$ and suppose $\operatorname{cocat} X = m+1$. Let $Q \xrightarrow{\eta} Y \xrightarrow{\beta} B$ be a fibration such that Q dominates X and $\operatorname{cocat} Y = m$.

Let $R \xrightarrow{\varphi} Z \xrightarrow{\psi} |S(B)|_0$ denote the fibration obtained by replacing the map $|\beta|_0: |S(Y)|_0 \to |S(B)|_0$ by a homotopy equivalent fibre map; the subscript 0 indicates restriction to the path-component of the base-point.

As easily seen, there is a map $r: R \to Q$ which, by the five lemma, induces isomorphisms

$$r_q : \pi_q(R) \approx \pi_q(Q) \quad \text{for all} \quad q \geqq 1 . \tag{13}$$

Let C be a connected covering space of $| S(B) |_0$ such that $\pi_1(C)$ maps isomorphically onto the subgroup $\psi_1 \pi_1(Z)$ of $\pi_1(| S(B) |_0)$ under the projection $f: C \to | S(B) |_0$. Since Z has the homotopy type of a connected CW-complex, the monodromy principle yields a map $g: Z \to C$ such that $f \circ g = \psi$. Let $T \xrightarrow{\varepsilon} W \xrightarrow{\gamma} C$ be the fibration obtained by replacing g by a homotopy equivalent fibre map. As above, there is a map $t: T \to R$ which, since f_1 is monomorphic, induces isomorphisms

$$t_q : \pi_q(T) \approx \pi_q(R) \quad \text{for all} \quad q \geqq 1 . \tag{14}$$

For the same reason and since $f_1 \pi_1(C) = \psi_1 \pi_1(Z)$, the homomorphism $\gamma_1 : \pi_1(W) \to \pi_1(C)$ is onto. Therefore, in $\pi_n(C)$ the subgroup

$$\gamma_n \pi_n(W) \textit{ is closed under the operations of } \pi_1(C). \tag{15}$$

Introduce the diagram

$$\begin{array}{ccccc}
U & \xrightarrow{\sigma} & (W, n) & \xrightarrow{\lambda} & D \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle d} \\
T & \xrightarrow{\varepsilon} & W & \xrightarrow{\gamma} & C \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle e} \\
V & \xrightarrow{\tau} & (n, W) & \xrightarrow{\mu} & E
\end{array}$$

The space E and the inclusion map e are obtained by attaching cells to C in such a way that

$$e_q : \pi_q(C) \approx \pi_q(E) \quad \text{if} \quad q < n ,$$

the sequence

$$\pi_n(W) \xrightarrow{\gamma_n} \pi_n(C) \xrightarrow{e_n} \pi_n(E) \to 0$$

be exact, and $\pi_q(E) = 0$ if $q > n$; according to [10; Th. 2.10.1] this is possible in view of (15). The space D and the map d are selected so that $\pi_q(D) = 0$ if $q < n$,

$$d_n : \pi_n(D) \approx \gamma_n \pi_n(W) \quad \text{and} \quad d_q : \pi_q(D) \approx \pi_q(C)$$

if $q > n$. Since W has the homotopy type of a connected CW-complex, so have, by assumption, (W, n) and (n, W), and standard arguments now yield maps λ and μ for which the two squares on the right are homotopy commutative. Without altering the homotopy types of (W, n) and (n, W), we may assume that λ and μ are fibre maps with U and V as fibres; the inclusion maps are denoted by σ and τ. Next, by means of the covering homotopy theorem, we may readjust the maps p and j within their own homotopy classes so as to obtain totally commutative squares on the right. Suppose this is so and let h and k be the maps defined by p and j respectively. Passing to homotopy groups, application of the five lemma, in the form given in [2; p. 16], to the resulting ladder yields

$$\pi_q(U) = 0 \quad \text{if} \quad q < n\,, \quad h_q : \pi_q(U) \approx \pi_q(T) \quad \text{if} \quad q \geqq n\,, \tag{16}$$

$$\pi_q(V) = 0 \quad \text{if} \quad q \geqq n\,, \quad k_q : \pi_q(T) \approx \pi_q(V) \quad \text{if} \quad q < n\,. \tag{17}$$

Since X is a connected CW-complex which is dominated by Q, X is also dominated by $|S(Q)|_0$. Since γ_1 is onto and W is 0-connected, T is 0-connected and, by (13) and (14), $|S(Q)|_0$ has the homotopy type of $|S(T)|$. It follows from (16) that $(|S(T)|, n)$ has the homotopy type of $|S(U)|$, while (17) implies that $(n, |S(T)|)$ has the homotopy type of $|S(V)|$. Since (X, n) and (n, X) have the homotopy type of CW-complexes, it follows easily that (X, n) is dominated by $|S(U)|$, and (n, X) by $|S(V)|$.

Since W, like Z, has the homotopy type of $|S(Y)|_0$, and since the component of the base-point in a CW-complex is a retract of the complex, by [3; Prop. 2.8] we have

$$\text{cocat}\, W = \text{cocat}\, |S(Y)|_0 \leqq \text{cocat}\, |S(Y)| \leqq \text{cocat}\, Y = m\,.$$

Since W has the homotopy type of a connected CW-complex, the induction hypothesis now implies that $\text{cocat}\,(W, n) \leqq m$ and $\text{cocat}\,(n, W) \leqq m$. By [3; Prop. 2.8] and 1.1 we obtain

$$\text{cocat}\, |S(U)| \leqq \text{cocat}\, U \leqq m + 1\,, \quad \text{cocat}\, |S(V)| \leqq \text{cocat}\, V \leqq m + 1\,,$$

and this clearly implies the desired result.

Appendix

(Added in proof)

The inductive arguments used in the proof of 1.4 are easily seen to yield the following more general result:

Let X be a $(p-1)$-connected CW-complex, $p \geqq 2$. If the set of all integers q for which $\pi_q(X) \neq 0$ is contained in the union of k closed linear intervals, each of length $p-2$, then cocat $X \leqq k + 1$.

We allow the linear intervals to be degenerate, i.e. to have length 0. The second part of 1.2 now follows as the set $\{q_1, \ldots, q_n\}$ is contained in the intervals $[q_j, q_j]$, $j = 1, \ldots, n$; Theorem 1.4 follows upon noticing that the integers between p and r are all contained in the intervals

$$[j(p-1)+1, j(p-1)+p-1], j = 1, \ldots, \left[\frac{r-1}{p-1}\right].$$

Also, the author wishes to acknowledge that a result equivalent to Lemma 2.1 above has been obtained independently and with a different proof by P. J. HILTON as Theorem 3 in his paper "Excision and principal fibrations", Comment. Math. Helv. 35 (1961).

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