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Autor: Ganea, Tudor
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Fibrations and Cocategory¹⁾

by TUDOR GANEA, Bucharest

1. Introduction and results

In a previous paper [3], I have defined a based homotopy type invariant, the cocategory, which appears as dual to LUSTERNIK-SCHNIRELMANN category within the framework of the ECKMANN-HILTON [1] duality in homotopy theory. The cocategory, $\text{cocat } X$, of an arbitrary topological space X with base-point is the (possibly infinite) strictly positive integer given by the following inductive

Definition 1.1. *$\text{cocat } X = 1$ if and only if X is contractible; $\text{cocat } X \leq n + 1$ whenever there exists a fibration $Q \rightarrow Y \rightarrow B$ such that the fibre Q dominates X and $\text{cocat } Y \leq n$. If the phrase $\text{cocat } X \leq n$ is false for all $n \geq 1$, we put $\text{cocat } X = \infty$.*

Evidently, $\text{cocat } X = n$ will mean that $\text{cocat } X \leq n$ is true but $\text{cocat } X \leq n - 1$ is false. It is assumed that all homotopies involved in 1.1 keep base-points fixed; the precise sense of the word fibration is explained in the next section.

In the ECKMANN-HILTON setting, fibrations are dual to cofibrations, i. e. sequences $Q \leftarrow Y \leftarrow B$ in which the first arrow results by pinching to a point the image of B in Y , while the second has the lowering homotopy property [5; p. 14]. It follows that cocategory is indeed dual to category since, provided X has the based homotopy type of a connected CW -complex, $\text{cat } X$ is equal to the invariant obtained by reversing the arrows and replacing in 1.1 the words *fibration* and *fibre* by *cofibration* and *cofibre* respectively [3; Th. 1.9]. The following relations between $\text{cocat } X$ and standard homotopy invariants of X have been established:

1.2. *If X is a 1-connected CW -complex with only n non-trivial Postnikov invariants k^{q+2} , then $\text{cocat } X \leq n + 2$; in particular, if the 1-connected CW -complex X has only n non-trivial homotopy groups, then $\text{cocat } X \leq n + 1$ [3; Th. 2.10 and Cor. 2.11].*

1.3. *If X has a non-trivial n -fold WHITEHEAD product, then $\text{cocat } X \geq n + 1$ [3; Cor. 2.13].*

¹⁾ Presented at the International Colloquium on Differential Geometry and Topology, Zürich, June 1960.

Also, for every $n \geq 1$ there exists a connected CW -complex X such that $\text{cocat } X = n$ [3; Remark 2.16].

The purpose of this paper is to present two further results.

Theorem 1.4. *If X is a $(p-1)$ -connected ($p \geq 2$) CW -complex such that $\pi_q(X) = 0$ for $q \geq r+1$, then $\text{cocat } X \leq [(r-1)/(p-1)] + 1$.*

Here $[a/b]$ stands for the largest integer $\leq a/b$. Theorem 1.4 dualizes a previous result by D. P. GROSSMAN [4] according to which $\text{cat } X \leq [r/p] + 1$ if X is a $(p-1)$ -connected complex of dimension $\leq r$. In fact, GROSSMAN's result may be restated for a 1-connected complex X such that $H^q(X; G) \neq 0$ only if $p \leq q \leq r$. The slight difference between the numerical estimation given in 1.4 and that of the GROSSMAN theorem agrees with the relations

$$\dim[\alpha, \beta] = \dim \alpha + \dim \beta - 1 \quad \text{and} \quad \dim u \smile v = \dim u + \dim v,$$

involving WHITEHEAD and cup products which are dual to each other. The proof of 1.4 is based on the extension, given in the next section, of a well known result concerning fibrations with a $K(\pi, n)$ as fibre.

Our next result refers to the cocategory of $(n-1)$ -connective spaces (X, n) over X , and to that of spaces (n, X) obtained by attaching cells to X so as to kill its homotopy groups in dimensions $\geq n$. When X is a CW -complex we shall assume, as we may, that both (X, n) and (n, X) have the based homotopy type of CW -complexes, and state

Theorem 1.5. *Let X be a connected CW -complex. Then, for all $n \geq 1$, $\text{cocat } (X, n) \leq \text{cocat } X$ and $\text{cocat } (n, X) \leq \text{cocat } X$.*

For $n = 2$ we have the

Corollary 1.6. *The simply connected covering space \tilde{X} of a connected CW -complex X satisfies $\text{cocat } \tilde{X} \leq \text{cocat } X$.*

2. A lemma on induced fibrations

All spaces, maps, and homotopies hereafter are assumed to possess, preserve, or keep fixed a base-point, generally denoted by $*$. A sequence $\mathcal{F}: Q \xrightarrow{\eta} Y \xrightarrow{\beta} B$ of spaces and maps is a fibration with fibre $Q = \beta^{-1}(*)$ and inclusion map η if for any space E , any homotopy $h_t: E \rightarrow B$ and any map $k: E \rightarrow Y$ satisfying $\beta \circ k = h_0$, there is a homotopy $H_t: E \rightarrow Y$ such that $H_0 = k$ and $\beta \circ H_t = h_t$. We do not require that β be onto. The space of paths in B emanating from the base-point is denoted by EB , the loop-space by ΩB . Consider the fibration \mathcal{F} above, and let $\Phi: C \rightarrow B$ be a map; the sequence $Q \xrightarrow{\xi} Z \xrightarrow{\gamma} C$ in which

$$Z = \{(c, y) \mid \Phi(c) = \beta(y)\} \subset C \times Y, \quad *_Z = (*_C, *_Y),$$

$$\zeta(q) = (*_C, \eta(q)) \quad \text{and} \quad \gamma(c, y) = c,$$

is the familiar fibration induced by \mathcal{F} via Φ . Suppose the rows in the diagram

$$\begin{array}{ccccc} Q_1 & \xrightarrow{\eta_1} & Y_1 & \xrightarrow{\beta_1} & B_1 \\ \downarrow h & & \downarrow g & & \downarrow f \\ Q_2 & \xrightarrow{\eta_2} & Y_2 & \xrightarrow{\beta_2} & B_2 \end{array}$$

are fibrations. The first is algebraically equivalent to the second if there are singular homotopy equivalences f, g, h such that each square commutes. A map $f: X \rightarrow Y$ is a singular homotopy equivalence if $f_q: \pi_q(X, x) \rightarrow \pi_q(Y, f(x))$ is isomorphic for all $q \geq 0$ and all $x \in X$; if X and Y are 0-connected, it suffices to take $x = *$. We shall often use the geometric realization $|S(X)|$ of the singular complex of an arbitrary space X and the canonical map $j_X: |S(X)| \rightarrow X$ which induces homotopy and homology isomorphisms in all dimensions [7].

Lemma 2.1. *Let $\mathcal{F}: Q \xrightarrow{\eta} Y \xrightarrow{\beta} B$ be a fibration with Y and B both having the based homotopy type of a CW-complex. Suppose that B is $(m-1)$ -connected and that $\pi_q(Q) \neq 0$ only if $n \leq q \leq n+m-2$, where $m \geq 2$ and $n \geq 1$. Suppose further that there exists a singular homotopy equivalence $\theta: Q \rightarrow \Omega W$, where W is a 1-connected space. Then, there exists a map $\Phi: B \rightarrow W$ such that \mathcal{F} is algebraically equivalent to the fibration induced by $\mathcal{G}: \Omega W \xrightarrow{i} EW \xrightarrow{p} W$ via Φ .*

Proof. Introduce the diagram

$$\begin{array}{ccccc} Q & \xleftarrow{j_Q} & |S(Q)| & \xrightarrow{f} & \Omega W \\ \eta \downarrow & & \downarrow |\eta| & & \downarrow i \\ Y & \xleftarrow{j_Y} & |S(Y)| & \xrightarrow{g} & EW \\ \beta \downarrow & & \downarrow |\beta| & & \downarrow p \\ B & \xleftarrow{j_B} & |S(B)| & \xrightarrow{k} & C(\gamma) \xrightarrow{h} W \\ & & \downarrow \epsilon & & \downarrow \psi \\ & & & & W \end{array} \quad (1)$$

$\begin{array}{ccc} & \nearrow \varphi & \\ & |S(Y)| // |S(Q)| & \\ & \nwarrow \gamma & \end{array}$

The maps $|\eta|$ and $|\beta|$ are induced by η and β respectively, and the two squares on the left commute; $|S(Q)|$ is a subcomplex of $|S(Y)|$ and $|\eta|$ is the inclusion map. Since $|S(*)| = *$ and $\beta \circ \eta(Q) = *$, we have

$$|\beta| \circ |\eta| (|S(Q)|) = *. \quad (2)$$

Let $|S(Y)| / |S(Q)|$ and φ result by pinching the subset $|S(Q)|$ of $|S(Y)|$ to a point, which will serve as base-point in $|S(Y)| / |S(Q)|$. It follows from (2) that $\gamma = |\beta| \circ \varphi^{-1}$ is single-valued, and hence continuous. According to [11; § 8], the space $|S(Y)| / |S(Q)|$ may be given a *CW*-structure and γ is easily seen to be cellular; therefore, its reduced mapping cylinder $C(\gamma)$ is a *CW*-complex in which $|S(Y)| / |S(Q)|$ and $|S(B)|$ are embedded as subcomplexes by means of the maps ε and k respectively. The standard retraction ϱ of $C(\gamma)$ onto $|S(B)|$ satisfies the relation $\varrho \circ \varepsilon = \gamma$. Let $f = \theta \circ j_Q$. Since the *CW*-pair $(|S(Y)|, |S(Q)|)$ has the homotopy extension property and since *EW* is contractible, there exists a map g such that

$$g \circ |\eta| = i \circ f. \quad (3)$$

Finally, since $p \circ i(\Omega W) = *$, (3) implies that $\psi = p \circ g \circ \varphi^{-1}$ is single-valued, and hence continuous.

We now prove that there is a map h such that

$$h \circ \varepsilon = \psi.$$

Since ε is an inclusion map, this amounts to extending ψ over the complex $C(\gamma)$. The diagram

$$\begin{array}{ccccc} H_q(Y, Q) & \xleftarrow{j_*} & H_q(|S(Y)|, |S(Q)|) & \xrightarrow{\varphi_*} & H_q(|S(Y)| / |S(Q)|, *) \\ \downarrow \beta_* & & \downarrow |\beta|_* & \swarrow \gamma_* & \downarrow \varepsilon_* \\ H_q(B, *) & \xleftarrow{(j_B)_*} & H_q(|S(B)|, *) & \xleftarrow{\varrho_*} & H_q(C(\gamma), *) \end{array} \quad (4)$$

in which j_* is induced by the map of pairs defined by j_Y , is obviously commutative. Consideration of the upper left square in (1) and the five lemma show j_* isomorphic in all dimensions; excision in the *CW*-pair $(|S(Y)|, |S(Q)|)$ implies that so is also φ_* , while $(j_B)_*$ and ϱ_* are standard isomorphisms. Since $\pi_q(Q) = 0$ for $q < n$ and $\pi_q(B) = 0$ for $q < m$, a well known result by SERRE [9; p. 469] implies that β_* , whence $|\beta|_*$ and ε_* , are monomorphic for $q \leq n + m - 1$ and epimorphic for

$q \leq n + m$. Passing to cohomology, the universal coefficient theorem yields

$$H^{q+1}(C(\gamma), |S(Y)| / |S(Q)|; G) = 0 \quad \text{for all } q \leq n + m - 1 \quad (5)$$

and all coefficient groups G . Since $\pi_q(\Omega W) \approx \pi_q(Q)$, we also have

$$\pi_q(W) = 0 \quad \text{for all } q \geq n + m. \quad (6)$$

From (5) and (6) we finally obtain

$$H^{q+1}(C(\gamma), |S(Y)| / |S(Q)|; \pi_q(W)) = 0 \quad \text{for all } q \geq 0,$$

and a standard obstruction argument now yields the desired map h .

Since Y and B have the based homotopy type of a CW -complex, there exist homotopy inverses e_Y and e_B of j_Y and j_B respectively. Select a homotopy

$$b_t: |S(B)| \rightarrow |S(B)| \quad \text{with } b_0 = \text{id}, \quad b_1 = e_B \circ j_B, \quad b_t(*) = *.$$

Notice next that there is a homotopy

$$k_t: |S(Y)| / |S(Q)| \rightarrow C(\gamma) \quad \text{with } k_0 = \varepsilon, \quad k_1 = k \circ \gamma, \quad k_t(*) = *.$$

Define a homotopy $H_t: |S(Y)| \rightarrow W$ by

$$\begin{aligned} H_t(y) &= h \circ k_{2t} \circ \varphi(y) && \text{if } 0 \leq t \leq \frac{1}{2}, \\ &= h \circ k \circ b_{2t-1} \circ |\beta|(y) && \text{if } \frac{1}{2} \leq t \leq 1. \end{aligned}$$

Taking (2) into account, we obtain

$$H_t(|S(Q)|) = * = p \circ g(|S(Q)|) \quad \text{and} \quad H_0(y) = p \circ g(y).$$

Therefore, by [6], there is a map $g_1: |S(Y)| \rightarrow EW$ such that

$$g_1 \circ |\eta| = g \circ |\eta| \quad \text{and} \quad h \circ k \circ e_B \circ j_B \circ |\beta| = p \circ g_1 \quad (7)$$

Let $\Phi = h \circ k \circ e_B$ and let $\mathcal{H}: \Omega W \xrightarrow{\zeta} Z \xrightarrow{\lambda} B$ be the fibration induced by \mathcal{G} via Φ . According to (7), a map $d: |S(Y)| \rightarrow Z$, satisfying

$$d \circ |\eta| = \zeta \circ f \quad \text{and} \quad j_B \circ |\beta| = \lambda \circ d, \quad (8)$$

is defined by setting $d(y) = (j_B \circ |\beta|(y), g_1(y))$. In the sequence

$$\pi_q(|S(Y)|, |S(Q)|) \xrightarrow{j_q} \pi_q(Y, Q) \xrightarrow{\beta_q} \pi_q(B, *) \xleftarrow{\lambda_q} \pi_q(Z, \Omega W),$$

where j_q is induced by the map of pairs defined by j_Y , the first arrow, as in (4), is isomorphic for all $q \geq 1$; since \mathcal{F} and \mathcal{H} are fibrations, so are also β_q and λ_q . Therefore, (8) and commutativity on the left in (1) imply that the map of pairs defined by d induces isomorphisms

$$\pi_q(|S(Y)|, |S(Q)|) \approx \pi_q(Z, \Omega W)$$

in all dimensions. Since $f = \theta \circ j_Q$ is a singular homotopy equivalence, the first of the relations (8) and the five lemma now imply that d also is a singular homotopy equivalence. As easily seen, the map $\lambda \circ d \circ e_Y: Y \rightarrow B$ is homotopic to β . Since \mathcal{H} is a fibration, the covering homotopy theorem yields a map $D: Y \rightarrow Z$, which is homotopic to $d \circ e_Y$, and satisfies $\lambda \circ D = \beta$; let $F: Q \rightarrow \Omega W$ be the map defined by D . Like $d \circ e_Y$, D is a singular homotopy equivalence; the five lemma implies that so is also F , and the required algebraic equivalence is now provided by the maps id_B , D , F .

Remark 2.2. Letting $m = 2$ in 2.1 we recover the well known result concerning fibrations with a $K(\pi, n)$ as fibre (see for instance [5; Th. 7.1, p. 43]).

Lemma 2.1 has a dual concerning induced cofibrations.

3. Proof of Theorem 1.4

It is well known that any $(n - 1)$ -connected CW -complex of dimension $< 2n$ has the homotopy type of a suspension. Dually, we have

Lemma 3.1. *Let X be an arbitrary space and let $n \geq 2$. If X is $(n - 1)$ -connected and $\pi_q(X) = 0$ for $q \geq 2n - 1$, then there exists a 1-connected space W and a singular homotopy equivalence $X \rightarrow \Omega W$.*

Proof. The space W is obtained by attaching cells to the reduced suspension ΣX so as to kill its homotopy groups in dimensions $\geq 2n$. Let $\sigma: \Sigma X \rightarrow W$ denote the inclusion map and consider the sequence

$$X \xrightarrow{e} \Omega \Sigma X \xrightarrow{\Omega \sigma} \Omega W,$$

in which e is the natural embedding. Evidently, $\Omega \sigma$ induces isomorphisms of homotopy groups in dimensions $\leq 2n - 2$; by the FREUDENTHAL theorem (see for instance [8; p. 05]), so does also e . Finally, for $q \geq 2n - 1$ we have $\pi_q(X) = \pi_q(\Omega W) = 0$.

Proof of 1.4. The result is obvious if $1 \leq r \leq p - 1$ since X then is contractible. Suppose $r \geq p$ and let X be an arbitrary $(p - 1)$ -connected

CW -complex such that $\pi_q(X) = 0$ for $q \geq r + 1$. Let the CW -complex B result by attaching cells to X so as to kill its homotopy groups in dimensions $\geq r - p + 2$. Replace the inclusion map $X \rightarrow B$ by a homotopy equivalent fibre map to obtain a fibration $\mathcal{F}: Q \rightarrow Y \rightarrow B$ such that

$$Y \text{ has the homotopy type of } X, \quad (9)$$

$$\pi_q(B) \neq 0 \quad \text{only if} \quad p \leq q \leq r - p + 1, \quad (10)$$

$$\pi_q(Q) \neq 0 \quad \text{only if} \quad \max(p, r - p + 2) \leq q \leq r. \quad (11)$$

Since $r - p + 1 < r$, we may assume as an induction hypothesis that (10) implies

$$\text{cocat } B \leq [(r - p)/(p - 1)] + 1. \quad (12)$$

It follows from (10), (11), 3.1, and Lemma 2.1 that there is a 1-connected space W and a map $\Phi: B \rightarrow W$ such that \mathcal{F} is algebraically equivalent to the fibration $\Omega W \rightarrow Z \rightarrow B$ induced by $\Omega W \rightarrow EW \rightarrow W$ via Φ . Therefore, Y has the homotopy type of the singular polytope of Z . By (9), [3; Prop. 2.8 and 2.9], and (12) we finally obtain

$$\text{cocat } X = \text{cocat } |S(Z)| \leq \text{cocat } Z \leq \text{cocat } B + 1 \leq [(r - 1)/(p - 1)] + 1.$$

4. Proof of Theorem 1.5

For any 0-connected space X and any $n \geq 1$ there is a space (X, n) and a map $p: (X, n) \rightarrow X$ such that $\pi_q(X, n) = 0$ if $q < n$ and $p_q: \pi_q(X, n) \approx \pi_q(X)$ if $q \geq n$. Similarly, there is a space (n, X) and a map $j: X \rightarrow (n, X)$ such that $\pi_q(n, X) = 0$ if $q \geq n$ and $j_q: \pi_q(X) \approx \pi_q(n, X)$ if $q < n$. When X has the homotopy type of a CW -complex, we shall assume, as we may, that the same holds for both (X, n) and (n, X) ; their homotopy type is then uniquely determined by that of X and n .

Proof of 1.5. If $\text{cocat } X = 1$, then X is contractible and so are both (X, n) and (n, X) . Suppose 1.5 is true for any connected CW -complex of cocategory $\leq m$ and suppose $\text{cocat } X = m + 1$. Let $Q \xrightarrow{\eta} Y \xrightarrow{\beta} B$ be a fibration such that Q dominates X and $\text{cocat } Y = m$.

Let $R \xrightarrow{\varphi} Z \xrightarrow{\psi} |S(B)|_0$ denote the fibration obtained by replacing the map $|\beta|_0: |S(Y)|_0 \rightarrow |S(B)|_0$ by a homotopy equivalent fibre map; the subscript 0 indicates restriction to the path-component of the base-point.

As easily seen, there is a map $r: R \rightarrow Q$ which, by the five lemma, induces isomorphisms

$$r_q: \pi_q(R) \approx \pi_q(Q) \quad \text{for all } q \geq 1. \quad (13)$$

Let C be a connected covering space of $|S(B)|_0$ such that $\pi_1(C)$ maps isomorphically onto the subgroup $\psi_1\pi_1(Z)$ of $\pi_1(|S(B)|_0)$ under the projection $f: C \rightarrow |S(B)|_0$. Since Z has the homotopy type of a connected CW -complex, the monodromy principle yields a map $g: Z \rightarrow C$ such that $f \circ g = \psi$. Let $T \xrightarrow{\varepsilon} W \xrightarrow{\gamma} C$ be the fibration obtained by replacing g by a homotopy equivalent fibre map. As above, there is a map $t: T \rightarrow R$ which, since f_1 is monomorphic, induces isomorphisms

$$t_q: \pi_q(T) \approx \pi_q(R) \quad \text{for all } q \geq 1. \quad (14)$$

For the same reason and since $f_1\pi_1(C) = \psi_1\pi_1(Z)$, the homomorphism $\gamma_1: \pi_1(W) \rightarrow \pi_1(C)$ is onto. Therefore, in $\pi_n(C)$ the subgroup

$$\gamma_n\pi_n(W) \text{ is closed under the operations of } \pi_1(C). \quad (15)$$

Introduce the diagram

$$\begin{array}{ccccc} U & \xrightarrow{\sigma} & (W, n) & \xrightarrow{\lambda} & D \\ \downarrow h & & \downarrow p & & \downarrow d \\ T & \xrightarrow{\varepsilon} & W & \xrightarrow{\gamma} & C \\ \downarrow k & & \downarrow j & & \downarrow e \\ V & \xrightarrow{\tau} & (n, W) & \xrightarrow{\mu} & E \end{array}$$

The space E and the inclusion map e are obtained by attaching cells to C in such a way that

$$e_q: \pi_q(C) \approx \pi_q(E) \quad \text{if } q < n,$$

the sequence

$$\pi_n(W) \xrightarrow{\gamma_n} \pi_n(C) \xrightarrow{e_n} \pi_n(E) \rightarrow 0$$

be exact, and $\pi_q(E) = 0$ if $q > n$; according to [10; Th. 2.10.1] this is possible in view of (15). The space D and the map d are selected so that $\pi_q(D) = 0$ if $q < n$,

$$d_n: \pi_n(D) \approx \gamma_n\pi_n(W) \quad \text{and} \quad d_q: \pi_q(D) \approx \pi_q(C)$$

if $q > n$. Since W has the homotopy type of a connected CW -complex, so have, by assumption, (W, n) and (n, W) , and standard arguments now

yield maps λ and μ for which the two squares on the right are homotopy commutative. Without altering the homotopy types of (W, n) and (n, W) , we may assume that λ and μ are fibre maps with U and V as fibres; the inclusion maps are denoted by σ and τ . Next, by means of the covering homotopy theorem, we may readjust the maps p and j within their own homotopy classes so as to obtain totally commutative squares on the right. Suppose this is so and let h and k be the maps defined by p and j respectively. Passing to homotopy groups, application of the five lemma, in the form given in [2; p. 16], to the resulting ladder yields

$$\pi_q(U) = 0 \quad \text{if} \quad q < n, \quad h_q: \pi_q(U) \approx \pi_q(T) \quad \text{if} \quad q \geq n, \quad (16)$$

$$\pi_q(V) = 0 \quad \text{if} \quad q \geq n, \quad k_q: \pi_q(T) \approx \pi_q(V) \quad \text{if} \quad q < n. \quad (17)$$

Since X is a connected CW -complex which is dominated by Q , X is also dominated by $|S(Q)|_0$. Since γ_1 is onto and W is 0-connected, T is 0-connected and, by (13) and (14), $|S(Q)|_0$ has the homotopy type of $|S(T)|$. It follows from (16) that $(|S(T)|, n)$ has the homotopy type of $|S(U)|$, while (17) implies that $(n, |S(T)|)$ has the homotopy type of $|S(V)|$. Since (X, n) and (n, X) have the homotopy type of CW -complexes, it follows easily that (X, n) is dominated by $|S(U)|$, and (n, X) by $|S(V)|$.

Since W , like Z , has the homotopy type of $|S(Y)|_0$, and since the component of the base-point in a CW -complex is a retract of the complex, by [3; Prop. 2.8] we have

$$\text{cocat } W = \text{cocat } |S(Y)|_0 \leq \text{cocat } |S(Y)| \leq \text{cocat } Y = m.$$

Since W has the homotopy type of a connected CW -complex, the induction hypothesis now implies that $\text{cocat } (W, n) \leq m$ and $\text{cocat } (n, W) \leq m$. By [3; Prop. 2.8] and 1.1 we obtain

$$\text{cocat } |S(U)| \leq \text{cocat } U \leq m + 1, \quad \text{cocat } |S(V)| \leq \text{cocat } V \leq m + 1,$$

and this clearly implies the desired result.

Appendix

(Added in proof)

The inductive arguments used in the proof of 1.4 are easily seen to yield the following more general result:

Let X be a $(p - 1)$ -connected CW -complex, $p \geq 2$. If the set of all integers q for which $\pi_q(X) \neq 0$ is contained in the union of k closed linear intervals, each of length $p - 2$, then $\text{cocat } X \leq k + 1$.

We allow the linear intervals to be degenerate, i.e. to have length 0. The second part of 1.2 now follows as the set $\{q_1, \dots, q_n\}$ is contained in the intervals $[q_j, q_j]$, $j = 1, \dots, n$; Theorem 1.4 follows upon noticing that the integers between p and r are all contained in the intervals

$$[j(p-1) + 1, j(p-1) + p - 1], j = 1, \dots, \left\lfloor \frac{r-1}{p-1} \right\rfloor.$$

Also, the author wishes to acknowledge that a result equivalent to Lemma 2.1 above has been obtained independently and with a different proof by P. J. HILTON as Theorem 3 in his paper "Excision and principal fibrations", *Comment. Math. Helv.* 35 (1961).

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Institute of Mathematics, R.P.R. Academy, Bucharest

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